Elastic backscattering of quantum spin Hall edge modes from Coulomb interactions with nonmagnetic impurities

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We demonstrate that electrostatic interactions between helical electrons at the edge of a quantum spin Hall insulator and a dynamical impurity can induce quasielastic backscattering. Modeling the impurity as a two-level system, we show that transitions between counterpropagating Kramers-degenerate electronic states can occur without breaking time-reversal symmetry, provided that the impurity also undergoes a transition. The associated electrical resistance has a weak temperature dependence down to a nonuniversal temperature scale. Our results extend the range of known backscattering mechanisms in helical edge modes to include scenarios where electron tunneling out of the system is absent.

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I. INTRODUCTION

The quantum spin Hall (QSH) effect [1–4] is a prototypical example of a symmetry protected topological phase [5], featuring helical edge modes that remain gapless and conducting as long as time-reversal symmetry is maintained and the bulk gap stays open. While these edge modes have been directly observed in a variety of solid state systems using photoemission spectroscopy [6], their characteristic conductance properties are found to be much less robust than those of chiral edge modes in the integer quantum Hall effect [7–11].

Aside from perturbations to the edge mode that break time-reversal symmetry—either explicitly or spontaneously [11–13]—a number of mechanisms have been put forward to account for the edge mode resistance seen in experiment. In systems with inhomogeneous doping, the bulk gap may vary with position and even close in certain regions, leading to the formation of metallic charge puddles. If helical electrons can tunnel into these gapless regions, then their protection against backscattering is compromised and the edge becomes resistive [14,15]. This effect is particularly strong for Kramers-degenerate impurities, wherein the resultant magnetic exchange interactions can facilitate *quasielastic* backscattering, i.e., involving only a small energy exchange (of the order of the Kondo temperature) [16–21].

In the above mechanisms, either the bulk gap or the protecting time-reversal symmetry (TRS) are compromised, and so the induced resistance can be strong. In contrast, if the bulk gap and TRS are maintained, the only known sources of resistance involve *inelastic* backscattering, which is less strong at low temperatures due to the need for energy exchange. Electron-electron and electron-phonon mediated backscattering lead to a resistance that is strongly suppressed as the temperature or voltage is decreased [22–27], which is inconsistent with the weak dependence seen in experiment [9–11]. One possible explanation is that the necessary energy is provided by external noise, which gives a weaker temperature dependence [28].

In this article, we identify a mechanism by which helical electrons can undergo quasielastic backscattering that does not involve tunneling into in-gap states or TRS breaking. Electrostatic interactions between helical electrons and a nonmagnetic dynamical impurity lead to transitions between degenerate counterpropagating states, as illustrated in Fig. 1. The resistance profile induced by this scattering process strongly resembles that of a magnetic impurity: for

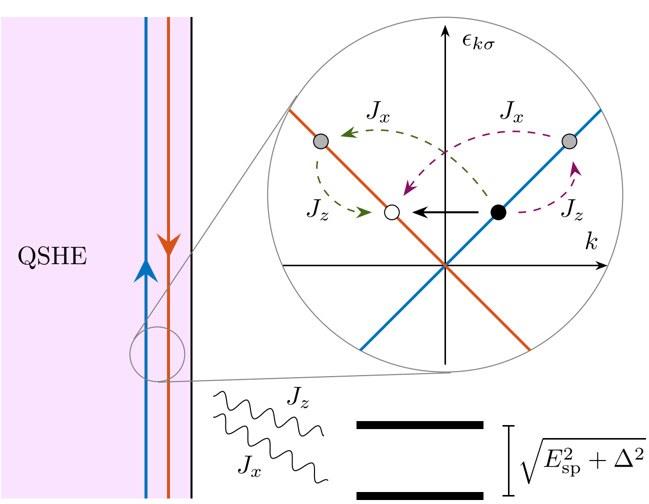

FIG. 1. Quantum spin Hall insulator coupled to an impurity (modeled as a two-level system) via electrostatic interactions [Eq. (3)]. Inset: the bare couplings J_z and J_x [Eq. (5)] combine to induce elastic backscattering between Kramers-degenerate states (black and white dots). The transition can proceed by two paths (green and violet) depending on which perturbation is applied first. Because the impurity pseudospin operators do not commute $[\hat{\sigma}^z, \hat{\sigma}^x] = 2i\hat{\sigma}^y$, destructive interference of these two paths is avoided. During this process, the impurity undergoes a simultaneous transition, thus requiring an energy transfer of $\epsilon := \sqrt{E_{sp}^2 + \Delta^2}$, which can be arbitrarily small.

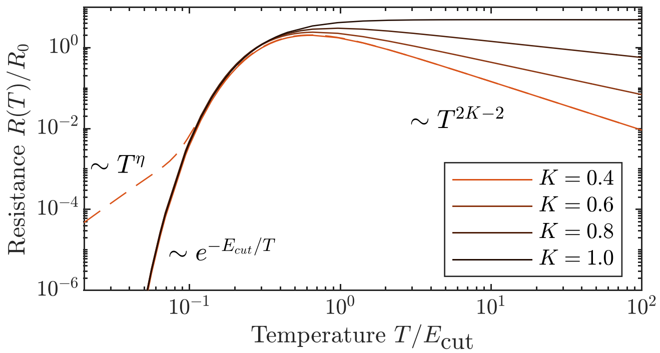

FIG. 2. Resistance of helical edge modes due to interaction with a two-level system impurity, in the regime $\epsilon \gtrsim E_y$ such that Eq. (9) applies with $E_{\rm cut} = \epsilon$, for various values of K. The resistance is plotted in units of $R_0 := (h/e^2) \times y_0^2(E_{\rm cut}/E_{\rm g})^{2K-2}$. For $E_y \gtrsim \epsilon$, the same qualitative form is expected with $E_{\rm cut} = E_y$. The dashed line includes the contributions from inelastic scattering, which are dominant for $T \ll E_{\rm cut}$.

temperatures T above some nonuniversal cutoff $E_{\rm cut}$, the resistance scales inversely with temperature as T^{2K-2}, where $K \leqslant 1$ is the Luttinger parameter (equal to unity for noninteracting electrons; see Fig. 2). This contrasts strongly with previously studied inelastic backscattering mechanisms. At temperatures below $E_{\rm cut}$, the dynamics of the impurity becomes frozen, leaving only the aforementioned processes which scale as T^{η} with $\eta > 0$.

Quasielastic backscattering is possible here because time-reversal symmetry does not act "locally" on the helical edge electrons, but on the composite system plus impurity, as we highlighted in Ref. [29]. Accordingly, transitions between Kramers-degenerate electron states—which would be forbidden by time-reversal symmetry in the absence of any extraneous degrees of freedom—are in fact allowed, provided that the impurity undergoes a simultaneous transition. In essence, the energy scale below which the helical edge modes are protected is not set by the QSH gap, but by the gap of the composite system, which can be arbitrarily small. Our results indicate that the elimination of charge puddles (e.g., by using QSH insulators with larger band gaps) will not necessarily restore conductance quantization.

II. SETUP

The boundary of a two-dimensional topological insulator hosts a single pair of counterpropagating modes in which the direction of motion is correlated with the electron spin. In a clean, dispersionless, noninteracting system in the absence of Rashba spin-orbit coupling, the Hamiltonian can be written as

$$\hat{H}_0 = \sum_{\sigma} \sigma v_F \int dx\, \hat{\psi}^\dagger_\sigma(x) i\partial_x \hat{\psi}_\sigma(x), \tag{1}$$

where $\hat{\psi}^\dagger_\sigma(x)$ creates a fermion with spin $\sigma \in \{+1, -1\}$ at a coordinate x along the edge, and v_F is the velocity of edge electrons. The TRS operator $\hat{T}$ (which is antiunitary $\hat{T}\, i\, \hat{T}^{-1} = -i$) exchanges the two spin species $\hat{T}\hat{\psi}_\sigma(x)\hat{T}^{-1} = \sigma\hat{\psi}_{-\sigma}(x)$ and squares to $\hat{T}^2 = (-1)^{\hat{N}_F}$, where $\hat{N}_F$ is the fermion number operator. Perturbations cannot couple counterpropagating states of the same energy unless TRS is broken explicitly or spontaneously [4]. This protection against elastic backscattering prevents the helical edge modes from being gapped or localized [3].

The simple Hamiltonian $\hat{H}_0$ can be refined by including electron-electron interactions. It then becomes convenient to employ bosonization techniques [30]. Within a fixed $\hat{N}_F$ sector, we write $\hat{\psi}_\sigma(x) = (2\pi\xi)^{-1/2} e^{i\sigma(k_F - \pi/L)x} e^{i\hat{\theta}(x) - i\sigma\hat{\phi}(x)}$, where k_F is the Fermi wavelength, L is the length of the edge, ξ is a short distance cutoff of the order $u/E_{\rm g}$ [16] ($E_{\rm g}$ is the bulk gap, and u is a renormalized velocity), and $\hat{\phi}(x)$, $\hat{\theta}(x)$ are bosonic fields satisfying the commutation relations $[\hat{\phi}(x), \nabla\hat{\theta}(x')] = i\pi\delta(x - x')$. The edge modes are then described by the helical Luttinger liquid (HLL) theory [3]

$$\hat{H}_{\rm HLL} = \frac{u}{2\pi} \int dx\, \frac{1}{K}(\nabla\hat{\phi})^2 + K(\nabla\hat{\theta})^2, \tag{2}$$

where the dimensionless Luttinger parameter K quantifies the strength of interactions ($K < 1$ for repulsive interactions). Stability of the edge mode against spontaneous TRS breaking requires $K > 1/4$ [16]. TRS acts as $\hat{T}\hat{\phi}(x)\hat{T}^{-1} = \hat{\phi}(x) + \pi/2$; $\hat{T}\hat{\theta}(x)\hat{T}^{-1} = -\hat{\theta}(x) + \pi/2$ [31].

Here, we consider generic helical edges without any additional symmetries. Due to Rashba spin-orbit coupling and other related effects, such edges will generically possess nontrivial spin textures in momentum space, i.e., the spin quantization axis depends on wave vector [32,33]. The HLL theory can still be used in this case, but the usual bosonization identity relating bare electron operators and bosonic variables cannot be directly used, since it assumes a fixed spin quantization axis. Nevertheless, the fields $\hat{\phi}(x)$, $\hat{\theta}(x)$ obey the same symmetry properties as before. One should think of Eq. (2) as a long-wavelength fixed point Hamiltonian whose symmetries are inherited from a more complicated microscopic theory.

It is well known that the perfect conductance of these helical edge modes can be compromised if the electrons can tunnel into magnetic impurities [3,16,17], which induces an exchange coupling $J\sum_{\alpha=x,y,z} \hat{S}^\alpha_{\rm el}(x) \otimes \hat{S}^\alpha_{\rm imp}$. The electron spin operators $\hat{S}^\alpha_{\rm el}(x) = \sum_{\sigma\sigma'} \hat{\psi}^\dagger_\sigma [\tau^\alpha]_{\sigma\sigma'} \hat{\psi}_{\sigma'}$ (τ^α are the Pauli matrices) are odd under time reversal [34], and so can induce elastic backscattering even though TRS is preserved overall. We will instead consider electrostatic interactions between the HLL and a nonmagnetic impurity, such that the Hamiltonian only features TRS-even, fermion-number-conserving operators acting on the system. Despite the absence of any tunneling or exchange processes, we will show that this nonmagnetic impurity can still give rise to quasielastic backscattering, leading to a deviation from quantized conductance that is in principle just as strong as a magnetic impurity.

For simplicity and concreteness, we model the impurity as a two-level system (TLS). We discuss possible physical manifestations of such TLSs below. For now, consider the impurity to have two low-energy configurations whose zero point energies differ by $E_{\rm sp}$, with a tunneling matrix element Δ between the two [35]. The Hamiltonian is $\hat{H}_{\rm TLS} = (E_{\rm sp}/2)\hat{\sigma}^z + (\Delta/2)\hat{\sigma}^x$, where $\hat{\sigma}^{x,y,z}$ are the Pauli matrices in the impurity Hilbert space. In this basis, time reversal acts as $\hat{T} = K$, the complex conjugation operator, which forbids a term proportional to $\hat{\sigma}^y$.

Electrostatic density-density interactions between the system and the TLS take the form

$$\hat{H}_{\text{int}} = \int d^2\vec{r}\, \hat{\rho}_{\text{el}}(\vec{r}) \otimes [\hat{\sigma}^x V_x(\vec{r}) + \hat{\sigma}^z V_z(\vec{r})], \tag{3}$$

where $\hat{\rho}_{\text{el}}(\vec{r})$ is the density operator for the bare electrons and $V_{x,z}(\vec{r})$ are arbitrary real functions of the 2D spatial coordinate $\vec{r}$, which we presume to be localized near some point $x = 0$ along the edge. This interaction captures the dependence of both the splitting E_{sp} and tunneling matrix element Δ on the distribution of electrons in the system. No term proportional to $\hat{\sigma}^y$ appears because the charge density operator for the degrees of freedom in the impurity must be TRS invariant.

Note that there is some freedom in choosing the basis for the impurity Hilbert space. However, generically it is not possible to find a basis in which the Hamiltonian only includes diagonal TLS operators. One could pick a coordinate $\vec{r}^*$ and perform a basis transformation using the unitary $\hat{U} = e^{i\theta\hat{\sigma}^y/2}$, with $\theta = \tan^{-1}[V_z(\vec{r}^*)/V_x(\vec{r}^*)]$. The contributions to (3) in the vicinity of $\vec{r} \approx \vec{r}^*$ would then be diagonal, but those further away from $\vec{r}^*$ will not be diagonal, because the ratio $V_z(\vec{r})/V_x(\vec{r})$ will generically vary. If we write $\hat{A}_\alpha = \int d^2\vec{r}\, \hat{\rho}_{\text{el}}(\vec{r}) V_\alpha(\vec{r})$ $(\alpha = x, z)$, then (3) becomes

$$\hat{H}_{\text{int}} = \hat{A}_x \otimes \hat{\sigma}^x + \hat{A}_z \otimes \hat{\sigma}^z. \tag{4}$$

As long as the operators $\hat{A}_x$ and $\hat{A}_z$ are linearly independent, we must consider both terms in the above. This will prove crucial for our subsequent analysis.

For concreteness, from hereon we will work in a basis in which the uncoupled impurity Hamiltonian is diagonal, so we can write $\hat{H}_{\text{TLS}} = \epsilon\hat{\sigma}^z$, where $\epsilon = \sqrt{E_{\text{sp}}^2 + \Delta^2}$.

III. EFFECTIVE LOW-ENERGY THEORY

We now analyze the low-energy properties of the full Hamiltonian $\hat{H}_{\text{tot}} = \hat{H}_{\text{HLL}} + \hat{H}_{\text{TLS}} + \hat{H}_{\text{int}}$. Our strategy is to consider which terms can appear in the bare microscopic Hamiltonian and study how they behave under renormalization.

As mentioned, if the helical modes have nontrivial spin texture then it is unclear how to express Eq. (3) in terms of the bosonic fields $\hat{\phi}(x)$, $\hat{\theta}(x)$. Nevertheless, we can determine which terms will generically arise by considering the symmetry properties of the operators acting on the system $\hat{A}_\alpha$. Evidently, $\hat{A}_\alpha$ are Hermitian, charge-conserving, TRS-invariant operators [36]. Furthermore, since the interaction between the system and the TLS is localized around the point $x = 0$, we can perform a gradient expansion of the bosonic fields about this point (which is well controlled at low energies), leaving only the fields $\hat{\phi}(x)$, $\hat{\theta}(x)$ and their spatial derivatives evaluated at $x = 0$.

There are still infinitely many terms that meet these criteria, but for illustrative purposes we will consider just two,

$$\hat{H}_{\text{int}} = J_z \nabla^2\hat{\phi} \otimes \hat{\sigma}^z + J_x : \nabla\hat{\theta}\, \cos[2\hat{\phi}] : \otimes\, \hat{\sigma}^x + \cdots, \tag{5}$$

with all fields evaluated at $x = 0$. (The colons denote normal ordering with respect to the product of $\nabla\hat{\theta}$ and $\cos[2\hat{\phi}]$.) The coefficients $J_{x,z}$ will depend in some complicated way on the microscopic details of the QSHE system in question as well as the profiles $V_{x,z}(\vec{r})$ in (3), but neither are constrained by time-reversal symmetry. Later, we will consider a specific microscopic model for the impurity and obtain expressions for the bare parameters J_x, J_z.

To provide some intuition, if we were to map these two perturbations back to fermionic operators using the bosonization identity then the first term would describe forward scattering of electrons. The second term corresponds to single-particle backscattering between nondegenerate states, which is allowed because counterpropagating states of different energies are not related by TRS. (See Fig. 1.)

At tree level, both operators in Eq. (5) are RG irrelevant, with scaling dimensions $\Delta_1 = 2$, $\Delta_2 = 1 + K$. Therefore, the deviation from quantized conductance at leading order in $J_{x,z}$ will decrease as the temperature of the system is lowered as T^{2K}. The reason we consider them here is that when perturbative loop corrections are included, the combination of these two operators will generate a new relevant operator,

$$\hat{H}_y = (yu/\xi)\cos[2\hat{\phi}] \otimes \hat{\sigma}^y, \tag{6}$$

where y is a dimensionless coupling constant. The operator $\cos[2\hat{\phi}]$ describes single-particle backscattering of the same kind that would be expected from a magnetic impurity [16], and the factor of $\hat{\sigma}^y$ indicates that the impurity undergoes a simultaneous transition. This is represented by the solid black arrow in Fig. 1.

Although the product $\hat{H}_y$ is invariant under TRS, the individual operator $\cos[2\hat{\phi}]$ is odd under TRS. This coupling is therefore forbidden in the bare theory due to the microscopic symmetries of the electrostatic coupling (3) (4). (Recall that the operators $\hat{A}_\alpha$ acting on the system are TRS invariant.) Nevertheless, the renormalized theory can have a nonzero value of y. Because $\cos[2\hat{\phi}]$ is TRS odd, this gives rise to quasielastic backscattering, which as we will see is not suppressed at low temperatures in the way that inelastic backscattering is.

To be more precise, in the Supplemental Material [37], we study an infinitessimal RG transformation in which the UV cutoff length scale ξ transforms to $\xi \to e^{\delta\ell}\xi$ for some $\delta\ell \ll 1$. The parameter y flows according to

$$\frac{dy}{d\ell} = (1 - K)y - J_x J_z/u^2\xi + \cdots. \tag{7}$$

The irrelevant parameters J_x, J_z also obey their own RG equations, under which they flow towards zero:

$$\frac{dJ_z}{d\ell} = -J_z + \mathcal{O}\big(J_{x,z}^2\big), \tag{8a}$$

$$\frac{dJ_x}{d\ell} = -KJ_x + \mathcal{O}\big(J_{x,z}^2\big). \tag{8b}$$

Even though J_x, J_z eventually drop out of the theory, in the early stages of the flow the second term in (7) drives the system away from the unstable fixed point $y = 0$. Then, since y is RG relevant (or marginal for noninteracting fermions $K = 1$), its value will increase (or stay constant) during the subsequent flow. Specifically, if we solve the above equations using $y = 0$ at $\ell = 0$, we find $y(\ell) = [-KJ_x(0)J_z(0)/2u^2\xi](e^{(1-K)\ell} - e^{-(1+K)\ell})$, which in the late stages of the flow tends towards $y(\ell) \to y_0 e^{(1-K)\ell}$, with $y_0 = -KJ_x(0)J_z(0)/2u^2\xi$. Thus, as long as both $J_x(0)$ and $J_z(0)$ are nonzero, the low-energy theory will feature the emergent relevant operator (6).

While we have focused on the two couplings given in Eq. (5), there are infinitely many more combinations of irrelevant operators that can combine to contribute to y, as represented by the ellipsis in Eq. (7). Thus solving the RG flow equations in full generality is prohibitively hard. However, the behavior of $y(\ell)$ at large ℓ will necessarily be independent of the initial values of these couplings, except for a modification of the multiplicative factor y_0. Thus one can treat y_0 as a single phenomenological parameter capturing the influence of all the other couplings on the long-wavelength physics.

Our derivation of the RG equation (7) (details of which can be found in the Supplemental Material [37]) is a variant of the Anderson-Yuval-Hamann approach to the Kondo model [38]. Because the underlying theory (2) is free, we can compute the operator product expansion of the two terms in Eq. (5), from which the one-loop beta function can be obtained using standard methods (see, e.g., Ref. [39]).

One can develop a more intuitive understanding of how the elastic backscattering term (6) arises by successively applying the two terms in Eq. (5) in a perturbative manner, as illustrated in the inset of Fig. 1 (working in a fermionic representation). An electron undergoes a transition proceeding via an intermediate virtual state, which will be either left or right moving depending on whether the forward- or backscattering term (J_z or J_x, respectively) is applied first. If the operators acting on the impurity were not included in Eq. (5), then the contributions from the two choices of ordering would destructively interfere due to TRS. However, because $\hat{\sigma}^x$ and $\hat{\sigma}^z$ anticommute, an additional relative phase of π between the two contributions is introduced. The result is elastic backscattering accompanied by a transition of the state of the impurity, as represented by the coupling (6).

These considerations imply that, at low energies, the helical modes will be governed by an effective Hamiltonian $\hat{H}_{\rm eff} = \hat{H}_{\rm HLL} + \hat{H}_{\rm TLS} + (y_0 u/\xi)\cos[2\hat{\phi}] \otimes \hat{\sigma}^y$ (keeping only relevant operators).

IV. CONDUCTANCE

Having derived the effective low-energy theory, we can now calculate the resistance induced by the HLL-TLS interactions by adapting the standard derivation for the conductance of a Luttinger liquid in the presence of a static impurity [40]. Details of the calculations can be found in the Supplemental Material [37].

At leading order in y_0, we find a linear dc resistance of

$$\frac{R(T)}{h/e^2} = \frac{\pi^2 y_0^2}{2}\left(\frac{2\pi T}{E_g}\right)^{2K-2} \operatorname{sech}\left(\frac{\epsilon}{2T}\right)\frac{|\Gamma(K+i\epsilon/2\pi T)|^2}{\Gamma(2K)}, \tag{9}$$

where $h = 2\pi\hbar$, $\Gamma(x)$ is the Euler gamma function, and $\epsilon = \sqrt{E_{\rm sp}^2 + \Delta^2}$ is the difference in energy of the eigenvalues of $\hat{H}_{\rm TLS}$. For $T \gtrsim \epsilon$, the resistance scales as a nonpositive power of temperature $R(T) \sim T^{2K-2}$, as could be anticipated from the scaling dimension of y. This is in stark contrast to the temperature dependence of inelastic scattering processes. For example, if we only included the leading-order effects of the terms in (5), we would find a resistance that decreases with temperature as a power law: $R(T) \propto T^{2K}$. The difference is that the operator $\cos[2\hat{\phi}]$ appearing in Eq. (6) can couple counterpropagating helical states of the *same* energy, and so in the noninteracting electron limit $K = 1$ the scattering rate is independent of T.

As T is lowered below ϵ, the impurity becomes frozen in its ground state, and so can no longer efficiently undergo a transition. The contribution (9) edge resistance then becomes thermally activated $R(T) \sim e^{-\epsilon/T}$. This reflects the quasielastic nature of the process described here. Once the impurity is frozen, the dominant sources of resistance will be the previously studied inelastic mechanisms, giving $R(T) \sim T^{\eta}$, with $\eta = \min(2K+2, 8K-2) > 0$ [24]. Note that this scale ϵ is nonuniversal, and so can be arbitrarily small in principle. If the impurity itself is degenerate, e.g., due to some symmetry, then $\epsilon = 0$ and this "frozen" regime is never reached.

Being based on perturbation theory in $\hat{H}_y$, Eq. (9) is valid provided that the renormalized $y(\ell)$ is small at the energy scale of interest. The expression is only invalidated if $T, \epsilon \ll E_y$, where we define $E_y = E_g y_0^{1/(1-K)}$, in which case $y(\ell)$ flows to strong coupling. By analogy to the Kondo effect in helical liquids [16], the impurity and bulk electrons will then hybridize and behave as a composite system with gap $\sim E_y$ and the remaining backscattering processes will be inelastic $R(T) \sim T^{\eta}$. Since we expect y to be perturbatively small for realistic systems [37], it is likely that this regime will arise at unrealistically low temperatures.

It is also possible to obtain expressions for the conductance at finite voltage V (i.e., beyond linear response) and nonzero frequencies ω. Outside of the frozen regime, the conductance will scale as E^{2K-2}, where $E \sim \min(T, V, \omega)$. See the Supplemental Material for details [37].

To conclude, the resistance (plotted in Fig. 2) scales as $R(T) \sim T^{2K-2}$ for T above some nonuniversal cutoff scale $E_{\rm cut} = \max(\epsilon, E_y)$ that depends on the details of the impurity, and so can be arbitrarily small, in principle. This form is in stark contrast with previously studied inelastic backscattering mechanisms, which quickly decrease in magnitude as T is lowered [22–25].

V. MICROSCOPIC MODEL

While the temperature dependence of $R(T)$ is universal [Eq. (9)], the overall scale set by y_0 depends on the specific details of the impurity, and thus so far we have treated it as a phenomenological parameter. Here, we construct a simple microscopic model for an impurity capable of inducing the backscattering processes studied above, which allows us to obtain an order-of-magnitude estimate for y_0 in certain experimentally relevant scenarios. The calculation also helps provide more intuition behind the form of the system-impurity interaction (3), (5).

Consider an electron sitting a distance d above the layer in which the QSH insulator resides. It is trapped in a double quantum well, with two potential minima a distance r apart, oriented at an angle $\pi/2 - \varphi$ from the edge channel direction (Fig. 3). At sufficiently low temperatures, the electron remains in the ground state of one of the two wells. If r is sufficiently small, then it is possible for the electron to tunnel between

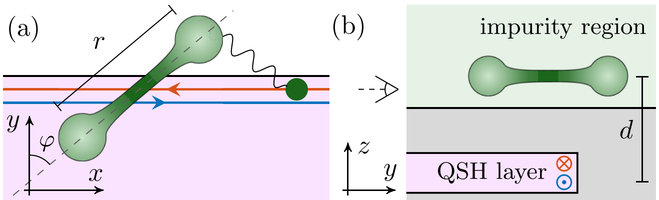

FIG. 3. Possible realization of an impurity capable of inducing quasielastic backscattering, from top view (a) and side view (b). An electron is confined within one of two potential wells (light green lobes) separated by distance r along a direction oriented at an angle φ from the normal to the edge. The impurity is a distance d above the edge mode, where d may be much larger than a tunneling length.

these two ground states; the impurity can then be treated as a TLS with splitting energy $E_{\rm sp}$ and tunneling amplitude Δ determined by the potential energy landscape felt by the impurity electron [35]. This scenario is particularly relevant to HgTe quantum well heterostructures [6,7,41], where dopant ions in the layers above the QSH insulator naturally give rise to local potential minima for electrons at random locations. There, the distance over which the impurity electron can tunnel is set by the effective Bohr radius of the doped layers, which can be on the order of tens of nanometers—comparable to modulation doping distance $d = 10$ nm used in the experiments of Refs. [6,7].

As a helical electron approaches the vicinity of the impurity, electrostatic interactions will modify the potential landscape felt by the impurity electron, and thus $E_{\rm sp}$ and Δ will fluctuate. This interaction can be described by the Hamiltonian (3). While we do not have full knowledge of the double well potential, which would in principle allow us to compute the profiles $V_{x,z}(\vec{r})$, we expect that their orders of magnitude will be set by the scale of Coulomb repulsion $E_C \sim e^2/4\pi\epsilon_0\kappa d$, where κ is the effective dielectric constant. The range over which the profiles drop off will also be set by both d and ξ, since the effective potential may be smeared over the penetration depth of the edge mode in the transverse direction ξ. Since the value $\xi \approx 36$ nm typical of HgTe quantum well experiments [42] is the same order of magnitude as r, d, we treat all length scales as comparable.

In the absence of inversion symmetry, the helical modes have a nontrivial spin-momentum texture [22,32,33], which can be parametrized by a typical wave vector k_0 over which the spin quantization axis varies appreciably. Thus the charge density operator will be coupled to spin degrees of freedom. To convert the interaction Hamiltonian (3) into bosonized variables, we can use an expression for the electron density operator given in Ref. [31]: $\hat{\rho}_{\rm el}(x) \approx \nabla\hat{\phi}(x) + \zeta : \nabla\hat{\theta}(x)\cos[2\hat{\phi}(x)] : + \mathcal{O}(\zeta^2)$, where $\zeta \approx 2k_F\xi^{-1}k_0^{-2}$ is a dimensionless parameter that characterizes the strength of Rashba spin-orbit coupling. Upon performing a gradient expansion, we see that the terms (5) do indeed arise, along with others that we include explicitly in the Supplemental Material [37].

Combining the various parameters, we find a resistance of the order $R(T) \sim \alpha_u^4\zeta^2(E_g/2\pi T)^{2-2K}$, where $\alpha_u = e^2/4\pi\epsilon_0\hbar u$ is a dimensionless parameter characterizing the strength of Coulomb interactions in the edge mode. For HgTe wells, $\alpha_u \approx 0.31$ [14]. A more detailed description of this estimate and the various experimental parameters used below are given in the Supplemental Material [37].

Overall, we find that a single impurity induces a rather small resistance due to the weak spin-momentum texture found in the edge modes of HgTe quantum wells. Based on an estimation of the number of dopant ions present in a given experiment, we find a total resistance of around one percent of h/e^2 for a $L_{\rm edge} = 1\ \mu$m long edge, increasing linearly with $L_{\rm edge}$. Therefore, it is unlikely that the present microscopic model can account for the resistance seen in experiment, which is typically larger [6,7] and so more likely to be due to backscattering from charge puddles [8]. Nevertheless, our calculation highlights an important point of principle: even if charge puddles were eliminated from the sample, one should not expect to see quantized conductance at low temperatures, since the helical modes are not protected against quasielastic backscattering in the presence of dynamical impurities.

There may be related effects not captured by the present calculation that could lead to a larger resistance. First, there may be dynamical impurities of a different origin that are more numerous. For instance, the impurities could have the same origin as the effective two-level systems used to account for the ubiquity of "$1/f$ noise" in solid state systems [43,44]. Secondly, we note that our analysis only captures a subset of the possible backscattering processes that can arise—specifically those in which the intermediate virtual state lies within the edge channel (gray circles in Fig. 1). There will also be transitions between counterpropagating modes that proceed via bulk excited states [29], which can have spin quantization axes that deviate appreciably from that of the edge mode, thanks to the strong spin-orbit coupling that is necessarily present in topological insulators. These processes could be studied in detail with more precise knowledge of the bulk band structure. Since such transitions are still quasielastic, we expect that the resistance will exhibit a similar temperature dependence and will not be suppressed in the same way as the intra-edge-mode processes studied here.

The expression for $R(T)$ quoted above will be modified if there is a hierarchy of length scales in the problem. For instance, if d is much larger than ξ and r, then the coupling strengths $J_{x,z}$ will depend inversely on d, giving an overall d^{-4} dependence. Thus we expect an algebraic dependence on the system-impurity distance, which contrasts with tunneling into in-gap states, whose magnitude decays exponentially with distance as $e^{-d/\ell_{\rm tun}}$, where $\ell_{\rm tun}$ is the tunneling length.

VI. CONCLUSIONS AND OUTLOOK

We have shown that electrostatic interactions between a dynamic impurity and helical electrons facilitate quasielastic backscattering, leading to an edge mode resistance that does not decrease as T is lowered, in contrast to previously studied inelastic backscattering mechanisms. The resistance increases (or remains constant for $K = 1$) as the temperature is lowered, following a power law T^{2K-2} down to a nonuniversal cutoff scale $E_{\rm cut} = \max(\epsilon, E_y)$. At temperatures below $E_{\rm cut}$, the TLS becomes frozen either by its own dynamics or by interactions with bulk electrons, and the resistance then scales as T^η (see Fig. 2).

The topological protection of helical edge modes is usually attributed to the fact that TRS-invariant Hamiltonian perturbations $\hat{H}$ cannot couple counterpropagating Kramers-degenerate states $|\psi\rangle$ and $|\bar{\psi}\rangle$, since $\langle\bar{\psi}|\hat{H}|\psi\rangle = 0$. However, such an argument only applies to situations where the quantum spin Hall insulator is isolated. Here, the system is coupled to additional degrees of freedom that make up the impurity. TRS then applies at the level of the composite system plus impurity, rather than just the system, and so transitions from $|\psi\rangle$ to $|\bar{\psi}\rangle$ are not forbidden, provided the environment undergoes a simultaneous transition. (See Ref. [29] for a similar example in 1D symmetry-protected topological phases.)

This same reasoning can be used to understand how quasielastic backscattering can occur in scenarios where helical electrons are tunnel coupled to magnetic impurities [16] or charge puddles tuned to resonance [14]. However, a key aspect that distinguishes the mechanism described here from those previous results is that the impurities need not be tunnel coupled to the edge mode. As such, the impurities that can contribute to the resistance are of a much broader class and do not need to be within a tunneling length ℓ_{tun} of the sample.

Unlike previously studied backscattering mechanisms involving tunneling out of the helical edge, whose strength decays exponentially with system-impurity distance d, here the dependence on d is algebraic due to the long-ranged nature of the Coulomb force [37]. This implies that the presence of mobile electrons in regions relatively far from the sample may still suppress conductance quantization. Therefore, even if the contribution from a single impurity is small, it is likely that many impurities will be involved in backscattering, potentially leading to a significant cumulative resistance (similar to Ref. [28]).

While we have adopted a two-level system model to describe the impurity, our arguments naturally generalize to multilevel impurities. The low-energy effective theory will generically contain all operators of the form (6) in which $\hat{\sigma}^y$ is replaced with other Hermitian, TRS-odd operators. The crossover scale ϵ below which the impurity dynamics is frozen is again set by the energy difference between the two lowest eigenstates. In certain scenarios, ϵ may be small or zero by symmetry, in which case the resistance persists down to correspondingly low temperatures. For instance, the impurity may be formed of an odd number of electrons, which gives $\epsilon = 0$ by Kramers theorem. Therefore, in regimes where ϵ is sufficiently small, our results are consistent with the weak temperature dependence of the edge mode resistance seen in experiment [10].

Finally, for the same general reasons, analogous effects should arise in other systems featuring topological modes where TRS plays an important rôle, e.g., 3D topological insulators and Dirac semimetals.

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