Bosonization study of a generalized statistics model with four Fermi points

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We study a one-dimensional lattice model of generalized statistics in which particles have next-nearest-neighbor hopping between sites which depends on the occupation number at the intermediate site and a statistical parameter ϕ. The model breaks parity and time-reversal symmetries and has four-fermion interactions if $\phi \neq 0$. We first analyze the model using mean field theory and find that there are four Fermi points whose locations depend on ϕ and the filling η. We then study the modes near the Fermi points using the technique of bosonization. Based on the quadratic terms in the bosonized Hamiltonian, we find that the low-energy modes form two decoupled Tomonaga-Luttinger liquids with different values of the Luttinger parameters which depend on ϕ and η; further, the right- and left-moving modes of each system have different velocities. A study of the scaling dimensions of the cosine terms in the Hamiltonian indicates that the terms appearing in one of the Tomonaga-Luttinger liquids will flow under the renormalization group and the system may reach a nontrivial fixed point in the long distance limit. We examine the scaling dimensions of various charge density and superconducting order parameters to find which of them is the most relevant for different values of ϕ and η. Finally, we look at two-particle bound states that appear in this system and discuss their possible relevance to the properties of the system in the thermodynamic limit. Our work shows that the low-energy properties of this model of generalized statistics have a rich structure as a function of ϕ and η.

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I. INTRODUCTION

The possibility of identical particles having generalized statistics in one dimension has been studied extensively over many years. Such generalizations can be introduced in many different ways, for instance, by modifying the conditions on the wave function and its derivative at the points when two of the particles have the same coordinate, modifying the commutation relations between the creation and annihilation operators in a second-quantized formalism, or modifying the form of the exclusion principle [1–15]. Several theoretical proposals have been made for realizing generalized statistics in one dimension [16–20].

A recent paper has studied a model of pseudofermions on a one-dimensional lattice in which the second quantized operators have a generalized statistics governed by a parameter ϕ [13]. The model has both nearest- and next-nearest-neighbor hoppings t_1 and t_2, and the latter is sensitive to ϕ. At half-filling, it has a rich phase diagram as a function of t_1/t_2 and ϕ. The model has two Fermi points when $|t_1/t_2| > 2$ and four Fermi points when $|t_1/t_2| < 2$, with a Lifshitz transition occurring between the two phases at $|t_1/t_2| = 2$. The phase with two Fermi points has been studied in detail using bosonization [13]. However, the phase with four Fermi points is more difficult to study as it requires the diagonalization of a model with two right-moving and two left-moving modes. In this paper, we aim to analyze this phase in detail for arbitrary values of the filling.

The plan of this paper is as follows. In Sec. II, we introduce our model of pseudofermions with generalized statistics. In order to focus on the phase with four Fermi points, we consider only next-nearest-neighbor hoppings which have a phase which depends on the particle number on the intermediate site and a parameter ϕ. In Sec. III, we analyze the model using mean field theory. This enables us to find the locations of the four Fermi points as a function of ϕ and the filling which is governed by a parameter η. In Sec. IV, we use the bosonization method to study the modes close to the Fermi points. We find that the bosonized Hamiltonian has terms which are quadratic in the bosonic fields and terms which involve cosines of those fields. We diagonalize the quadratic part of the Hamiltonian, thereby finding that the model consists of two decoupled Tomonaga-Luttinger liquids with separate Luttinger parameters K_1 and K_2 and velocities of right- and left-moving modes. We then calculate the scaling dimensions of the cosine terms and discuss what these may imply about the long-distance properties of the model. We also find the scaling dimensions of various charge density and superconducting order parameters to determine which of them is likely to dominate the long-distance properties. In Sec. V, we study a system with only two particles and show that this has both continuum and bound states. We examine the implication of the bound states for the properties of the system with a large number of particles. In Sec. VI, we summarize our results, point out some directions for future studies, and mention possible realizations of our model. In the Appendixes, we discuss some technical details like a Bogoliubov transformation for bosonic fields with unequal right- and left-moving velocities and a nonlocal mapping between models with ϕ and $\pi + \phi$.

II. GENERALIZED STATISTICS IN ONE DIMENSION, HAMILTONIAN, AND SYMMETRIES

In this section, we will study a lattice model for generalized statistics which was introduced in Ref. [13]. The generalized algebra of creation and annihilation operators of pseudofermions on sites j and k is given by

$$\begin{aligned} a_j a_k \,+\, a_k a_j e^{i\phi \mathrm{sgn}(k-j)} &= 0,\\ a_j a_k^\dagger \,+\, a_k^\dagger a_j e^{-i\phi \mathrm{sgn}(k-j)} &= \delta_{jk},\\ [N_j, a_k] &= -\,\delta_{jk}\, a_k,\\ [N_j, a_k^\dagger] &= \delta_{jk}\, a_k^\dagger, \end{aligned} \tag{1}$$

where $N_j = a_j^\dagger a_j$ is the occupation number of pseudofermions on site j. The definition sgn(0) = 0 generates the algebra of pseudofermions for $j = k$ which is consistent with the algebra of usual fermions. In contrast, the algebra for $j \neq k$ is different and it can be tuned from ordinary fermions to hard core bosons by tuning the statistical phase from $\phi = 0$ to $\phi = \pi$.

It is clear from Eq. (1) that changing $\phi \to \phi + 2\pi$ makes no difference. Hence, it is enough to study values of ϕ lying in the range $[-\pi, \pi]$. We also see that the system remains unchanged if we change $\phi \to -\phi$ and do a parity transformation $j \to -j$ for all j. We will therefore only consider the range $0 \leqslant \phi \leqslant \pi$ in this paper.

A. Hamiltonian

We consider the following Hamiltonian for a model of pseudofermions,

$$H = -\sum_j \,[(t_1 a_j^\dagger a_{j+1} + t_2 a_j^\dagger a_{j+2} + \mathrm{H.c.}) + \mu a_j^\dagger a_j], \tag{2}$$

where t_1 and t_2 are the nearest- and next-nearest-neighbor hopping amplitudes respectively, and μ is the chemical potential. (Throughout this paper, we will set both $\hbar$ and the lattice spacing a to unity.) We can map this to a Hamiltonian of ordinary (spinless) fermions by the fractional Jordan-Wigner transformation,

$$c_j = K_j a_j, \quad c_j^\dagger = a_j^\dagger K_j^\dagger, \quad \text{where} \quad K_j = e^{-i\phi \sum_{k<j} n_k}, \tag{3}$$

and c_j and $c_j^\dagger$ are the creation and annihilation operators of fermions with the usual anticommuting algebra. Equation (2) is then mapped into a Hamiltonian of ordinary fermions

$$H = -\sum_j [(t_1 c_j^\dagger c_{j+1} + t_2 e^{i\phi n_{j+1}} c_j^\dagger c_{j+2} + \mathrm{H.c.}) + \mu c_j^\dagger c_j], \tag{4}$$

where $n_j = c_j^\dagger c_j$. We note that the number operator N_j for pseudofermions is mapped to the number operator of the ordinary fermions n_j by this transformation. The first term, nearest-neighbor hopping with amplitude t_1, remains unaffected by the phase ϕ. However, the next-nearest-neighbor hopping carries the information of the statistical phase: The fermions hop with a phase which 0 if the intermediate phase is empty and is $\pm\phi$ if the intermediate site is filled and the hopping is to occurs to the left (right) respectively. This dependence of the next-nearest-neighbor hopping on the statistical phase ϕ makes it evident that a nonzero finite t_2 is necessary to obtain nontrivial phases in this model. This motivates us to explore the limit where we have a finite t_2, but t_1 is set equal to zero. We will therefore consider the Hamiltonian

$$H = -\sum_j \,[(t_2 e^{i\phi n_{j+1}} c_j^\dagger c_{j+2} + \mathrm{H.c.}) + \mu c_j^\dagger c_j]. \tag{5}$$

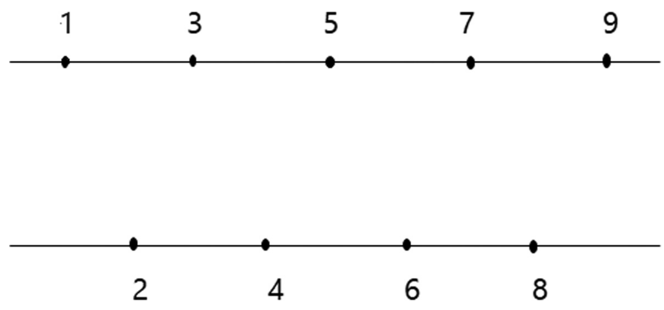


FIG. 1. Schematic picture of the system showing the two sublattices corresponding to odd and even values of the site label.

Note that Eq. (5) remains invariant under $t_2 \to -t_2$ since we can change the sign of those terms by carrying out the transformation

$$c_j \to e^{ij\pi/2} c_j \;\text{ and }\; c_j^\dagger \to e^{-ij\pi/2} c_j^\dagger. \tag{6}$$

A schematic picture of our model is shown in Fig. 1. Hopping only occurs between nearest-neighbor sites on the same sublattice, corresponding to either odd or even values of j; the hopping amplitude depends on the occupation number of the intermediate site which belongs to the other sublattice. We note here that a system with $t_1 = 0$ can be physically realized if the sublattices are replaced by the two spin components of a spin-1/2 particle; such a system naturally has $t_1 = 0$ if the hopping conserves the spin component. A system similar to this has been experimentally studied in Ref. [21].

We will now do the following transformation on the creation and annihilation operators,

$$c_j \;\to\; c_j e^{-ij\phi/4}, \quad c_j^\dagger \;\to\; c_j^\dagger e^{ij\phi/4}. \tag{7}$$

Then Eq. (5) takes the form

$$H = -\sum_j \,[(t_2 e^{i\phi(n_{j+1}-1/2)} c_j^\dagger c_{j+2} + \mathrm{H.c.}) + \mu c_j^\dagger c_j]. \tag{8}$$

This is the Hamiltonian that we will study in the rest of this paper. The choice of the term $n_{j+1} - 1/2$, rather than just n_{j+1}, in the phase is motivated by a particle-hole transformation which will be discussed later.

Note that the number of particles on each sublattice (either even- or odd-numbered sites) is a conserved quantity since the hopping only occurs within each sublattice separately.

B. Symmetries

Next, we will the symmetries of our model. First, we examine how the Hamiltonian in Eq. (8) behaves under a particle-hole transformation. To discuss that, we add a constant to turn the Hamiltonian into

$$H = -t_2 \sum_j [e^{i\phi(n_{j+1}-1/2)} c_j^\dagger c_{j+2} + \text{H.c.}] -\mu \sum_j [c_j^\dagger c_j - 1/2]. \tag{9}$$

Under a particle-hole transformation, we have

$$c_j \to c_j^\dagger \quad \text{and} \quad c_j^\dagger \to c_j. \tag{10}$$

As a result of this transformation, we find that $n_j - 1/2 \to -(n_j - 1/2)$. The Hamiltonian Eq. (9) then flips sign and we obtain

$$H = t_2 \sum_j [e^{i\phi(n_{j+1}-1/2)} c_j^\dagger c_{j+2} + \text{H.c.}] +\mu \sum_j [c_j^\dagger c_j - 1/2]. \tag{11}$$

We then carry out the transformation given in Eq. (6) to change the Hamiltonian in Eq. (11) to

$$H = -t_2 \sum_j [e^{i\phi(n_{j+1}-1/2)} c_j^\dagger c_{j+2} + \text{H.c.}] +\mu \sum_j [c_j^\dagger c_j - 1/2]. \tag{12}$$

Comparing Eqs. (9) and (12), we see that the Hamiltonian remains invariant under a particle-hole transformation provided that we also change $\mu \to -\mu$. (We note that the Hamiltonian has this invariance only if $t_1 = 0$.)

We will now discuss parity (P) and time-reversal (T) transformations. Under P, the creation and annihilation operators transform as

$$c_j \to c_{-j}, \quad c_j^\dagger \to c_{-j}^\dagger. \tag{13}$$

The Hamiltonian in Eq. (8) then becomes

$$H = -\sum_j [t_2 e^{i\phi(n_{-(j+1)}-1/2)} c_{-j}^\dagger c_{-(j+2)} + \text{H.c.}) + \mu c_{-j}^\dagger c_{-j}], \tag{14}$$

which can be written as

$$H = -\sum_j [t_2 (e^{i\phi(n_{j+1}-1/2)} c_{j+2}^\dagger c_j + \text{H.c.}) + \mu c_j^\dagger c_j]. \tag{15}$$

Thus, the Hamiltonian in Eq. (8) is not invariant under P, unless we also flip $\phi \to -\phi$. Similarly, under time-reversal T, we complex conjugate the Hamiltonian, which implies that Eq. (8) transforms into Eq. (15). Hence, the Hamiltonian is not invariant under T. However, the Hamiltonian is invariant under PT.

III. MEAN FIELD THEORY

We begin our discussion by considering the special case $\phi = 0$ which describes a system of noninteracting fermions. The energy-momentum dispersion is then given by

$$E_k = -2t_2 \cos(2k) - \mu, \tag{16}$$

where k lies in the range $[-\pi, \pi]$. We see that this system has four Fermi points if the chemical potential lies in the range $-2t_2 < \mu < 2t_2$. For $\mu = 0$, the Fermi points lie at $k = \pm\pi/4$ and $\pm 3\pi/4$.

We now discuss a mean field treatment for general ϕ to include the effect of the statistical interaction. In the rest of this paper, we will set $t_2 = -1$ for convenience. The Hamiltonian is then

$$H = \sum_j [e^{i\phi(n_{j+1}-1/2)} c_j^\dagger c_{j+2} + \text{H.c.} - \mu c_j^\dagger c_j]. \tag{17}$$

Now, the exponential factor can be written in a more convenient form by noting that n_j can only take the values zero or 1. Hence,

$$\begin{aligned} &(n_j - 1/2)^p = (1/2)^p, \quad \text{if } p \text{ is even} \\ &(n_j - 1/2)^p = (1/2)^{p-1}(n_j - 1/2) \text{ if } p \text{ is odd.} \end{aligned} \tag{18}$$

The phase factor can therefore be written as

$$e^{i\phi(n_j-1/2)} = \cos(\phi/2) + 2i\sin(\phi/2)(n_j - 1/2). \tag{19}$$

The Hamiltonian in Eq. (17) can now be written as the sum of a noninteracting part H_0 (which is quadratic in the fermion operators) and an interacting part H_{int} (quartic),

$$\begin{aligned} H &= H_0 + H_{\text{int}}, \\ H_0 &= \sum_j [\cos(\phi/2)(c_j^\dagger c_{j+2} + c_{j+2}^\dagger c_j) \\ &\quad - i\sin(\phi/2)(c_j^\dagger c_{j+2} - c_{j+2}^\dagger c_j) - \mu c_j^\dagger c_j], \\ H_{\text{int}} &= 2i\sin(\phi/2)\sum_j n_{j+1}(c_j^\dagger c_{j+2} - c_{j+2}^\dagger c_j). \end{aligned} \tag{20}$$

Following a mean field treatment, the interacting part becomes

$$\begin{aligned} &\sum_j n_{j+1}(c_j^\dagger c_{j+2} - c_{j+2}^\dagger c_j) \\ &\to \sum_j [\langle n_{j+1}\rangle(c_j^\dagger c_{j+2} - c_{j+2}^\dagger c_j) + \langle c_i^\dagger c_{j+2} - c_{j+2}^\dagger c_j\rangle c_j^\dagger c_j]. \end{aligned} \tag{21}$$

In the first term in Eq. (21), we replace $\langle n_{j+1}\rangle \to 1/2 + \eta$, where η denotes the deviation from half-filling and lies in the range $[-1/2, 1/2]$, where $\eta = -1/2$ and $1/2$ correspond to a completely empty and completely filled band respectively. The second term in Eq. (21) corresponds to a shift in the chemical potential. Its effect can be absorbed by introducing a new chemical potential μ'

$$2i\sin(\phi/2)\langle c_j^\dagger c_{j+2} - c_{j+2}^\dagger c_j\rangle - \mu = -\mu'. \tag{22}$$

[Note that in the mean field treatment, we are setting expectation values of nearest-neighbor terms $\langle c_j^\dagger c_{j\pm1}\rangle = 0$. This is because we are analyzing a model with nearest-neighbor hopping $t_1 = 0$. Hence the number of fermions on the sublattices corresponding to even and odd values of j are separately conserved; this means that $\langle c_j^\dagger c_{j\pm1}\rangle = 0$ in any state independently of the mean field approximation.] The final form of the mean field Hamiltonian is given by

$$\begin{aligned} H_{\rm MF} = \sum_j \ & [\alpha(c_j^\dagger c_{j+2} + c_{j+2}^\dagger c_j) \\ & + i\beta(c_j^\dagger c_{j+2} - c_{j+2}^\dagger c_j) - \mu' n_j], \\ \alpha &= \cos(\phi/2), \\ \beta &= 2\eta\sin(\phi/2). \end{aligned} \tag{23}$$

Transforming to momentum space, we find that the dispersion for the mean field Hamiltonian in Fourier space is given by

$$E_k = 2\alpha\cos(2k) - 2\beta\sin(2k) - \mu'. \tag{24}$$

We now see that the mean field Hamiltonian in Eq. (23) again has four Fermi points for any value of ϕ, provided that μ' lies in the range $-2\sqrt{\alpha^2+\beta^2} < \mu' < 2\sqrt{\alpha^2+\beta^2}$. We denote the Fermi points by k_1', k_2', k_3', and k_4'. The filling $1/2+\eta$ leads to the following condition on the Fermi points,

$$\frac{k_1' - k_2' + k_3' - k_4'}{2\pi} = \frac{1}{2} + \eta. \tag{25}$$

The Fermi points can be calculated using Eqs. (24) and (25). We find that

$$\begin{aligned} k_1' &= \frac{3\pi}{4} + \frac{\pi\eta}{2} - \frac{\theta}{2}, \\ k_2' &= \frac{\pi}{4} - \frac{\pi\eta}{2} - \frac{\theta}{2}, \\ k_3' &= -\frac{\pi}{4} + \frac{\pi\eta}{2} - \frac{\theta}{2}, \\ k_4' &= -\frac{3\pi}{4} - \frac{\pi\eta}{2} - \frac{\theta}{2}, \end{aligned} \tag{26}$$

where θ and μ' are given by

$$\begin{aligned} \theta &= \tan^{-1}(\beta/\alpha) \ = \ \tan^{-1}(2\eta\tan(\phi/2)), \\ \mu' &= 2\sqrt{\alpha^2+\beta^2}\sin(\pi\eta). \end{aligned} \tag{27}$$

Note that all the Fermi points shift by the same amount given by $\theta/2$. A plot of the dispersion in Eq. (24) along with the four Fermi points k_1', k_2', k_3', and k_4' from right to left is shown in Fig. 2 for $\phi = \pi/2$ and $\eta = 0.25$.

We can check if any neighboring pair of Fermi points can cross each other as the filling η and the statistical phase ϕ are varied. Let us discuss this crossing with respect to the Fermi point at k_1'. Equation (26) implies that the momentum difference $|k_1' - k_2'| = \pi/2 + \pi\eta$ and $|k_1' - k_4'| = \pi/2 - \pi\eta$ (to calculate the latter we have used the 2π periodicity of k). Since η lies in the range $[-1/2, 1/2]$, we see that k_1' and k_2' can at most touch each other at $\eta = -1/2$ but not cross. Similarly, k_1' and k_4' can at most touch at $\eta = 1/2$ but not cross.

An important point to note from Eq. (26) is that

$$k_1' - k_3' = k_2' - k_4' = \pi \tag{28}$$

independently of η and ϕ. This is due to the fact that the Hamiltonian in Eq. (8) remains invariant under the transformation

$$c_j \to (-1)^j c_j, \quad c_j^\dagger \to (-1)^j c_j^\dagger. \tag{29}$$

Since $(-1)^j = e^{ij\pi}$, Eq. (29) corresponds to shifting the momentum $k \to k + \pi$. As a result of this invariance, each Fermi

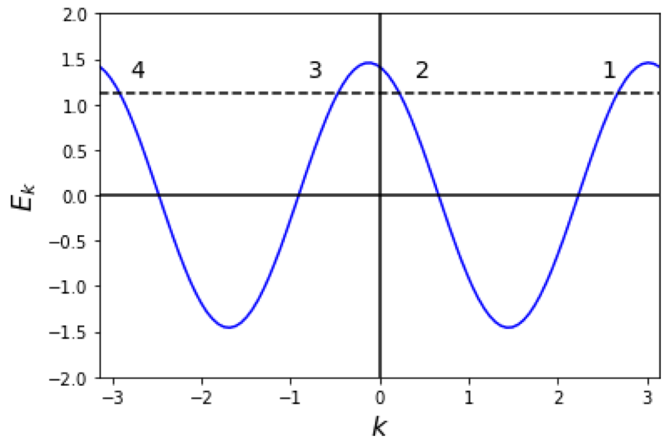


FIG. 2. Plot of the energy-momentum dispersion for $\phi = \pi/2$ and $\eta = 0.25$. The four Fermi points are labeled 1, 2, 3, and 4 from right to left, and the dotted line shows the value of μ'.

point is accompanied by another Fermi point with a momentum difference π. [This symmetry of our model will be lost if we turn on a nearest-neighbor hopping.]

The Fermi velocities dE_k/dk at the four Fermi points can be calculated in terms of α and β using Eqs. (24), (26), and (27),

$$\begin{aligned} v_{k_1'} &= v_{k_3'} = v, \\ v_{k_2'} &= v_{k_4'} = -v, \\ \text{where} \quad v &= 4\sqrt{\alpha^2+\beta^2}\cos(\pi\eta). \end{aligned} \tag{30}$$

Note that v depends on both ϕ and η and it vanishes at the limits $\eta = \pm 1/2$. Equation (30) implies that fermions near both k_1' and k_3' are right-moving while those near k_2' and k_4' are left-moving.

IV. BOSONIZATION

We will now systematically develop a theory of fluctuations about the mean field theory using the technique of bosonization [22–26]. Bosonization involves mapping noninteracting fermionic systems to systems of noninteracting bosons in one dimension. It has been widely used to construct the low-energy theory of one-dimensional systems with two Fermi points, but our model requires us to use it for the case of four Fermi points.

We begin by linearizing the spectrum near the four Fermi points in Eq. (26) to obtain a low-energy effective description of the fluctuations. We write the fermionic annihilation operators as

$$\begin{aligned} c_j = e^{ik_1'j}\,\psi_1(j) \ + \ e^{ik_2'j}\,\psi_2(j) \\ + \ e^{ik_3'j}\,\psi_3(j) \ + \ e^{ik_4'j}\,\psi_4(j), \end{aligned} \tag{31}$$

where k_1', k_2', k_3', and k_4' are the four Fermi points, and ψ_1, ψ_2, ψ_3, and ψ_4 are the slowly varying fields with momentum components lying near those points. We now rewrite the full Hamiltonian $:H_{\rm MF} + H_{\rm int}:$ (where :: denotes normal ordering) in terms of these slowly varying fields. We consider the mean field part first. Using Eq. (23) and using the Taylor expansion $\psi(j+2) = \psi(j) + 2\partial_x\psi(j) +$ higher order terms, the mean field Hamiltonian is given by

$$\begin{aligned} H = 4i \int dx\, ([\alpha \sin(2k_1') + \beta \cos(2k_1')]\, \psi_1^\dagger \partial_x \psi_1 \\ + [\alpha \sin(2k_2') + \beta \cos(2k_2')]\, \psi_2^\dagger \partial_x \psi_2 \\ + [\alpha \sin(2k_3') + \beta \cos(2k_3')]\, \psi_3^\dagger \partial_x \psi_3 \\ + [\alpha \sin(2k_4') + \beta \cos(2k_4')]\, \psi_4^\dagger \partial_x \psi_4 \\ + [\alpha \cos(2k_1') - \beta \sin(2k_1') - \mu']\, \psi_1^\dagger \psi_1 \\ + [\alpha \cos(2k_2') - \beta \sin(2k_2') - \mu']\, \psi_2^\dagger \psi_2 \\ + [\alpha \cos(2k_3') - \beta \sin(2k_3') - \mu']\, \psi_3^\dagger \psi_3 \\ + [\alpha \cos(2k_4') - \beta \sin(2k_4') - \mu']\, \psi_4^\dagger \psi_4). \end{aligned} \quad (32)$$

This can be simplified by using Eqs. (26), (27), and (30). The Hamiltonian in Eq. (32) then reduces to

$$\begin{aligned} H_{\rm MF} = -iv \int dx\, [\psi_1^\dagger \partial_x \psi_1 \;-\; \psi_2^\dagger \partial_x \psi_2 \\ + \psi_3^\dagger \partial_x \psi_3 \;-\; \psi_4^\dagger \partial_x \psi_4]. \end{aligned} \quad (33)$$

[Note that the coefficients of $\psi_1^\dagger \psi_1$, $\psi_2^\dagger \psi_2$, $\psi_3^\dagger \psi_3$, and $\psi_4^\dagger \psi_4$ in Eq. (32) vanish due to Eqs. (26) and (27)].

The Hamiltonian in Eq. (33) can be bosonized using the usual rules of bosonization. We use the following convention for the mapping between the fermionic and bosonic fields, $\psi_j(x)$ and $\phi_j(x)$,

$$\begin{aligned} \psi_1 = F_1 \frac{e^{-i2\sqrt{\pi}\phi_1}}{\sqrt{2\pi\alpha}}, \quad \psi_2 = F_2 \frac{e^{i2\sqrt{\pi}\phi_2}}{\sqrt{2\pi\alpha}}, \\ \psi_3 = F_3 \frac{e^{-i2\sqrt{\pi}\phi_3}}{\sqrt{2\pi\alpha}}, \quad \psi_4 = F_4 \frac{e^{i2\sqrt{\pi}\phi_4}}{\sqrt{2\pi\alpha}}, \end{aligned} \quad (34)$$

where α is a short distance cutoff and F_j ($j = 1, 2, 3, 4$) are the Klein factors which ensure the correct anticommutation relations between the fermionic operators. They have the following properties:

$$\begin{aligned} F_j^\dagger F_j = F_j F_j^\dagger = 1 \quad \text{for} \quad j = 1, 2, 3, 4, \\ \{F_j, F_k\} = \{F_j, F_k^\dagger\} = 0, \quad \text{for} \quad j \neq k. \end{aligned} \quad (35)$$

Using bosonization, we find that Eq. (33) turns into

$$\begin{aligned} H_{\rm MF} = v \int dx\, [(\partial_x \phi_1)^2 \;+\; (\partial_x \phi_2)^2 \\ + (\partial_x \phi_3)^2 \;+\; (\partial_x \phi_4)^2]. \end{aligned} \quad (36)$$

We will now bosonize the interacting part of the Hamiltonian in Eq. (20), namely, $: H_{\rm int} :$. The bosonization of this part is more complicated since there are many nonoscillatory terms possible with four fermionic fields. (We will ignore all the rapidly oscillating terms since they average to zero when we integrate over x.) To examine these terms systematically, we will divide them into two categories, momentum-conserving terms where the change in momentum $\Delta k = 0$, and umklapp terms where $\Delta k = 2\pi$.

We will consider the momentum-conserving terms first. These terms can be further separated into two groups, interaction terms which involve only one Fermi point (called diagonal density-density interactions below) and interaction terms which involve two Fermi points (called off-diagonal density-density interactions). Using Eqs. (31) and (34), we get the following diagonal density-density interactions

$$\begin{aligned} H_{\rm diag} \;=\; 4 \sin(\phi/2) \int dx \big[\big(\rho_1^2 + \rho_3^2\big) \cos(\pi\eta - \theta) \\ - \big(\rho_2^2 + \rho_4^2\big) \cos(\pi\eta + \theta)\big], \end{aligned} \quad (37)$$

where $\rho_j = \psi_j^\dagger \psi_j$ ($j = 1, 2, 3, 4$), is the density operator in terms of fermionic fields. This Hamiltonian can be bosonized using the relation between the fermionic and bosonic fields

$$\rho_j = -\frac{\partial_x \phi_j}{\sqrt{\pi}}, \quad \text{where} \quad j = 1, 2, 3, 4. \quad (38)$$

Equation (37) therefore has the bosonic form

$$\begin{aligned} H_{\rm diag} = \frac{4 \sin(\phi/2)}{\pi} \int dx [((\partial_x \phi_1)^2 + (\partial_x \phi_3)^2) \cos(\pi\eta - \theta) \\ - ((\partial_x \phi_2)^2 + (\partial_x \phi_4)^2) \cos(\pi\eta + \theta)]. \end{aligned} \quad (39)$$

Next we consider the off-diagonal density-density interactions which involve two different fields. There are six terms possible which are given by $\rho_1\rho_2$, $\rho_3\rho_4$, $\rho_1\rho_3$, $\rho_2\rho_3$, $\rho_1\rho_3$, and $\rho_2\rho_4$. Using Eq. (31) and the fact that $k_1' + k_2' = \pi - \theta$ and $k_1' - k_2' = \pi/2 + \pi\eta$, we find that the off-diagonal interactions are given by

$$\begin{aligned} H_{\text{off-diag}} = 8 \sin(\phi/2) \sin(\theta) \int dx\, ([\sin(\pi\eta) + 1](\rho_1\rho_2 + \rho_3\rho_4) + [\sin(\pi\eta) - 1](\rho_1\rho_4 + \rho_2\rho_3)) \\ + 16 \sin(\phi/2) \int dx\, [\cos(\pi\eta - \theta)\rho_1\rho_3 - \cos(\pi\eta + \theta)\rho_2\rho_4]. \end{aligned} \quad (40)$$

Using Eq. (38), the bosonized form of Eq. (40) is found to be

$$\begin{aligned} H_{\text{off-diag}} = \frac{8 \sin(\phi/2) \sin(\theta)}{\pi} \int dx ([\sin(\pi\eta) + 1][(\partial_x \phi_1)(\partial_x \phi_2) + (\partial_x \phi_3)(\partial_x \phi_4)] \\ + [\sin(\pi\eta) - 1][(\partial_x \phi_1)(\partial_x \phi_4) + (\partial_x \phi_2)(\partial_x \phi_3)]) \\ + \frac{16 \sin(\phi/2)}{\pi} \int dx [\cos(\pi\eta - \theta)(\partial_x \phi_1)(\partial_x \phi_3) - \cos(\pi\eta + \theta)(\partial_x \phi_2)(\partial_x \phi_4)]. \end{aligned} \quad (41)$$

There is another kind of momentum-conserving interaction possible in our system, which is a term involving all the four Fermi points. In the fermionic language, these are given by

$$H_{\text{four-Fermi-pt}} = -4i\sin(\phi/2)e^{-i\theta}\int dx\,[\psi_1^\dagger\psi_2\psi_4^\dagger\psi_3 + \psi_2^\dagger\psi_1\psi_3^\dagger\psi_4 + \sin(\pi\eta)(\psi_1^\dagger\psi_3\psi_4^\dagger\psi_2 + \psi_2^\dagger\psi_4\psi_3^\dagger\psi_1)] + \text{H.c.} \tag{42}$$

To convert this into the bosonic language, we use Eq. (34). We now get products of Klein factors. For instance, the first two terms in Eq. (42) have the products $F_1^\dagger F_2 F_4^\dagger F_3$ and $F_2^\dagger F_1 F_3^\dagger F_4$ respectively. Equations (35) imply that these two products commute with each other and can therefore be simultaneously diagonalized. Hence, we can ignore the Klein factor products in the following [25]. We then find that in the bosonic language, Eq. (42) becomes

$$H_{\text{four-Fermi-pt}} = \frac{16\sin(\phi/2)\sin(\theta)[\sin(\pi\eta)-1]}{(2\pi\alpha)^2} \times \int dx\cos[2\sqrt{\pi}(\phi_1+\phi_2-\phi_3-\phi_4)]. \tag{43}$$

We now discuss the various umklapp terms which appear. These describe scattering processes in which the momentum difference between the initial and final states is 2π; these terms are allowed in a lattice system since since a momentum transfer $\Delta k = 2\pi$ is equivalent to $\Delta k = 0$. In the bosonization of a system with two Fermi points (left- and right-moving points denoted as R and L respectively), umklapp terms of the form $\psi_R^\dagger\psi_R^\dagger\psi_L\psi_L$ and its Hermitian conjugate are allowed at half-filling, i.e., with $k_F = \pi/2$, since $4k_F = 2\pi$. In our system, however, there are two kinds of umklapp terms possible which are quite different from the umklapp terms which appear for the case of two Fermi points. The first kind of umklapp term appears due to scattering between two Fermi points with the same chirality, both right- or both left-moving in same direction. This term is given by

$$H_{\text{umklapp},1} = 2i\sin(\phi/2)\int dx[e^{2ik_3'}\psi_1^\dagger\psi_3\psi_1^\dagger\psi_3 + e^{2ik_1'}\psi_3^\dagger\psi_1\psi_3^\dagger\psi_1 + e^{2ik_4'}\psi_2^\dagger\psi_4\psi_2^\dagger\psi_4 + e^{2ik_2'}\psi_4^\dagger\psi_2\psi_4^\dagger\psi_2]. + \text{H.c.} \tag{44}$$

(Each term in this equation changes the momentum by $\pm 2\pi$.) When we bosonize this, we again find that the products of Klein factors [such as $(F_1^\dagger)^2(F_3)^2$, $(F_3^\dagger)^2(F_1)^2$, etc.] all commute with each other as well as with the products of Klein factors which appear in Eq. (43). Hence, we can ignore all these product terms. We then find that in the bosonic language, Eq. (44) becomes

$$H_{\text{umklapp},1} = -\frac{8\sin(\phi/2)\sin(\theta)}{(2\pi\alpha)^2}\int dx(\cos[4\sqrt{\pi}(\phi_1-\phi_3)] + \cos[4\sqrt{\pi}(\phi_2-\phi_4)]). \tag{45}$$

The second kind of umklapp terms appears due to interaction among four Fermi points. This interaction is given by

$$H_{\text{umklapp},2} = 2i\sin(\phi/2)\int dx[\psi_1^\dagger\psi_2^\dagger\psi_3\psi_4(e^{-i2k_1'}+e^{-i2k_2'}) + \psi_3^\dagger\psi_4^\dagger\psi_1\psi_2(e^{-2ik_3'}+e^{-2ik_4'})] + \text{H.c.} \tag{46}$$

[Each term in Eq. (46) changes the momentum by $\pm 2\pi$, unlike the terms in Eq. (42) which conserve momentum.] Upon bosonizing, we again find that the products of Klein factors for the different terms commute with each other and with all the products which appeared above; hence, we ignore all these products. In the bosonic language, Eq. (46) then becomes

$$H_{\text{umklapp},2} = \frac{16\sin(\phi/2)\sin(\theta)\sin(\pi\eta)}{(2\pi\alpha)^2} \times \int dx\cos[2\sqrt{\pi}(\phi_1-\phi_2-\phi_3+\phi_4)]. \tag{47}$$

Note that in our system, umklapp terms appear at any filling due to Eq. (28), unlike systems with two Fermi points where umklapp terms appear only at half-filling where $4k_F = 2\pi$.

The total Hamiltonian of our model is now given by

$$H_{\text{total}} = H_{\text{diag}} + H_{\text{off-diag}} + H_{\text{four-Fermi-point}} + H_{\text{umklapp},1} + H_{\text{umklapp},2}. \tag{48}$$

Next, we use the Bogoliubov transformation to diagonalize the quadratic part of the Hamiltonian given by $H_{\text{diag}} + H_{\text{off-diag}}$. It is convenient to write the quadratic part of the Hamiltonian in a matrix form. To do so, we choose the basis of bosonic fields

$$\Phi = (\phi_1 \quad \phi_2 \quad \phi_3 \quad \phi_4)^T, \tag{49}$$

where the subscript T denotes the transpose of the row, so that Φ is a column. In this basis, the quadratic part of Hamiltonian has the form

$$H_{\text{quad}} = \int dx\partial_x\Phi^T M\partial_x\Phi, \tag{50}$$

where M is 4×4 matrix. We now define

$$\begin{aligned}
\alpha_1 &= v + \frac{4}{\pi}\sin(\phi/2)\cos(\pi\eta-\theta),\\
\alpha_2 &= v - \frac{4}{\pi}\sin(\phi/2)\cos(\pi\eta+\theta),\\
\alpha_3 &= \frac{4}{\pi}\sin(\phi/2)\sin(\theta)[\sin(\pi\eta)+1],\\
\alpha_4 &= \frac{4}{\pi}\sin(\phi/2)\sin(\theta)[\sin(\pi\eta)-1],\\
\alpha_5 &= \frac{8}{\pi}\sin(\phi/2)\cos(\pi\eta-\theta),\\
\alpha_6 &= -\frac{8}{\pi}\sin(\phi/2)\cos(\pi\eta+\theta).
\end{aligned} \tag{51}$$

In terms of these parameters, the matrix M in Eq. (50) is given by

$$M = \begin{pmatrix} \alpha_1 & \alpha_3 & \alpha_5 & \alpha_4 \\ \alpha_3 & \alpha_2 & \alpha_4 & \alpha_6 \\ \alpha_5 & \alpha_4 & \alpha_1 & \alpha_3 \\ \alpha_4 & \alpha_6 & \alpha_3 & \alpha_2 \end{pmatrix}. \tag{52}$$

We will now choose the following linear combinations of the bosonic fields,

$$\phi_{R1} = \frac{\phi_1+\phi_3}{\sqrt{2}}, \quad \phi_{R2} = \frac{\phi_1-\phi_3}{\sqrt{2}},$$
$$\phi_{L1} = \frac{\phi_2+\phi_4}{\sqrt{2}}, \quad \phi_{L2} = \frac{\phi_2-\phi_4}{\sqrt{2}}. \tag{53}$$

In terms of creation and annihilation operators, the fields in Eq. (53) are given by

$$\phi_{Ri} = \chi_{Ri} + \chi_{Ri}^\dagger - \frac{\sqrt{\pi}x}{L}\hat{N}_{Ri}, \quad i = 1,2,$$
$$\phi_{Li} = \chi_{Li} + \chi_{Li}^\dagger - \frac{\sqrt{\pi}x}{L}\hat{N}_{Li}, \quad i = 1,2,$$
$$\chi_{Ri} = \frac{i}{2\sqrt{\pi}}\sum_{q>0}\frac{1}{\sqrt{n_q}}b_{Ri,q}e^{iqx-\alpha q/2},$$
$$\chi_{Ri}^\dagger = -\frac{i}{2\sqrt{\pi}}\sum_{q>0}\frac{1}{\sqrt{n_q}}b_{Ri,q}^\dagger e^{-iqx-\alpha q/2},$$
$$\chi_{Li} = -\frac{i}{2\sqrt{\pi}}\sum_{q>0}\frac{1}{\sqrt{n_q}}b_{Ri,q}e^{-iqx-\alpha q/2},$$
$$\chi_{Li}^\dagger = -\frac{i}{2\sqrt{\pi}}\sum_{q>0}\frac{1}{\sqrt{n_q}}b_{Ri,q}^\dagger e^{iqx-\alpha q/2}, \tag{54}$$

where L denotes the length of system (we are assuming periodic boundary conditions so that a momentum q can be defined as $q = 2\pi n_q/L$, where n_q is an integer). We now define a new basis

$$\Phi' = (\phi_{R1} \quad \phi_{L1} \quad \phi_{R2} \quad \phi_{L2})^T. \tag{55}$$

In this basis, the quadratic part of the Hamiltonian takes the form

$$H_{\rm quad} = \int dx\, \partial_x\Phi'^T M' \partial_x\Phi', \tag{56}$$

where M' is given by

$$M' = \begin{pmatrix} \alpha_1+\alpha_5 & \alpha_3+\alpha_4 & 0 & 0 \\ \alpha_3+\alpha_4 & \alpha_2+\alpha_6 & 0 & 0 \\ 0 & 0 & \alpha_1-\alpha_5 & \alpha_3-\alpha_4 \\ 0 & 0 & \alpha_3-\alpha_4 & \alpha_2-\alpha_6 \end{pmatrix}. \tag{57}$$

We see that M' has a block diagonal form. We will now diagonalize each block separately using the Bogoliubov transformation.

A remarkable point to note at this stage is that $\alpha_3 = \alpha_4 = 0$ if either the statistical phase $\phi = 0$ or $\eta = 0$ (i.e., half-filling). The matrix M' is then diagonal. We therefore have a nontrivial interacting theory only if both ϕ and η are nonzero (i.e., away from half-filling).

A. Diagonalization of Hamiltonian and Luttinger parameters

We begin our analysis by considering the upper block of the Hamiltonian in Eq. (56). This can be written as

$$H_{\rm quad,1} = \int dx\, \partial_x\Phi_1'^T M_1' \partial_x\Phi_1', \tag{58}$$

where Φ_1' is given by

$$\Phi_1' = (\phi_{R1} \quad \phi_{L1})^T, \tag{59}$$

and

$$M_1' = \begin{pmatrix} \alpha_1+\alpha_5 & \alpha_3+\alpha_4 \\ \alpha_3+\alpha_4 & \alpha_2+\alpha_6 \end{pmatrix}. \tag{60}$$

Next we define

$$v_1^{(1)} = \alpha_1+\alpha_5, \quad v_2^{(1)} = \alpha_2+\alpha_6, \quad \lambda^{(1)} = \alpha_3+\alpha_4. \tag{61}$$

We then find that in momentum space, the Hamiltonian in Eq. (58) is given, up to a constant, by

$$H_{\rm quad,1} = \sum_{q>0} q\big[v_1^{(1)} b_{q,R1}^\dagger b_{q,R1} + v_2^{(1)} b_{q,L1}^\dagger b_{q,L1} + \lambda^{(1)}(b_{q,R1}^\dagger b_{q,L1}^\dagger + b_{q,L1}b_{q,R1})\big], \tag{62}$$

where $b_{q,R1}$ and $b_{q,L1}$ are bosonic operators obeying the usual commutation relations. This Hamiltonian can be diagonalized using the Bogoliubov transformation

$$\tilde{b}_{q,R1} = \frac{b_{q,R1}+\gamma_1 b_{q,L1}^\dagger}{\sqrt{1-\gamma_1^2}}, \quad \tilde{b}_{q,L1} = \frac{b_{q,L1}+\gamma_1 b_{q,R1}^\dagger}{\sqrt{1-\gamma_1^2}},$$
$$\gamma_1 = \frac{1-K_1}{1+K_1}. \tag{63}$$

The old and new ϕ fields in this block are related as

$$\phi_{R1} = \frac{(1+K_1)\tilde{\phi}_{R1} - (1-K_1)\tilde{\phi}_{L1}}{2\sqrt{K_1}},$$
$$\phi_{L1} = \frac{(1+K_1)\tilde{\phi}_{L1} - (1-K_1)\tilde{\phi}_{R1}}{2\sqrt{K_1}}. \tag{64}$$

The Hamiltonian now takes the diagonalized form

$$H_{\rm quad,1} = \sum_{q>0} q\bigg[\bigg(v_F^{(1)} + \frac{v_1^{(1)}-v_2^{(1)}}{2}\bigg)\tilde{b}_{q,R1}^\dagger\tilde{b}_{q,R1} + \bigg(v_F^{(1)} - \frac{v_1^{(1)}-v_2^{(1)}}{2}\bigg)\tilde{b}_{q,L1}^\dagger\tilde{b}_{q,L1}\bigg], \tag{65}$$

up to a constant, and $v_F^{(1)}$ and K_1 are given by

$$v_F^{(1)} = \frac{v_1^{(1)}+v_2^{(1)}}{2}\sqrt{1-\bigg(\frac{2\lambda^{(1)}}{v_1^{(1)}+v_2^{(1)}}\bigg)^2},$$
$$= \frac{1}{2}\sqrt{\bigg(2v + \frac{24}{\pi}\sin(\phi/2)\sin(\pi\eta)\sin\theta\bigg)^2 - \bigg(\frac{16}{\pi}\sin(\phi/2)\sin(\pi\eta)\bigg)^2},$$
$$K_1 = \sqrt{\frac{v_1^{(1)}+v_2^{(1)}-2\lambda^{(1)}}{v_1^{(1)}+v_2^{(1)}+2\lambda^{(1)}}} = \sqrt{\frac{2v+\frac{8}{\pi}\sin(\phi/2)\sin(\theta)\sin(\pi\eta)}{2v+\frac{40}{\pi}\sin(\phi/2)\sin(\theta)\sin(\pi\eta)}}. \tag{66}$$

We note from Eq. (66) that the right- and left-moving bosonic fields do not have equal velocities; this is because our model breaks parity symmetry ($x \to -x$) when $\phi \neq 0$. We can also find the condition for the ground state to be well defined; as shown in Appendix A, the condition turns out to be $v_1^{(1)} v_2^{(1)} > (\lambda^{(1)})^2$.

We now consider the lower block of M' in Eq. (57), which leads to the Hamiltonian

$$H_{\rm quad,2} = \int dx\, \partial_x \Phi_2'^T M_2' \partial_x \Phi_2', \tag{67}$$

where

$$\Phi_2' = (\phi_{R2} \quad \phi_{L2})^T, \tag{68}$$

and M_2' has the form

$$M_2' = \begin{pmatrix} \alpha_1 - \alpha_5 & \alpha_3 - \alpha_4 \\ \alpha_3 - \alpha_4 & \alpha_2 - \alpha_6 \end{pmatrix}. \tag{69}$$

We now define

$$v_1^{(2)} = \alpha_1 - \alpha_5, \quad v_2^{(2)} = \alpha_2 - \alpha_6, \quad \lambda^{(2)} = \alpha_3 - \alpha_4. \tag{70}$$

In momentum space, the Hamiltonian in Eq. (67) takes the form

$$H_{\rm quad,2} = \sum_{q>0} q\big[v_1^{(2)} b_{q,R2}^\dagger b_{q,R2} + v_2^{(2)} b_{q,L2}^\dagger b_{q,L2} + \lambda^{(2)}\big(b_{q,R2}^\dagger b_{q,L2}^\dagger + b_{q,L2} b_{q,R2}\big)\big] \tag{71}$$

up to a constant, where $b_{q,R2}$ and $b_{q,L}$ are bosonic fields obeying the usual commutation relations. We then do a Bogoliubov transformation to obtain a Hamiltonian in a diagonal form

$$H_{\rm quad,2} = \sum_{q>0} q\Bigg[\left(v_F^{(2)} + \frac{v_1^{(2)} - v_2^{(2)}}{2}\right)\tilde{b}_{q,R2}^\dagger \tilde{b}_{q,R2} + \left(v_F^{(2)} - \frac{v_1^{(2)} - v_2^{(2)}}{2}\right)\tilde{b}_{q,L2}^\dagger \tilde{b}_{q,L2}\Bigg], \tag{72}$$

up to a constant, where the Bogoliubov transformation is

$$\tilde{b}_{q,R2} = \frac{b_{q,R2} + \gamma_2 b_{q,L2}^\dagger}{\sqrt{1-\gamma_2^2}}, \quad \tilde{b}_{q,L2} = \frac{b_{q,L2} + \gamma_2 b_{q,R2}^\dagger}{\sqrt{1-\gamma_2^2}}, \quad \gamma_2 = \frac{1-K_2}{1+K_2}. \tag{73}$$

The old and new ϕ fields are related as

$$\phi_{R2} = \frac{(1+K_2)\tilde{\phi}_{R2} - (1-K_2)\tilde{\phi}_{L2}}{2\sqrt{K_2}}, \quad \phi_{L2} = \frac{(1+K_2)\tilde{\phi}_{L2} - (1-K_2)\tilde{\phi}_{R2}}{2\sqrt{K_2}}, \tag{74}$$

where

$$\begin{aligned} v_F^{(2)} &= \frac{v_1^{(2)} + v_2^{(2)}}{2}\sqrt{1 - \left(\frac{2\lambda^{(2)}}{v_1^{(2)} + v_2^{(2)}}\right)^2}, \\ &= \frac{1}{2}\sqrt{\left(2v - \frac{8}{\pi}\sin(\phi/2)\sin(\pi\eta)\sin\theta\right)^2 - \left(\frac{16}{\pi}\sin(\phi/2)\sin(\pi\eta)\right)^2}, \\ K_2 &= \sqrt{\frac{v_1^{(2)} + v_2^{(2)} - 2\lambda^{(2)}}{v_1^{(2)} + v_2^{(2)} + 2\lambda^{(2)}}} = \sqrt{\frac{2v - \frac{8}{\pi}\sin(\phi/2)\sin(\theta)(\sin(\pi\eta)+2)}{2v - \frac{8}{\pi}\sin(\phi/2)\sin(\theta)(\sin(\pi\eta)-2)}}. \end{aligned} \tag{75}$$

We again see that the right- and left-moving bosonic fields have unequal velocities. As before the condition for the ground state to be well-defined turns out to be $v_1^{(2)} v_2^{(2)} > (\lambda^{(2)})^2$.

Figures 3(a) and 3(b) show K_1 and K_2 as functions of η for various values of ϕ. [We have not shown the values of K_1 and K_2 very close to $\eta = \pm 1/2$ since the analytical expressions in Eqs. (66) and (75) become singular in that limit. In particular, $v \to 0$ as $\eta \to \pm 1/2$.] We see from the figures that K_1 remains unchanged while $K_2 \to 1/K_2$ if we flip $\eta \to -\eta$. The reason for these symmetries is discussed in Sec. IV B. We also see from Fig. 3 that $K_1 \leqslant 1$ for all values of η and ϕ, while $K_2 \geqslant 1$ ($\leqslant 1$) for $\eta \leqslant 0$ ($\geqslant 0$) for all values of ϕ. This is in agreement with the expressions given in Eqs. (66) and (75).

We conclude that the diagonalization of the quadratic parts of the bosonized Hamiltonian gives two decoupled Tomonaga-Luttinger liquids. Since each of this is described by a conformal field theory with central charge $c = 1$, the full system has $c = 2$.

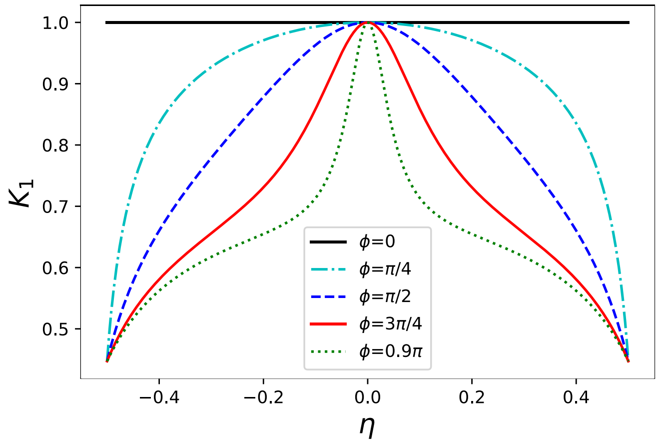
B. Implications of particle-hole and parity transformations for K_1 and K_2

In this section, we will discuss two kinds of transformations which leave the Hamiltonian in Eq. (8) invariant, and what these imply for the Luttinger liquid parameters K_1 and K_2.

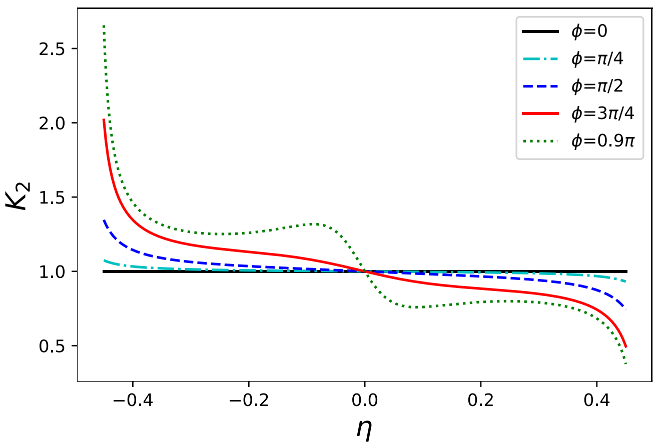
The first transformation that we will consider is particle-hole transformation, keeping the statistical phase ϕ unchanged. We find that transforming

$$c_j \to e^{-ij\pi/2} c_j^\dagger \tag{76}$$

flips the sign of $n_{j+1} - 1/2$. This leaves the hopping part of the Hamiltonian in Eq. (8) invariant but flips the sign of the



(a)



(b)

FIG. 3. Luttinger parameters K_1 and K_2 vs η for $\phi = 0,\ \pi/4,\ \pi/2,\ 3\pi/4$ and 0.9π. They satisfy the symmetries $K_1(\phi, -\eta) = K_1(\phi, \eta)$ and $K_2(\phi, -\eta) = 1/K_2(\phi, \eta)$.

chemical potential term, thus transforming the filling from $1/2 + \eta$ to $1/2 - \eta$. Looking at Eq. (31), we see that Eq. (76) transforms the Fermi momenta as

$$k_i' \to -k_i' + \frac{\pi}{2}. \tag{77}$$

Further, $\eta \to -\eta$ implies that $\theta \to -\theta$ in Eq. (27). Equation (77) then means that k_1' and k_3' get interchanged, and k_2' and k_4' remain as they are. Equations (31) and (34) then imply that ϕ_1 and ϕ_3 get interchanged, and ϕ_2 and ϕ_4 remain unchanged. Turning to the parameters v and α_i in Eqs. (30) and (51), we see that v, α_1, α_2, α_5, and α_6 remain unchanged, and α_3 and α_4 get interchanged. We then see that the matrix M_1' remains unchanged in Eq. (60) while in Eq. (69), the off-diagonal terms flip sign. Equation (66) then means that K_1 remains unchanged while Eq. (75) means that K_2 transforms to $1/K_2$ under $\eta \to -\eta$.

The second transformation we look at is $\phi \to -\phi$ combined with parity, $j \to -j$ in Eq. (8) or $x \to -x$ in the continuum, keeping the filling (η) unchanged. This leaves Eq. (8) unchanged. Equation (27) implies that $\theta \to -\theta$ while Eq. (26) means that k_1' and k_4' get interchanged as do k_2' and k_3'. Equations (31) and (34) then imply that $\phi_1 \leftrightarrow -\phi_4$ and $\phi_2 \leftrightarrow -\phi_3$. Further, the parameters v, α_3, and α_4 remain unchanged, and α_1 and α_2 get interchanged as do α_5 and α_6. Equations (60) and (69) then show that the diagonal entries of M_1' get interchanged while the off-diagonal entries do not change, and similarly for M_2'. This means that K_1 and K_2 remain unchanged under $\phi \to -\phi$ as we can see from Eqs. (66) and (75).

C. Scaling dimensions of the various four-fermion interaction terms

In Sec. IV A, we diagonalized the quadratic part of the Hamiltonian and found the relation between the old and new bosonic fields. We will now discuss the scaling dimension of the various terms involving cosines of the bosonic fields (arising from four-fermion interacting terms) with respect to the new vacuum obtained after the Bogoliubov transformations.

In general, two-point correlation functions of exponentials of bosonic fields decay as power laws,

$$\langle\tilde{0}|Te^{i2\sqrt{\pi}\beta\tilde{\phi}_{R1}}e^{-i2\sqrt{\pi}\beta\tilde{\phi}_{R1}}|\tilde{0}\rangle \sim \left(\frac{\alpha}{v_R^{(1)}t - x - i\alpha\,\mathrm{sgn}(t)}\right)^{\beta^2},$$

$$\langle\tilde{0}|Te^{i2\sqrt{\pi}\beta\tilde{\phi}_{R2}}e^{-i2\sqrt{\pi}\beta\tilde{\phi}_{R2}}|\tilde{0}\rangle \sim \left(\frac{\alpha}{v_R^{(2)}t - x - i\alpha\,\mathrm{sgn}(t)}\right)^{\beta^2},$$

$$\langle\tilde{0}|Te^{i2\sqrt{\pi}\beta\tilde{\phi}_{L1}}e^{-i2\sqrt{\pi}\beta\tilde{\phi}_{L1}}|\tilde{0}\rangle \sim \left(\frac{\alpha}{v_L^{(1)}t + x - i\alpha\,\mathrm{sgn}(t)}\right)^{\beta^2},$$

$$\langle\tilde{0}|Te^{i2\sqrt{\pi}\beta\tilde{\phi}_{L2}}e^{-i2\sqrt{\pi}\beta\tilde{\phi}_{L2}}|\tilde{0}\rangle \sim \left(\frac{\alpha}{v_L^{(2)}t + x - i\alpha\,\mathrm{sgn}(t)}\right)^{\beta^2},$$

$$\text{where}\quad v_R^{(1)} = v_F^{(1)} + \frac{v_1^{(1)} - v_2^{(1)}}{2}, \qquad v_R^{(2)} = v_F^{(2)} + \frac{v_1^{(2)} - v_2^{(2)}}{2},$$

$$v_L^{(1)} = v_F^{(1)} - \frac{v_1^{(1)} - v_2^{(1)}}{2}, \qquad v_L^{(2)} = v_F^{(2)} - \frac{v_1^{(2)} - v_2^{(2)}}{2}. \tag{78}$$

Here $\tilde{\phi}_{Ri}$ and $\tilde{\phi}_{Li}$ are the new fields obtained after the Bogoliubov transformation and $|\tilde{0}\rangle$ denotes the new vacuum.

We first consider the operator in Eq. (43) given by

$$A_1 = \cos[2\sqrt{2\pi}(\phi_{R2} + \phi_{L2})]. \tag{79}$$

Equation (74) implies that

$$\phi_{R2} + \phi_{L2} = \sqrt{K_2}(\tilde{\phi}_{R2} + \tilde{\phi}_{L2}). \tag{80}$$

We find the scaling dimension of the operator in Eq. (79) by calculating the correlation function $\langle\tilde{0}|Te^{i2\sqrt{2\pi}(\phi_{R2}+\phi_{L2})}$ $e^{-i2\sqrt{2\pi}(\phi_{R2}+\phi_{L2})}|\tilde{0}\rangle$. Using Eq. (78), we find that

$$\begin{aligned}
&\langle\tilde{0}|Te^{i2\sqrt{2\pi}(\phi_{R2}+\phi_{L2})}e^{-i2\sqrt{2\pi}(\phi_{R2}+\phi_{L2})}|\tilde{0}\rangle \\
&\quad = \langle\tilde{0}|Te^{i2\sqrt{2\pi K_2}(\tilde{\phi}_{R2}+\tilde{\phi}_{L2})}e^{-i2\sqrt{2\pi K_2}(\tilde{\phi}_{R2}+\tilde{\phi}_{L2})}|\tilde{0}\rangle, \\
&\quad \sim \left(\frac{\alpha}{v_R^{(2)}t - x - i\alpha\,\mathrm{sgn}(t)}\right)^{2K_2}\left(\frac{\alpha}{v_L^{(2)}t + x - i\alpha\,\mathrm{sgn}(t)}\right)^{2K_2}.
\end{aligned} \tag{81}$$

Setting $t = 0$, we conclude that at large spatial separation the correlation function falls off as $(\alpha/x)^{4K_2}$. This means that the operator in Eq. (79) has scaling dimension $2K_2$. This term is relevant in the renormalization group (RG) sense if the scaling dimension is less than 2 which requires $K_2 < 1$.

Next, we find the scaling dimension of the umklapp terms from the appropriate correlation functions. The operators in the first umklapp term in Eq. (45) are

$$A_2 = \cos(4\sqrt{2\pi}\phi_{R2}) + \cos(4\sqrt{2\pi}\phi_{L2}). \tag{82}$$

To find the scaling dimension of these operators, we calculate the correlation functions $\langle\tilde{0}|Te^{i4\sqrt{2\pi}\phi_{R2}}e^{-i4\sqrt{2\pi}\phi_{R2}}|\tilde{0}\rangle$ and $\langle\tilde{0}|Te^{i4\sqrt{2\pi}\phi_{L2}}e^{-i4\sqrt{2\pi}\phi_{L2}}|\tilde{0}\rangle$ respectively. We find that

$$\begin{aligned}
&\langle\tilde{0}|Te^{i4\sqrt{2\pi}\phi_{R2}}e^{-i4\sqrt{2\pi}\phi_{R2}}|\tilde{0}\rangle \\
&\quad \sim \left(\frac{\alpha}{v_R^{(2)}t - x - i\alpha\,\mathrm{sgn}(t)}\right)^{\frac{2(1+K_2)^2}{K_2}} \\
&\quad\quad \times \left(\frac{\alpha}{v_L^{(2)}t + x - i\alpha\,\mathrm{sgn}(t)}\right)^{\frac{2(1-K_2)^2}{K_2}}, \\
&\langle\tilde{0}|Te^{i4\sqrt{2\pi}\phi_{L2}}e^{-i4\sqrt{2\pi}\phi_{L2}}|\tilde{0}\rangle \\
&\quad \sim \left(\frac{\alpha}{v_L^{(2)}t + x - i\alpha\,\mathrm{sgn}(t)}\right)^{\frac{2(1+K_2)^2}{K_2}{}_2} \\
&\quad\quad \times \left(\frac{\alpha}{v_R^{(2)}t - x - i\alpha\,\mathrm{sgn}(t)}\right)^{\frac{2(1-K_2)^2}{K_2}}.
\end{aligned} \tag{83}$$

So both the operators in Eq. (82) have the scaling dimension $2(K_2 + \frac{1}{K_2})$. Since this is equal to or larger than 4 for all values of K_2, this term is always irrelevant in the RG sense.

Similarly we can calculate the scaling dimension of the operator in the second umklapp term in Eq. (47) given by

$$A_3 = \cos[2\sqrt{2\pi}(\phi_{R2} - \phi_{L2})], \tag{84}$$

by looking at the correlation function $\langle\tilde{0}|Te^{i2\sqrt{2\pi}(\phi_{R2}-\phi_{L2})}e^{-i2\sqrt{2\pi}(\phi_{R2}-\phi_{L2})}|\tilde{0}\rangle$. This is given by

$$\begin{aligned}
&\langle\tilde{0}|Te^{i2\sqrt{2\pi}(\phi_{R2}-\phi_{L2})}e^{-i2\sqrt{2\pi}(\phi_{R2}-\phi_{L2})}|\tilde{0}\rangle \\
&\quad \sim \left(\frac{\alpha}{v_R^{(2)}t - x - i\alpha\,\mathrm{sgn}(t)}\right)^{\frac{2}{K_2}}\left(\frac{\alpha}{v_L^{(2)}t + x - i\alpha\,\mathrm{sgn}(t)}\right)^{\frac{2}{K_2}}.
\end{aligned} \tag{85}$$

The scaling dimension of this umklapp term is therefore $2/K_2$. This term is relevant if the dimension is less than 2 which requires $K_2 > 1$.

Given the scaling dimensions of the three operators A_1, A_2, and A_3, we can use RG equations to examine the effect that they would have on the long-distance properties of the model. Note that all of them involve fields belonging to only the second block given in Eq. (67). Following Eqs. (43), (45), and (47), we can write the contributions of these operators to the Hamiltonian as

$$\begin{aligned}
\delta H &= \int dx[\lambda_1 A_1 + \lambda_2 A_2 + \lambda_3 A_3], \\
\lambda_1 &= \frac{16\sin(\phi/2)\sin(\theta)[\sin(\pi\eta) - 1]}{(2\pi\alpha)^2}, \\
\lambda_2 &= -\frac{8\sin(\phi/2)\sin(\theta)}{(2\pi\alpha)^2}, \\
\lambda_3 &= \frac{16\sin(\phi/2)\sin(\theta)\sin(\pi\eta)}{(2\pi\alpha)^2}.
\end{aligned} \tag{86}$$

Given the scaling dimensions $2K_2$, $2(K_2 + 1/K_2)$, and $2/K_2$ of the operators A_1, A_2, and A_3, we find that the coefficients λ_i in Eq. (86) effectively become functions of the length scale L and satisfy the RG equations

$$\begin{aligned}
\frac{d\lambda_1}{dl} &= (2 - 2K_2)\lambda_1, \\
\frac{d\lambda_2}{dl} &= \left(2 - 2K_2 - \frac{2}{K_2}\right)\lambda_2, \\
\frac{d\lambda_3}{dl} &= \left(2 - \frac{2}{K_2}\right)\lambda_3
\end{aligned} \tag{87}$$

to first order in the λ_i's, where $l = \ln(L/a)$ and a is the lattice spacing. These equations have to be solved with the initial values of the λ_i's at $l = 0$ (i.e., $L = a$) given in Eq. (86). Equation (87) implies that the operator A_2 is always irrelevant, i.e., $\lambda_2 \to 0$ as $L \to \infty$ for any value of K_2. The operator A_1 is relevant if $K_2 < 1$, i.e., if $\eta > 0$, while the operator A_3 is relevant if $K_2 > 1$, i.e., if $\eta < 0$. Hence, depending on the sign of η, either $\lambda_1 \to \infty$ and $\lambda_3 \to 0$ or vice versa, as $L \to \infty$. Correspondingly, one of the operators, A_1 or A_3, would get pinned to its minimum value, and small oscillations around that pinned value would then describe excitations with a gap [26].

If either ϕ or η is close to zero (the latter means that we are close to half-filling), then we see from Eqs. (75) and (86) that K_2 is close to 1 and the λ_i's are close to zero. Then the RG equations in Eq. (87) imply that the relevant coupling will grow and become of order 1 only at enormous values of the length scale L/a. For instance, suppose that K_2 is less than but close to 1. Then the first equation in Eq. (87) along with the value of $\lambda_1(0)$ given in Eq. (86) would imply that $\lambda(l) \sim 1$ will occur at a length scale of the order of

$$\frac{L}{a} \sim \left(\frac{1}{\lambda_1(0)}\right)^{1/(2-2K_2)}, \tag{88}$$

which will be very large number if $\lambda_1(0)$ and $1 - K_2$ are small.

However, all the above statements about RG flows are based only on the first-order RG equations in Eq. (87). When the relevant coupling grows, one should consider second-order terms and see if those can lead to a nontrivial but gapless fixed point. More accurately, one should consider all the three operators $A_1,\ A_2,\ A_3$, derive RG equations up to second order in $\lambda_1,\ \lambda_2, \lambda_3$, and K_2, and then study what these equations imply about the fate of the second block at long distances [27–29]. We note, however, that there are no perturbations in the first block of Tomonaga-Luttinger liquids. Hence, this block is expected to remain gapless and be described by a $c = 1$ conformal field theory.

D. Scaling dimensions of charge density and superconducting order parameters

In this section, we will calculate the two-point correlation functions of charge density and superconducting order parameters and thereby find their scaling dimensions. We first discuss the charge density order parameter; this corresponds to the oscillating part of the density $\rho = c^\dagger c$. In systems with two Fermi points, the charge density wave (CDW) order parameter has the form

$$O_{\text{CDW}} = \psi_R^\dagger \psi_L, \tag{89}$$

where ψ_R and ψ_L are the right- and left-moving fermions. In our model, the CDW order parameter is more complicated since we have four Fermi points which implies that $\rho = \sum_{lm} \psi_l^\dagger \psi_m e^{i(k'_m - k'_l)x}$ has oscillating terms whenever $l \neq m$. The CDW order parameter is therefore given by a sum of six terms,

$$\begin{aligned} O_{\text{CDW}} &= O_1 + O_2 + O_3 + O_4 + O_5 + O_6,\\ O_1 &= \psi_1^\dagger \psi_2, \quad O_2 = \psi_1^\dagger \psi_4,\\ O_3 &= \psi_3^\dagger \psi_2, \quad O_4 = \psi_3^\dagger \psi_4,\\ O_5 &= \psi_1^\dagger \psi_3, \quad O_6 = \psi_2^\dagger \psi_4, \end{aligned} \tag{90}$$

where ψ_1, ψ_3 are right-moving fermions and ψ_2, ψ_4 are left-moving fermions. We need to calculate six correlation functions to find the scaling dimensions of $O_1, \dots, O_6$. We find that the correlation functions $\langle \tilde{0}|O_1^\dagger(x,t) O_1(0,0)|\tilde{0}\rangle, \dots,$ $\langle \tilde{0}|O_6^\dagger(x,t) O_6(0,0)|\tilde{0}\rangle$ are given by

$$\begin{aligned} \langle \tilde{0}|T O_1^\dagger(x,t) O_1(0,0)|\tilde{0}\rangle &\sim \langle \tilde{0}|T O_4^\dagger(x,t) O_4(0,0)|\tilde{0}\rangle\\ &\sim \left(\frac{\alpha}{v_L^{(1)}t + x - i\alpha\,\text{sgn}(t)}\right)^{\frac{K_1}{2}} \left(\frac{\alpha}{v_R^{(1)}t - x - i\alpha\,\text{sgn}(t)}\right)^{\frac{K_1}{2}}\\ &\quad\times \left(\frac{\alpha}{v_L^{(2)}t + x - i\alpha\,\text{sgn}(t)}\right)^{\frac{K_2}{2}} \left(\frac{\alpha}{v_R^{(2)}t - x - i\alpha\,\text{sgn}(t)}\right)^{\frac{K_2}{2}},\\ \langle \tilde{0}|T O_2^\dagger(x,t) O_2(0,0)|\tilde{0}\rangle &\sim \langle \tilde{0}|T O_3^\dagger(x,t) O_3(0,0)|\tilde{0}\rangle\\ &\sim \left(\frac{\alpha}{v_L^{(1)}t + x - i\alpha\,\text{sgn}(t)}\right)^{\frac{K_1}{2}} \left(\frac{\alpha}{v_R^{(1)}t - x - i\alpha\,\text{sgn}(t)}\right)^{\frac{K_1}{2}}\\ &\quad\times \left(\frac{\alpha}{v_L^{(2)}t + x - i\alpha\,\text{sgn}(t)}\right)^{\frac{1}{2K_2}} \left(\frac{\alpha}{v_R^{(2)}t - x - i\alpha\,\text{sgn}(t)}\right)^{\frac{1}{2K_2}},\\ \langle \tilde{0}|T O_5^\dagger(x,t) O_5(0,0)|\tilde{0}\rangle &\sim \left(\frac{\alpha}{v_L^{(2)}t + x - i\alpha\,\text{sgn}(t)}\right)^{\frac{(1-K_2)^2}{2K_2}} \left(\frac{\alpha}{v_R^{(2)}t - x - i\alpha\,\text{sgn}(t)}\right)^{\frac{(1+K_2)^2}{2K_2}},\\ \langle \tilde{0}|T O_6^\dagger(x,t) O_6(0,0)|\tilde{0}\rangle &\sim \left(\frac{\alpha}{v_L^{(2)}t + x - i\alpha\,\text{sgn}(t)}\right)^{\frac{(1+K_2)^2}{2K_2}} \left(\frac{\alpha}{v_R^{(2)}t - x - i\alpha\,\text{sgn}(t)}\right)^{\frac{(1-K_2)^2}{2K_2}}. \end{aligned} \tag{91}$$

From the power law fall-offs at large spatial separations (setting $t = 0$), we see that O_1 and O_4 have scaling dimension $\frac{K_1}{2} + \frac{K_2}{2}$ and are relevant if $K_1 + K_2 < 4$, O_2 and O_3 have scaling dimension $\frac{K_1}{2} + \frac{1}{2K_2}$ and are relevant if $K_1 + \frac{1}{K_2} < 4$, and O_5 and O_6 have scaling dimension $\frac{K_2}{2} + \frac{1}{2K_2}$ and are relevant if $K_2 + \frac{1}{K_2} < 4$.

We now discuss the superconducting order parameters and their scaling dimensions. In our model, there are six such terms whose sum is given by

$$\begin{aligned} O_{\text{SC}} &= O'_1 + O'_2 + O'_3 + O'_4 + O'_5 + O'_6,\\ O'_1 &= \psi_1^\dagger \psi_2^\dagger, \quad O'_2 = \psi_1^\dagger \psi_4^\dagger,\\ O'_3 &= \psi_3^\dagger \psi_2^\dagger, \quad O'_4 = \psi_3^\dagger \psi_4^\dagger,\\ O'_5 &= \psi_1^\dagger \psi_3^\dagger, \quad O'_6 = \psi_2^\dagger \psi_4^\dagger. \end{aligned} \tag{92}$$

Calculating the correlation functions of these order parameters similarly, we find that

$$\begin{aligned}
\langle\tilde{0}|TO_1'^{\dagger}(x,t)O_1'(0,0)|\tilde{0}\rangle &\sim \langle\tilde{0}|TO_3'^{\dagger}(x,t)O_3'(0,0)|\tilde{0}\rangle \\
&\sim \left(\frac{\alpha}{v_L^{(1)}t+x-i\alpha\,\mathrm{sgn}(t)}\right)^{\frac{1}{2K_1}}\left(\frac{\alpha}{v_R^{(1)}t-x-i\alpha\,\mathrm{sgn}(t)}\right)^{\frac{1}{2K_1}} \\
&\quad\times\left(\frac{\alpha}{v_L^{(2)}t+x-i\alpha\,\mathrm{sgn}(t)}\right)^{\frac{1}{2K_2}}\left(\frac{\alpha}{v_R^{(2)}t-x-i\alpha\,\mathrm{sgn}(t)}\right)^{\frac{1}{2K_2}}, \\
\langle\tilde{0}|TO_2'^{\dagger}(x,t)O_2'(0,0)|\tilde{0}\rangle &\sim \langle\tilde{0}|TO_4'^{\dagger}(x,t)O_4'(0,0)|\tilde{0}\rangle \\
&\sim \left(\frac{\alpha}{v_L^{(1)}t+x-i\alpha\,\mathrm{sgn}(t)}\right)^{\frac{1}{2K_1}}\left(\frac{\alpha}{v_R^{(1)}t-x-i\alpha\,\mathrm{sgn}(t)}\right)^{\frac{1}{2K_1}} \\
&\quad\times\left(\frac{\alpha}{v_L^{(2)}t+x-i\alpha\,\mathrm{sgn}(t)}\right)^{\frac{K_2}{2}}\left(\frac{\alpha}{v_R^{(2)}t-x-i\alpha\,\mathrm{sgn}(t)}\right)^{\frac{K_2}{2}}, \\
\langle\tilde{0}|TO_5'^{\dagger}(x,t)O_5'(0,0)|\tilde{0}\rangle &\sim \left(\frac{\alpha}{v_L^{(1)}t+x-i\alpha\,\mathrm{sgn}(t)}\right)^{\frac{(1-K_1)^2}{2K_1}}\left(\frac{\alpha}{v_R^{(1)}t-x-i\alpha\,\mathrm{sgn}(t)}\right)^{\frac{(1+K_1)^2}{2K_1}}, \\
\langle\tilde{0}|TO_6'^{\dagger}(x,t)O_6'(0,0)|\tilde{0}\rangle &\sim \left(\frac{\alpha}{v_L^{(1)}t+x-i\alpha\,\mathrm{sgn}(t)}\right)^{\frac{(1+K_1)^2}{2K_1}}\left(\frac{\alpha}{v_R^{(1)}t-x-i\alpha\,\mathrm{sgn}(t)}\right)^{\frac{(1-K_1)^2}{2K_1}}.
\end{aligned} \tag{93}$$

The power law fall-offs at large spatial separations imply that O_1' and O_3' have scaling dimension $\frac{1}{2K_1}+\frac{1}{2K_2}$ and are relevant if $\frac{1}{K_1}+\frac{1}{K_2}<4$, O_2' and O_4' have scaling dimension $\frac{1}{2K_1}+\frac{K_2}{2}$ and are relevant if $\frac{1}{K_1}+K_2<4$, and O_5' and O_6' have scaling dimension $\frac{K_1}{2}+\frac{1}{2K_1}$ and are relevant if $K_1+\frac{1}{K_1}<4$.

When both K_1 and K_2 are close to 1, all six order parameters (three charge density and three superconducting) have scaling dimension close to 1 and are therefore relevant. However, it is interesting to see which of them has the smallest scaling dimension (i.e., the two-point correlation function decays with the smallest power) and is thus the most relevant. In Fig. 4, we plot the six scaling dimensions as functions of η for $\phi=\pi/2$. We find that in this case, one of the charge density order parameters has the smallest scaling dimension for all values of η: for $\eta<0$, the most relevant order parameter is $\psi_1^\dagger\psi_4$ and $\psi_3^\dagger\psi_2$ with scaling dimension $\frac{K_1}{2}+\frac{1}{2K_2}$, while for $\eta>0$, the most relevant order parameter is $\psi_1^\dagger\psi_2$ and $\psi_3^\dagger\psi_4$ with scaling dimension $\frac{K_1}{2}+\frac{K_2}{2}$.

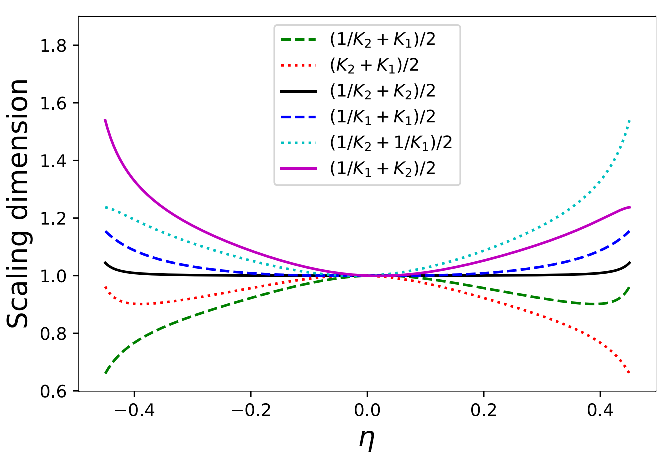


FIG. 4. Plots of scaling dimensions of the charge density and superconducting order parameters vs η for $\phi=\pi/2$. We see that the most relevant operator is one of the charge density order parameters with scaling dimension $\frac{K_1}{2}+\frac{1}{2K_2}$ for $\eta<0$ and $\frac{K_1}{2}+\frac{K_2}{2}$ for $\eta>0$.

V. TWO-PARTICLE BOUND STATES

In this section, we will study what happens if the system has only two particles, in particular, if there are two-particle bound states. From Eq. (8), it is clear that there are no interactions between the two particles if they are on the same sublattice (i.e., both have j even or j odd). We will therefore consider the case where one particle is at site n_1 which is odd and the other particle is at site n_2 which is even. We will define two-particle states as

$$|n_1,n_2\rangle = c_{n_1}^\dagger c_{n_2}^\dagger|\mathrm{vacuum}\rangle, \tag{94}$$

regardless of whether $n_1<n_2$ or $n_1>n_2$. Next, we consider states of the form

$$|\psi\rangle = \sum_{n_1\,\mathrm{odd}}\sum_{n_2\,\mathrm{even}}\psi(n_1,n_2)|n_1,n_2\rangle. \tag{95}$$

Since the center of mass is insensitive to interactions, we will consider wave functions of the form

$$\psi(n_1,n_2) = e^{iP(n_1+n_2)/2}f(n_2-n_1), \tag{96}$$

where P is the center-of-mass momentum, and the relative coordinate wave function $f(n_2-n_1)$ can depend on P. Since the wave function only changes by a minus sign if P is shifted by 2π (since n_1+n_2 is an odd integer), we can take P to lie in the range $[-\pi,\pi]$.

Given the Hamiltonian in Eq. (8) (we will set $\mu = 0$ in this section), we find that the eigenvalue condition $H|\psi\rangle = E|\psi\rangle$ implies that the wave function $f(n)$ (where $n = n_2 - n_1$ is an odd integer) must satisfy

$$\begin{aligned}
&-2t_2 \cos\left(P - \frac{\phi}{2}\right)[f(n-2) + f(n+2)] \\
&\quad = Ef(n) \text{ for } |n| \geqslant 3, \\
&-2t_2\left[\cos\left(P - \frac{\phi}{2}\right)f(3) + \cos\left(P + \frac{\phi}{2}\right)f(-1)\right] \\
&\quad = Ef(1), \\
&-2t_2\left[\cos\left(P - \frac{\phi}{2}\right)f(-3) + \cos\left(P + \frac{\phi}{2}\right)f(1)\right] \\
&\quad = Ef(-1).
\end{aligned} \tag{97}$$

Equations (97) describe a particle moving on a lattice with only odd-numbered sites, where the hopping amplitude between sites labeled -1 and $+1$ is $-2t_2\cos(P + \phi/2)$ and the hopping amplitude between all other neighboring sites is $-2t_2\cos(P - \phi/2)$. This system clearly has scattering states for which

$$f(n_2 - n_1) = e^{ik(n_2-n_1)/2} \tag{98}$$

for $|n_2 - n_1| \gg 1$, where k lies in the range $[-\pi, \pi]$. The energy of such a state is

$$E(P, k) = -4t_2 \cos\left(P - \frac{\phi}{2}\right)\cos k, \tag{99}$$

which is simply the sum of the energies $E(k_1) = -2t_2\cos(2k_1 - \phi/2)$ and $E(k_2) = -2t_2\cos(2k_2 - \phi/2)$ of two particles moving independently on odd- and even-numbered sites with momenta

$$k_1 = \frac{P-k}{2} \quad \text{and} \quad k_2 = \frac{P-k}{2} \tag{100}$$

respectively. These form a band with energies going from $-|4t_2\cos(P - \phi/2)|$ to $+|4t_2\cos(P - \phi/2)|$.

We now examine if this relative coordinate also has bound states in addition to the continuum of scattering states discussed above. The wave function of such states must go to zero exponentially as $|n_2 - n_1| \to \infty$. We therefore make the ansatz that the bound state wave function and energy are given by

$$\begin{aligned}
f(n_2 - n_1) &= e^{-\chi(n_2-n_1)/2} \quad \text{for} \quad n_2 - n_1 \geqslant 1, \\
&= \pm e^{\chi(n_2-n_1)/2} \quad \text{for} \quad n_2 - n_1 \leqslant -1, \\
\text{and} \quad E(P, \chi) &= -4t_2\cos\left(P - \frac{\phi}{2}\right)\cosh\chi,
\end{aligned} \tag{101}$$

where the real part of χ is positive. We then find that such bound states exist if

$$\left|\frac{\cos\left(P + \frac{\phi}{2}\right)}{\cos\left(P - \frac{\phi}{2}\right)}\right| > 1, \tag{102}$$

in which case χ is equal to either r or $r + i\pi$, where r is a positive real number in both cases, i.e., $\cosh\chi$ is >0 or <0, and

$$e^r = \left|\frac{\cos\left(P + \frac{\phi}{2}\right)}{\cos\left(P - \frac{\phi}{2}\right)}\right|. \tag{103}$$

Further, if bound states exist, they appear in pairs with equal and opposite energies given by

$$E = \pm 4t_2\cos(P - \phi/2)\cosh r. \tag{104}$$

The $\pm$ sign in the second line of Eq. (101) and the sign of $\cosh\chi$ depend on the signs of $\cos(P + \phi/2)$, $\cos(P - \phi/2)$ and E.

The condition in Eq. (102) implies that there are no bound states if $\phi = 0$ or π. If $0 < \phi < \pi$, Eq. (102) means that there are bound states if either $-\pi/2 < P < 0$ or $\pi/2 < P < \pi$.

Figures 5(a)–5(e) shows the energy levels of a system with 50 sites (for the relative coordinate problem) as a function of P for various values of ϕ, with $t_2 = -1$. In each of the figures, we see a continuum of states lying in the range $[-|4\cos(P - \phi/2)|, |4\cos(P - \phi/2)|]$, in agreement with the discussion above. We also see pairs of bound states with opposite energies given by the isolated black solid lines; these appear in the regions given by Eq. (102). Figure 5(f) shows the maximum value of the inverse participation ratio (IPR) as a function of P for different values of ϕ. Given a normalized eigenstate $\psi_a(n_2 - n_1)$ of the Hamiltonian, the IPR is defined as $\sum_{n_2-n_1} |\psi_a(n_2 - n_1)|^4$. It is known that this is a useful diagnostic for the presence of bound states. As the system size is taken to infinity, the IPRs of extended states (whose energies form a continuum) go to zero, while the IPRs of localized states (i.e., bound states) remain finite. We indeed see that the ranges of P in Fig. 5(f) where the maximum value of the IPR is large coincides precisely with the ranges in Figs. 5(a)–5(e) where there are bound states.

It is interesting to consider what happens if we shift $P \to P + \pi$; this changes the center-of-mass wave function by a factor of $e^{i\pi(n_1+n_2)/2}$. We then see from the discussion following Eq. (97) that all the hoppings of the relative coordinate problem flip sign. We then find that the energies of both continuum and bound states remain the same but the relative coordinate wave functions change by a factor of $e^{i\pi(n_2-n_1)/2}$ (we recall that $n_2 - n_1$ can only change by multiples of 2 when the particle hop). Combining the changes in the center-of-mass and relative coordinate wave functions, we see that the total wave function changes by $e^{i\pi n_2}$ which is equal to $+1$ since n_2 is even. We therefore see that the wave functions remain the same for all values of n_1 and n_2. Thus, the energy spectrum and eigenstates do not change if P is shifted by π. Both the energy levels and IPR values shown in Fig. 5 are consistent with this observation.

A. Implications for low-density limit

The existence of two-particle bound states may have significant implications for the nature of the ground state in the thermodynamic limit, i.e., the limit in which the number of particles N and the number of sites L are both taken to infinity, keeping the particle density $\rho = N/L = 1/2 + \eta$ fixed. For simplicity, we consider the low-density limit where $\rho \ll 1$, so that interactions between more than two particles can be ignored (hence we are ignoring the possibility of bound states

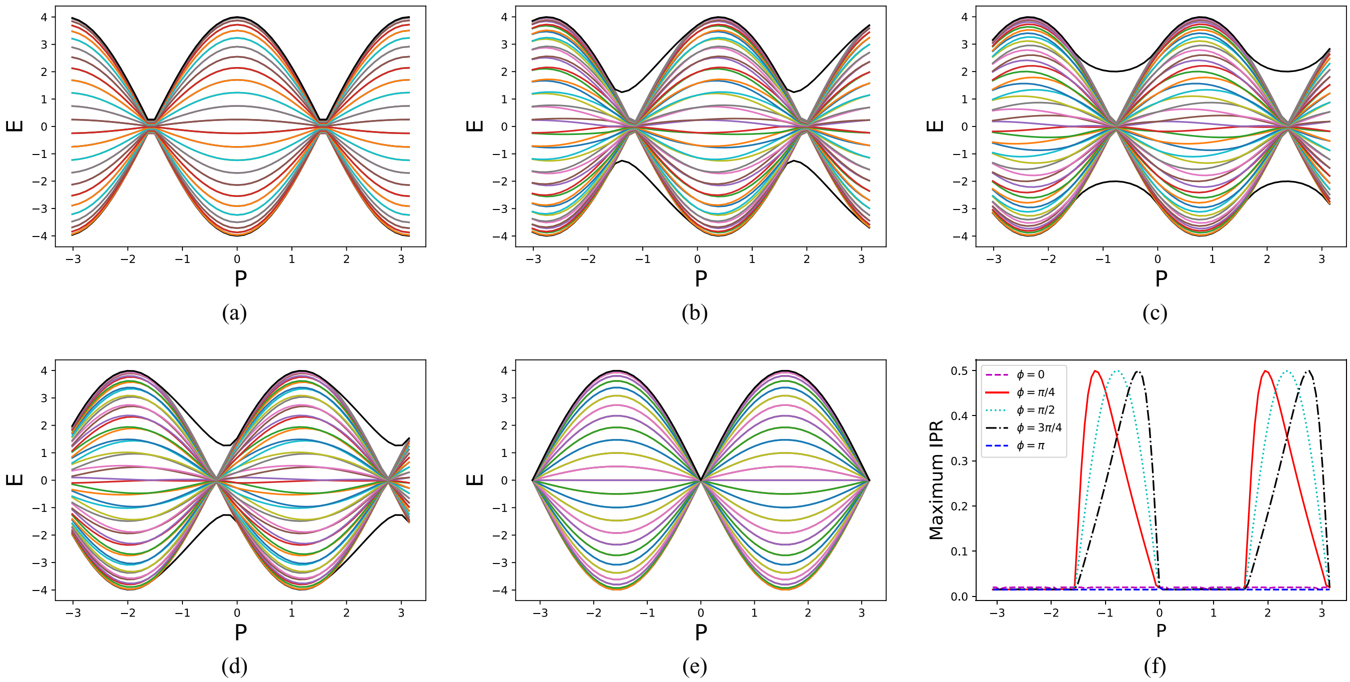


FIG. 5. [(a)–(e)] Energy levels of two-particle states vs the center-of-mass momentum P for $\phi = 0,\ \pi/4,\ \pi/2,\ 3\pi/4$, and π, found numerically for a relative coordinate system with 50 sites, $t_2 = -1$ and $\mu = 0$. The isolated lines (black solid) correspond to two-particle bound states, while all the other lines correspond to the two-particle continuum of states. There are no bound states for $\phi = 0$ and π. (f) Maximum IPR vs P for the same values of ϕ for a relative coordinate system with 100 sites. Whenever the maximum IPR is nonzero (for an infinitely large system), it corresponds to a two-particle bound state.

of three or more particles). In this limit, it may be preferable for pairs of particles to occupy the two-particle bound states with negative energy (which lie below the two-particle continuum as shown in Fig. 5) rather than for the particles to occupy single-particle states independently of each other. We will now briefly examine the values of density and ϕ where this is likely to happen.

In the low-density limit, pairs of particles will only occupy states near the minima of the energy levels shown in Eq. (99), namely, near $P - \phi/2 = 0$ and $k = 0$ or near $P - \phi/2 = \pi$ and $k = \pi$, i.e., near $P = \phi/2$ or $\pi + \phi/2$. On the other hand, Eq. (102) shows that bound states can appear only if $-\pi/2 < P < 0$ or $\pi/2 < P < \pi$. Thus, P must deviate from the minima at $\phi/2$ or $\pi + \phi/2$ by at least $-\phi/2$ for bound states to start appearing. As a result, the single-particle momenta k_1 and k_2 must deviate from their minimum possible value by $-\phi/4$, following Eq. (100). Now, since particles on a particular sublattice can only move in multiple of two sites, the ranges of k_1 and k_2 are equal to π and they are quantized in units of $2\pi/L$ (hence each of them can take $L/2$ values). Thus, a deviation of $\phi/4$ from the minimum of the energy means that the system must have at least $(\phi/4)/(2\pi/L) = \phi L/(8\pi)$ particles on each sublattice occupying the range of $\phi/4$ near $P = 0$ and an equal number of particles occupying the same range near $P = \pi$. Hence, the total number of particles must be equal to at least $\phi L/(2\pi)$, implying that the particle density must be at least $\phi/(2\pi)$ before bound states start appearing in the ground state. To be consistent with the low-density limit, we see that ϕ should be much smaller than π. We therefore see that if ϕ is small, we require the density to be of the order of $\phi/(2\pi)$ before bound states can start playing a role in the ground states of the system. When the density is larger than this amount, we may have to reanalyze the mean field theory done in Sec. III to take the bound states into account.

In conclusion, the possible effects of two-particle bound states on the ground state may be an interesting problem for detailed studies in the future.

VI. DISCUSSION

We first summarize our results. We have studied a one-dimensional model of spinless fermions in which particles have only next-nearest-neighbor hoppings, where the phase of the hopping depends on a statistical phase ϕ and the number of fermions (0 or 1) on the intermediate site. (This model is related, by a unitary transformation, to a model of particles which satisfy a generalized statistics which is governed by the parameter ϕ.) This kind of hopping leads to four-fermion interactions between particles living on the even and odd sublattices. We looked at the properties of the model under particle-hole, parity, and time-reversal transformations. We find that the model is not invariant under P and T separately but is invariant under the product PT. We then studied a mean field theory of the model and found that, for a range of values of the chemical potential, there are four Fermi points; the locations of these points depend on ϕ and the filling which is described by a parameter η. The Fermi points correspond to two right-moving and two left-moving points.

We then developed a bosonized theory of the excitations involving modes near the Fermi points; this theory involves four bosonic fields. We find that the theory has nontrivial interactions only if $\phi \neq 0$ and, more remarkably, only if we are away from half-filling (i.e., if $\eta \neq 0$). The original fermionic theory turns out to lead to a variety of terms in the bosonic language. Some of the terms are quadratic in the bosonic fields while others involve cosines of linear combinations of the bosonic fields. We diagonalized the quadratic terms using Bogoliubov transformations. It turns out that the four bosonic fields decouple into two sets of pairs of bosonic fields, thus giving rise to two separate Tomonaga-Luttinger liquids with different Luttinger parameters K_1 and K_2 and different velocities. (The right- and left-moving bosonic fields turn out to have different velocities because of the lack of parity symmetry.) In terms of these parameters, we found the scaling dimensions of the cosine terms mentioned above and the regimes of parameters ϕ and η where they are relevant or irrelevant. Based on these scaling dimensions and RG flow arguments to first order in the couplings, we found that in one of the Tomonaga-Luttinger liquids, one of the couplings may grow at long distances and may thereby produce a gap. However, we need to examine the effects of higher order terms in the RG equations to understand if this really occurs. The other Tomonaga-Luttinger liquid always remains gapless. Next, we calculated the correlation functions of the 12 different charge density and superconducting order parameters that exist in this model, and found that they all decay as power laws. As a function of ϕ and η, we found which of these order parameters is the most relevant (i.e., has the smallest scaling dimension) and therefore will dominate the correlations in the long-distance limit. We emphasize that exactly at half-filling ($\eta = 0$), the system is noninteracting for all values of ϕ and is described by two Tomonaga-Luttinger liquids which form a conformal field theory with $c = 2$.

Finally, we studied the energy spectrum of two particles, one living on each sublattice. We found that there can be a bound state of the two particles depending on the value of their center-of-mass momentum P and ϕ. Interestingly, the energies of some of the bound states, when they exist, lie below the two-particle continuum. This implies that these bound states can play a role in the form of the ground state in the thermodynamic limit, and we made an estimate of the minimum particle density when this might occur.

We now list some problems which may be useful to study in the future.

(i) In the second block of the two Tomonaga-Luttinger liquids, the RG equations for λ_1, λ_2, λ_3, and K_2 need to be found up to second order to obtain a better understanding of the fixed point that the system may reach at long distances [26–29]. In particular, we would like to know if the fixed point is gapless or gapped.

(ii) It may be interesting to examine what happens if the fillings in the even and odd sublattice are not the same. This would require us to take the chemical potentials to be different on the two sublattices in order to develop a mean field theory followed by bosonization.

(iii) The effect of the two-particle bound states on the ground state of the system in the thermodynamic limit needs to be understood [30,31]. For example, we can investigate if the ground state is a condensate of pairs of particles.

(iv) We can study if there are bound states of three or more particles for some values of the center-of-mass momentum and ϕ.

(v) We may ask what happens when a nearest-neighbor hopping t_1 is present in addition to the next-nearest-neighbor hopping t_2 [see Eq. (4)]. Such a model is considerably more complicated to analyze since it does not conserve the number of particles on the two sublattices separately, and the models with filling fractions $1/2 + \eta$ and $1/2 - \eta$ are no longer related to each other by a particle-hole transformation. We find that for small values of t_1/t_2, there is no significant change in the results, either for the bosonization analysis or the two-particle bound states, compared to the results for $t_1 = 0$ that we have presented in this paper. For $|t_1/t_2| > 2$, however, a different phase appears in which there are only two Fermi points. This phase has been studied in detail at half-filling in Ref. [13].

We conclude by discussing possible realizations of the model considered in this paper. Apart from theoretical ideas for realizing generalized statistics in one dimension [16–20], systems of fermionic or bosonic atoms with density-dependent hoppings have been proposed theoretically [32–40] and realized experimentally [21,41]. The system studied in Ref. [21] is particularly promising since the phase of the hopping of spin-1/2 fermions in one spin state is dependent on the density of fermions in the opposite spin state, analogous to our model where sublattice plays the role of spin.

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APPENDIX A: BOGOLIUBOV TRANSFORMATION OF BOSONS WITH OPPOSITE CHIRALITIES AND UNEQUAL VELOCITIES

In this Appendix, we will discuss the Bogoliubov transformation which was used to diagonalize the Hamiltonians in Eqs. (62) and (71). We consider a model with two bosonic fields with opposite chiralities and unequal velocities v_1 and v_2 and a coupling λ between them. The Hamiltonian of this system is given by

$$H = \sum_{q>0} q[v_1 b^{\dagger}_{q,R} b_{q,R} + v_2 b^{\dagger}_{q,L} b_{q,L} + \lambda(b^{\dagger}_{q,R} b^{\dagger}_{q,L} + b_{q,L} b_{q,R})], \tag{A1}$$

where b_q and $b^{\dagger}_q$ are bosonic annihilation and creation operators which satisfy

$$[b_{q,\nu}, b^{\dagger}_{q',\nu'}] = \delta_{q,q'}\delta_{\nu,\nu'},$$
$$[b_{q,\nu}, b_{q',\nu'}] = 0,$$
$$[b^{\dagger}_{q,\nu}, b^{\dagger}_{q',\nu'}] = 0, \tag{A2}$$

where $\nu, \nu' = R, L$. We will discuss the diagonalization of the Hamiltonian in Eq. (A1) by a Bogoliubov transformation for a particular value of q. The Bogoliubov transformation is given by

$$b_{q,R} = \alpha\tilde{b}_{q,R} + \beta\tilde{b}^{\dagger}_{q,L},$$
$$b_{q,L} = \alpha\tilde{b}_{q,L} + \beta\tilde{b}^{\dagger}_{q,R},$$
$$\alpha = \cosh\theta, \quad \beta = \sinh\theta. \tag{A3}$$

We have chosen α and β to have these forms to satisfy the commutation relations given in Eq. (A2) for the $\tilde{b}$ operators also. The Hamiltonian for a particular q is then given by

$$H = q\Big(\frac{v_1+v_2}{2}(\alpha^2+\beta^2) + 2\alpha\beta\lambda\Big)(\tilde{b}^{\dagger}_{q,R}b_{q,R} + \tilde{b}^{\dagger}_{q,L}\tilde{b}_{q,L}) + q\frac{v_1-v_2}{2}(\tilde{b}^{\dagger}_{q,R}b_{q,R} - \tilde{b}^{\dagger}_{q,L}\tilde{b}_{q,L}) + q[(v_1+v_2)\alpha\beta + \lambda(\alpha^2+\beta^2)](\tilde{b}^{\dagger}_{q,R}\tilde{b}^{\dagger}_{q,L} + \tilde{b}_{q,L}b_{q,R}). \tag{A4}$$

To have a diagonal Hamiltonian, α and β must satisfy $(v_1 + v_2)\alpha\beta + \lambda(\alpha^2+\beta^2) = 0$ which implies

$$\tanh(2\theta) = -\frac{2\lambda}{v_1+v_2}. \tag{A5}$$

Using this in Eq. (A4), we obtain

$$H = q\Big(v + \frac{v_1-v_2}{2}\Big)\tilde{b}^{\dagger}_{q,R}b_{q,R} + q\Big(v - \frac{v_1-v_2}{2}\Big)\tilde{b}^{\dagger}_{q,L}\tilde{b}_{q,L},$$
$$v = \frac{v_1+v_2}{2}\sqrt{1-\frac{4\lambda^2}{(v_1+v_2)^2}}. \tag{A6}$$

From Eq. (A6), we see that the system has a well-defined ground state if v is real and larger than $|v_1 - v_2|/2$. We find that these conditions hold if

$$v_1 v_2 > \lambda^2. \tag{A7}$$

The new bosonic fields have the forms

$$\tilde{b}_{q,R} = \frac{b_{q,R} + \gamma\, b^{\dagger}_{q,L}}{\sqrt{1-\gamma^2}},$$
$$\tilde{b}_{q,L} = \frac{b_{q,L} + \gamma\, b^{\dagger}_{q,R}}{\sqrt{1-\gamma^2}},$$
$$\gamma = \frac{1-K}{1+K}, \tag{A8}$$

where

$$K = \sqrt{\frac{v_1+v_2-2\lambda}{v_1+v_2+2\lambda}}. \tag{A9}$$

We note that the parameters v and K in Eqs. (A6) and (A9) do not depend on the value of q.

APPENDIX B: MAPPING BETWEEN ϕ AND $\pi + \phi$

In this Appendix, we will show that the systems defined by Eq. (8) for ϕ and $\pi + \phi$ can be mapped to each other by transforming the fermionic operators in a particular way. We will first consider an infinite system since the transformation is easier to discuss in that case. We consider the Hamiltonian H given in Eq. (20), which we rewrite as

$$H = \sum_j [\cos(\phi/2)(c^{\dagger}_j c_{j+2} + c^{\dagger}_{j+2}c_j) + i\sin(\phi/2)(2n_{j+1}-1) \times (c^{\dagger}_j c_{j+2} - c^{\dagger}_{j+2}c_j) - \mu c^{\dagger}_j c_j]. \tag{B1}$$

We now use the fact that $2n_j - 1$ is a Hermitian operator with eigenvalues equal to ± 1; further, it anticommutes with c_j and $c^{\dagger}_j$ but commutes with c_k and $c^{\dagger}_k$ for all $k \neq j$. We define new fermionic operators

$$\tilde{c}_j = c_j \prod_{l=0}^{\infty}(2n_{j+1+2l} - 1) \quad \text{if } j \text{ is even},$$
$$\tilde{c}_j = c_j \prod_{l=0}^{\infty}(2n_{j-1-2l} - 1) \quad \text{if } j \text{ is odd}. \tag{B2}$$

In words, $\tilde{c}_j$ is equal to c_j multiplied by a string of $2n_k - 1$ on its right on all the sites of the odd sublattice if j lies on the even sublattice and by a string of $2n_k - 1$ on its left on all the sites of the even sublattice if j lies on the odd sublattice. The crucial point to note is that the transformations in Eq. (B2) maintain the anticommutation relations $\{\tilde{c}_j, \tilde{c}_k\} = 0$ and $\{\tilde{c}_j, \tilde{c}^{\dagger}_k\} = \delta_{jk}$ for all values of j, k, and $\tilde{c}^{\dagger}_j\tilde{c}_j = c^{\dagger}_j c_j = n_j$. In terms of the new operators, the Hamiltonian in Eq. (B1) takes the form

$$H = \sum_j [\cos(\phi/2)(2n_{j+1}-1)(\tilde{c}^{\dagger}_j\tilde{c}_{j+2} + \tilde{c}^{\dagger}_{j+2}\tilde{c}_j) + i\sin(\phi/2)(\tilde{c}^{\dagger}_j\tilde{c}_{j+2} - \tilde{c}^{\dagger}_{j+2}\tilde{c}_j) - \mu\tilde{c}^{\dagger}_j\tilde{c}_j]. \tag{B3}$$

Next, we do another transformation

$$\tilde{c}_j \to e^{ij\pi/4}\tilde{c}_j \quad \text{and} \quad \tilde{c}^{\dagger}_j \to e^{-ij\pi/4}\tilde{c}^{\dagger}_j. \tag{B4}$$

Then Eq. (B3) turns into

$$H = \sum_j [-\sin(\phi/2)(\tilde{c}^{\dagger}_j\tilde{c}_{j+2} + \tilde{c}^{\dagger}_{j+2}\tilde{c}_j) + i\cos(\phi/2) \times (2n_{j+1}-1)(\tilde{c}^{\dagger}_j\tilde{c}_{j+2} - \tilde{c}^{\dagger}_{j+2}\tilde{c}_j) - \mu\tilde{c}^{\dagger}_j\tilde{c}_j]. \tag{B5}$$

Comparing Eqs. (B1) and (B5), we see that ϕ has effectively changed to $\pi + \phi$ so that $\cos(\phi/2) \to -\sin(\phi/2)$ and $\sin(\phi/2) \to \cos(\phi/2)$.

We now discuss how the above transformations work for a finite-sized system with periodic boundary conditions. We assume that the total number of sites N is even so that each sublattice has $N/2$ sites and the site indices in Eq. (B1) can only go from 1 to N. Then the string of $2n_k - 1$ in the first line of Eq. (B2) ends on the right at $k = N - 1$ and the string in the second line ends on the left at $k = 2$. Then if we look at the hopping between sites 1 and $N - 1$ or between 2 and N, we find that they will satisfy periodic boundary conditions only if

$$\prod_{l=1}^{N/2}(2n_{2l-1} - 1) = 1 \quad \text{and} \quad \prod_{l=1}^{N/2}(2n_{2l} - 1) = 1. \tag{B6}$$

These conditions imply that the number of unoccupied sites (which have $n_k = 0$) must be even on both even and odd sublattices, namely, $N/2$ minus the number of particles must be even on both sublattices. Next, we see that the transformation in Eq. (B4) will satisfy periodic boundary conditions if $e^{iN\pi/4} = 1$, i.e., if N is a multiple of 8. Hence, $N/2$ is an even number and therefore the previous condition implies that the number of particles on each sublattice should be even so that the mapping from ϕ to $\pi + \phi$ can work with periodic boundary conditions.

We note that the transformations given in Eq. (B2) between the old and new fermionic operators are highly nonlocal. Perhaps for this reason, the symmetry between ϕ and $\pi + \phi$ is not evident in the results obtained by bosonization. Namely, the expressions for various quantities in Sec. IV, such as θ, v, K_1, and K_2 in Eqs. (27), (30), (66), and (75), are not invariant under $\phi \to \pi + \phi$.

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