


Topology of an anti-parity-time symmetric non-Hermitian Su-Schrieffer-Heeger model

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We propose an anti-parity-time (anti- \mathcal{PT}) symmetric non-Hermitian Su-Schrieffer-Heeger (SSH) model, where the large non-Hermiticity constructively creates nontrivial topology and greatly expands the topological phase. In the anti- \mathcal{PT} -symmetric SSH model, the gain and loss are alternatively arranged in pairs under the inversion symmetry. The appearance of the degenerate point at the center of the Brillouin zone determines the topological phase transition, whereas the exceptional points unaffected the band topology. The large non-Hermiticity leads to unbalanced wave-function distribution in the broken anti- \mathcal{PT} -symmetric phase and induces the nontrivial topology. Our findings can be verified through introducing dissipations in every other two sites of the standard SSH model even in its trivial phase where the nontrivial topology is solely induced by the dissipations.

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I. INTRODUCTION

The discrete symmetries classify the Hermitian topological phases into 10 folds [1] and classify the non-Hermitian topological phases into 38 folds [2]. The many interesting topological properties of non-Hermitian phases have been reported [3], including the topological phase transitions [4,5], non-Hermitian band theory [6–11], topological insulators [12–19], topological metals [20–22], topological semimetals [23–28], topological invariants [29–37], and topological edge modes [38–46]. The bulk boundary correspondence (BBC) and bulk topological invariant play important roles in the topological characterization. However, in non-Hermitian systems with skin effect [47–61], the spectra between the open-boundary condition (OBC) and the periodic-boundary condition (PBC) can be dramatically distinct, and the conventional BBC is invalid because of the Aharonov-Bohm effect with imaginary magnetic flux [57]. To correctly describe the spectrum under the OBC, the non-Bloch band theory is developed [58,62–64], the quasimomentum becomes complex and varies on a generalized Brillouin zone (GBZ). A universal analytical method to obtain the GBZ is given for one-dimensional non-Hermitian systems [65]. In the presence of the non-Hermitian skin effect, the damping becomes unidirectional [66]. Zener tunneling becomes chiral at the non-Bloch collapse point [67]. The relation between edge modes and bulk topology is formulated using the Green's-function method [68]. The non-Hermitian topological systems can be implemented in many experimental platforms including the passive (active) photonic crystals of coupled

waveguides [69–75], coupled resonators [76–81], optical lattice [82–84], electronic circuits [85–88], and acoustic lattices [89,90].

The progress on the non-Hermitian Su-Schrieffer-Heeger (SSH) models [88,91–109], Aubry-André-Harper models [110–115], and Rice-Mele (RM) models [116–118] provides fundamental understanding of the non-Hermitian topological phase of matter. In the non-Hermitian SSH model with asymmetric couplings, nonzero imaginary magnetic flux [57], persistent current [63], and non-Hermitian skin effect exist. In the parity-time- (\mathcal{PT} -) symmetric non-Hermitian SSH model with gain and loss [92–98], the \mathcal{PT} symmetry prevents nonzero imaginary magnetic flux and ensures the BBC. In the exact \mathcal{PT} -symmetric region with a real spectrum, the Berry phase for each separable band is quantized; in the broken \mathcal{PT} -symmetric region with complex spectrum [119], the Berry phase for each separable band is not quantized [97]. Topological interface states are experimentally observed in \mathcal{PT} -symmetric non-Hermitian SSH lattices [120–124].

The anti- \mathcal{PT} symmetry can also protect the validity of the BBC. In this paper, we propose an anti- \mathcal{PT} -symmetric non-Hermitian SSH model through alternatively incorporating the balanced gain and loss under the inversion symmetry in the standard SSH model. The band spectrum becomes partially complex in the presence of non-Hermiticity, indicating the thresholdless anti- \mathcal{PT} -symmetry breaking. The gain and loss help creating the nontrivial topology. The topological characterization and the geometric picture of the topological phases are elaborated. The topological phase transition occurs when the band gap closes and reopens. The degenerated topological edge states have zero energy with a net gain and localized at two lattice boundaries, respectively. Exciting the edge states enable topological lasing [122–128].

II. MODEL

The schematic of the non-Hermitian SSH model is shown in Fig. 1(a), which describes a one-dimensional coupled

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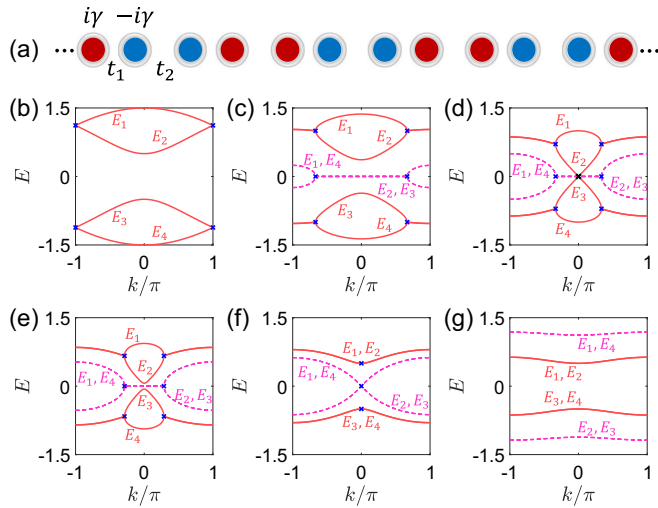


FIG. 1. (a) Schematic of the anti- \mathcal{PT} -symmetric non-Hermitian lattice of a coupled resonator array. The gain (loss) is indicated in red (blue). The spectra are shown in (b)–(g) with system parameters $t_1 = 1$, $t_2 = 1/2$, and $\gamma = 0$ for (b), $\gamma = 1/2$ for (c), $\gamma = \sqrt{3}/2$ for (d), $\gamma = 9/10$ for (e), $\gamma = 1$ for (f), and $\gamma = 3/2$ for (g). The red lines indicate the real part of energy, and the magenta dashed lines indicate the imaginary part of energy where the blue crosses indicate the edge degenerate point (DP) for $\gamma = 0$ or exceptional points (EPs) for $\gamma \neq 0$, and the black cross indicates a central DP in (d).

resonator array. All the resonators have identical resonant frequency. The staggered distance between the nearest-neighbor resonators determines the lattice couplings t_1 and t_2 [120–124], which classify two sublattices in the SSH model,

$$H_0 = \sum_j (t_1 a_j^\dagger b_j + t_2 b_j^\dagger a_{j+1} + \text{H.c.}),$$

where a_j^\dagger (b_j^\dagger) and a_j (b_j) are the creation and annihilation operators for the sublattice site indexed j . To create the anti- \mathcal{PT} symmetry [129–134], the gain and loss are introduced in the resonators under the inversion symmetry in the form of $\{i\gamma, -i\gamma, -i\gamma, i\gamma\}$ in the four-site unit cell,

$$H_1 = i\gamma \sum_j (a_{2j-1}^\dagger a_{2j-1} - b_{2j-1}^\dagger b_{2j-1} - a_{2j}^\dagger a_{2j} + b_{2j}^\dagger b_{2j}). \quad (1)$$

The Hamiltonian of the anti- \mathcal{PT} -symmetric non-Hermitian SSH model reads

$$H = H_0 + H_1. \quad (2)$$

As shown in Fig. 1(a), the anti- \mathcal{PT} -symmetric non-Hermitian SSH model is invariant under a π rotation of the left (right) non-Hermitian dimer and glide half of the unit cell in the translational direction [135].

In comparison to the \mathcal{PT} -symmetric non-Hermitian SSH model with the gain and loss $\{i\gamma, -i\gamma, i\gamma, -i\gamma\}$ in the four-site unit cell, the only difference between the anti- \mathcal{PT} -symmetric and \mathcal{PT} -symmetric SSH models is that the arrangement of the even number of gain and loss pairs; the anti- \mathcal{PT} -symmetric SSH model has the inversion symmetry, whereas the \mathcal{PT} -symmetric SSH model does not.

Interestingly, the role played by the non-Hermiticity γ is completely different in these two models. The non-Hermiticity γ in the anti- \mathcal{PT} -symmetric SSH model constructively create nontrivial topology. The topologically trivial phase changes into the topologically nontrivial phase as the increasing in non-Hermiticity γ . The nontrivial topology of the anti- \mathcal{PT} -symmetric SSH model can be directly verified in many experimental platforms that used to demonstrate the \mathcal{PT} -symmetric SSH model. The topological aspect of the anti- \mathcal{PT} -symmetric SSH model completely differs from that of the nonsymmorphic RM model [136].

Applying the Fourier transformation, the Bloch Hamiltonian of the lattice is obtained as

$$H_k = \begin{pmatrix} i\gamma & t_1 & 0 & t_2 e^{-ik} \\ t_1 & -i\gamma & t_2 & 0 \\ 0 & t_2 & -i\gamma & t_1 \\ t_2 e^{ik} & 0 & t_1 & i\gamma \end{pmatrix}. \quad (3)$$

H_k has the anti- \mathcal{PT} -symmetry $(\mathcal{PT})H_k(\mathcal{PT})^{-1} = -H_k$ with $\mathcal{P} = i\sigma_x \otimes \sigma_y$, and $\mathcal{T} = \mathcal{K}$ is the complex-conjugation operation. H_k also has $\text{TRS}^\dagger \mathcal{C}_+ H_k^\dagger \mathcal{C}_+^{-1} = H_{-k}$ with $\mathcal{C}_+ = \sigma_0 \otimes \sigma_0$, $\text{PHS}^\dagger \mathcal{T}_- H_k^* \mathcal{T}_-^{-1} = -H_{-k}$ with $\mathcal{T}_- = \sigma_0 \otimes \sigma_z$, and chiral symmetry (pseudo-anti-non-Hermiticity) $\Gamma H_k^\dagger \Gamma^{-1} = -H_k$ with $\Gamma = \mathcal{C}_+ \mathcal{T}_-$, where σ_0 and $\sigma_{x,y,z}$ are the two-by-two identical matrix and Pauli matrix. The system belongs to the BDI^\dagger class in the 38-fold topological classifications of non-Hermitian systems, and the topological phase-transition of the BDI^\dagger class is determined by the closure of the band gap of the real part of energy bands [2]. The interplay between the couplings and the non-Hermiticity alters the band topology and generates the nontrivial topology; furthermore, the loss can solely induce the nontrivial topology if we consider a common gain term $i\gamma$ is removed from H_k . This greatly simplifies the verification of the anti- \mathcal{PT} -symmetric SSH model in experiments.

In contrast to the \mathcal{PT} symmetry ensures the energy levels to be conjugate in pairs, the anti- \mathcal{PT} symmetry ensures the energy levels in pairs with identical imaginary part and opposite real part. The four energy bands are

$$E_{\pm, \pm} = \pm \sqrt{t_1^2 + t_2^2 - \gamma^2 \pm 2t_2 \sqrt{t_1^2 \cos^2(k/2) - \gamma^2}}. \quad (4)$$

In the absence of the gain and loss ($\gamma = 0$), the lattice is the Hermitian SSH model. At the topological phase-transition point $t_1 = t_2$, two bands $\pm 2t_1 \cos(k/2)$ of the SSH model are connected at the DP $k = \pm\pi$; the four-band spectrum $E_{\pm, \pm}$ of H_k can be regarded as the spectrum of the SSH model folded at $k = \pm\pi/2$ and stretched to the entire BZ. Thus, the central band gap closes at the DP at the center of the BZ $k = 0$, and the band folding generates another DP at the edge of the BZ $k = \pm\pi$. At $t_1 \neq t_2$, the SSH model is gapped, and the central gap is open as shown in Fig. 1(b); however, the spectrum of H_k still has a DP at the edge of BZ protected by the nonsymmorphic symmetry in the four-site unit cell of the SSH model [135].

In the presence of the gain and loss ($\gamma \neq 0$), the non-Hermiticity splits the edge DP into two EPs associated with the anti- \mathcal{PT} -symmetry breaking [Fig. 1(c)]. As the increase in the non-Hermiticity, the two EPs gradually move, and the complex energy region expands from the edge to the center of

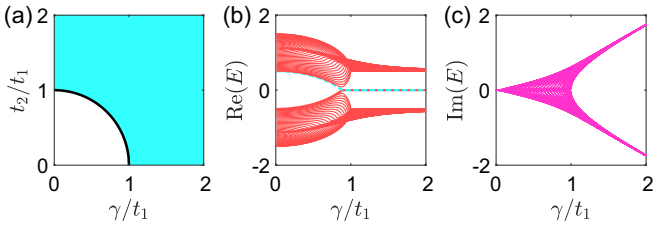


FIG. 2. (a) Phase diagram for the anti- \mathcal{PT} -symmetric non-Hermitian SSH model. The real part (b) and the imaginary part (c) of the energy spectrum as a function of the non-Hermiticity; the parameters are $t_1 = 1$ and $t_2 = 1/2$.

the BZ as shown in Figs. 1(d) and 1(e). When $\gamma^2 = t_1^2$, two EPs merge to one EP at the center of the BZ [Fig. 1(f)] and disappear for $\gamma^2 > t_1^2$ [Fig. 1(g)].

The band gap of the central two bands closes at $E = 0$ as presented in Fig. 1(d), which requires $(\gamma^2 + t_2^2 - t_1^2)^2 + 4t_1^2 t_2^2 \sin^2(k/2) = 0$. The central two bands touch at the DP $k = 0$ associated with the topological phase transition at the critical non-Hermiticity,

$$\gamma^2 + t_2^2 = t_1^2. \quad (5)$$

III. PHASE DIAGRAM

The anti- \mathcal{PT} -symmetric non-Hermitian SSH model reduces to the SSH model for $\gamma = 0$, which has the topologically nontrivial phase for $t_2^2 > t_1^2$ and the topologically trivial phase for $t_2^2 < t_1^2$. However, the situation changes in the presence of non-Hermiticity as shown in the phase diagram Fig. 2(a). The non-Hermiticity creates the nontrivial topology and the topological region expands in the anti- \mathcal{PT} -symmetric non-Hermitian SSH model, where $\gamma^2 + t_2^2 < t_1^2$ is the topologically trivial phase and $\gamma^2 + t_2^2 > t_1^2$ is the topologically nontrivial phase. The nontrivial topology of the anti- \mathcal{PT} -symmetric SSH model can be solely created by the non-Hermiticity because large non-Hermiticity induces unbalanced distributions of the wave-function probability. The non-Hermiticity generates nontrivial topology in the uniform chain at $t_1^2 = t_2^2$ [137] and even in the trivial phase of the Hermitian SSH model at $t_2^2 < t_1^2$.

The real part and the imaginary part of the energy bands under the OBC as a function of the non-Hermiticity are shown in Figs. 2(b) and 2(c), respectively. The anti- \mathcal{PT} -symmetric non-Hermitian SSH lattice under the OBC has one pair of edge states in the topologically nontrivial phase. To further elucidate the band structure, the energy bands on the complex energy plane for the non-Hermitian SSH lattice under the OBC are plotted as shown in Figs. 3(a)–3(f); the corresponding PBC spectra are shown in Figs. 1(b)–1(g). For $\gamma^2 \leq t_1^2$ and $\gamma^2 + t_2^2 \neq t_1^2$, the real part of the energy bands is gapped (the central gap is open). However, the energy bands E_1 and E_2 (E_3 and E_4) are inseparable [6] because of the existence of edge DP or EPs (blue crosses) as shown in Figs. 1(b), 1(c), 1(e), and 1(f); in this sense, the four energy bands can be regarded as two energy bands E_r (cyan) and E_l (magenta) according to the real part of energy bands as shown in Figs. 3(a), 3(b), 3(d), and 3(e). For $\gamma^2 + t_2^2 = t_1^2$ (topological phase transition), the real gap is closed (the central gap is closed) and the two

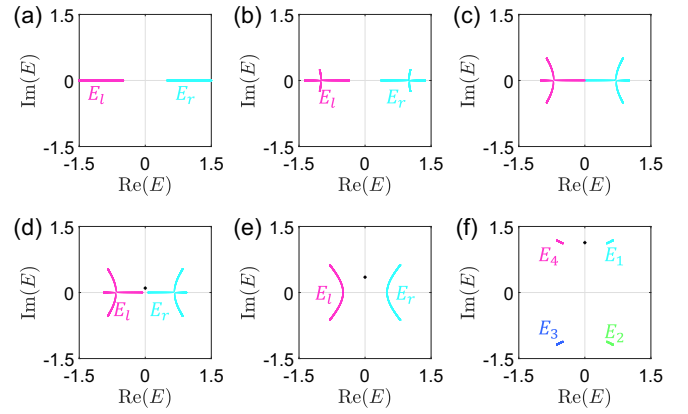


FIG. 3. Energy bands on the complex energy plane under the OBC. The parameters in (a)–(f) are similar as those in Figs. 1(b)–1(g). The black stars indicate a pair of edge states for topological nontrivial phases as shown in (d)–(f).

energy bands E_r and E_l become single band [Fig. 3(c)]. For $\gamma^2 > t_1^2$, the EP disappears between E_1 and E_2 (E_3 and E_4), and the four energy bands are separated [Fig. 3(f)]. In the topologically nontrivial region $\gamma^2 + t_2^2 > t_1^2$, there exist one pair of degenerated zero modes (black stars) with gain for the lattice under the OBC as shown in Figs. 3(d)–3(f).

IV. EDGE STATE

The inversion symmetry and TRS^\dagger ensure the validity of the conventional BBC because both of them lead to real wave numbers for the anti- \mathcal{PT} -symmetric non-Hermitian lattice under the OBC [2,57]. The lattice under the OBC supports two degenerate edge states localized at the left and right boundaries of the lattice, respectively. The eigenvalues of edge states are imaginary,

$$\varepsilon = i\{\lambda + [4\gamma^2 - 3(t_2^2 + t_1^2)]/\lambda - \gamma\}/3, \quad (6)$$

where $\lambda = \sqrt[3]{\alpha + \sqrt{\beta}}$ with $\alpha = (18t_2^2 + 8\gamma^2 - 9t_1^2)\gamma$ and $\beta = 27\{[4(2t_2^2 + 4\gamma^2 - 5t_1^2)t_2^2 - t_1^4]\gamma^2 + (t_1^2 + t_2^2)^3\}$. The left edge state localizes at the left boundary of the lattice. Without loss of generality, the wave functions in the j th unit cell of the left edge state $|\psi_L\rangle$ can be expressed as $\chi^{j-1}\{1, (\varepsilon - i\gamma)/t_1, -\chi t_2/t_1, 0\}$ with $\chi = (i\gamma - \varepsilon)/(i\gamma + \varepsilon)$. For $\gamma = 0$, the edge states reduce to zero modes with $\varepsilon = 0$ and the wave functions in the j th unit cell as $(t_1/t_2)^{j-1}\{1, 0, -t_1/t_2, 0\}$. The right edge state $|\psi_R\rangle$ is the mirror reflection of the left edge state $|\psi_L\rangle$. The edge states have net gain rate and are useful for topological lasing.

V. GEOMETRIC PICTURE OF BAND TOPOLOGY

The topology of the anti- \mathcal{PT} -symmetric non-Hermitian SSH model relates to the geometry of the Bloch Hamiltonian winding around the degeneracy points in a two-dimensional parameter space, we show how the non-Hermiticity creates the nontrivial topology. For the Bloch Hamiltonian Eq. (3), we replace e^{ik} by $h_x + ih_y$ to create a two-dimensional parameter

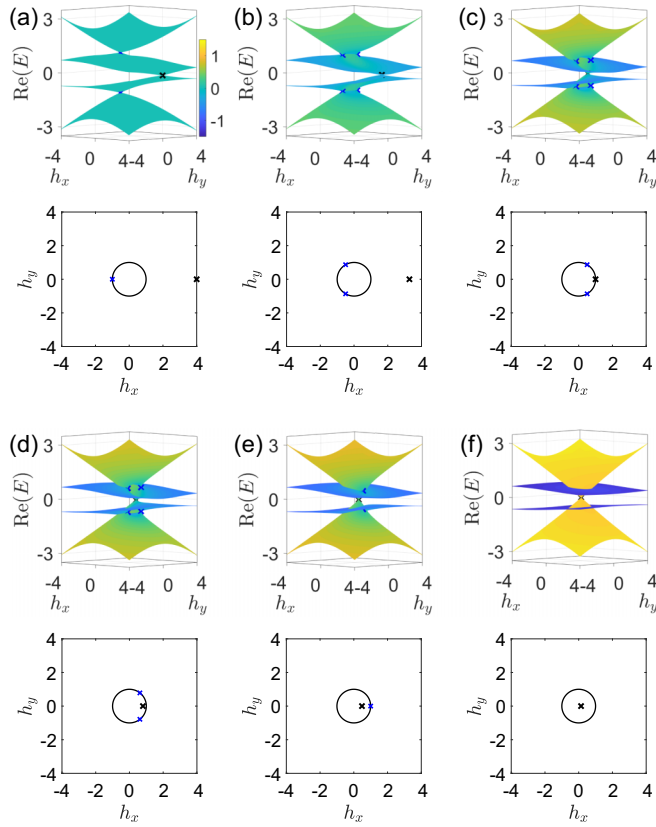


FIG. 4. Complex energy bands and geometric picture of the topology in the two-dimensional parameter space (h_x, h_y) for the anti- \mathcal{PT} -symmetric non-Hermitian SSH model. The upper panels depict the complex energy bands where the value indicates the real part and the color indicates the imaginary part of the energy; the lower panels depict the geometric picture of the topology. The unit circle indicates the anti- \mathcal{PT} -symmetric non-Hermitian SSH model, the blue crosses indicate the edge DP or EPs, and the black crosses indicate the central DP. The parameters are (a) $\gamma = 0$, (b) $\gamma = 1/2$, (c) $\gamma = \sqrt{3}/2$, (d) $\gamma = 9/10$, (e) $\gamma = 1$, and (f) $\gamma = 3/2$; other parameters are $t_1 = 1$ and $t_2 = 1/2$.

space (h_x, h_y) ,

$$H(h_x, h_y) = \begin{pmatrix} i\gamma & t_1 & 0 & t_2(h_x - ih_y) \\ t_1 & -i\gamma & t_2 & 0 \\ 0 & t_2 & -i\gamma & t_1 \\ t_2(h_x + ih_y) & 0 & t_1 & i\gamma \end{pmatrix}, \quad (7)$$

where the k -dependent Bloch Hamiltonian Eq. (3) corresponds to a unit circle $h_x^2 + h_y^2 = 1$ in the two-dimensional parameter space (h_x, h_y) .

Figure 4 depicts the complex energy bands extended to the two-dimensional parameter space (h_x, h_y) for the anti- \mathcal{PT} -symmetric non-Hermitian SSH model at fixed parameters $t_1 = 1$ and $t_2 = 1/2$ for the different non-Hermiticity γ . The edge DP (blue cross) on the unit circle at $\gamma = 0$ splits into two EPs on the unit circle at nonzero non-Hermiticity $\gamma < 1$; and the two EPs are symmetrically distributed about $h_y = 0$ because the EPs are symmetrically distributed about $k = 0$ in the BZ [see Figs. 1(c)–1(e)]. As the non-Hermiticity increases, the two EPs move on the unit circle from $(h_x, h_y) = (-1, 0)$ to

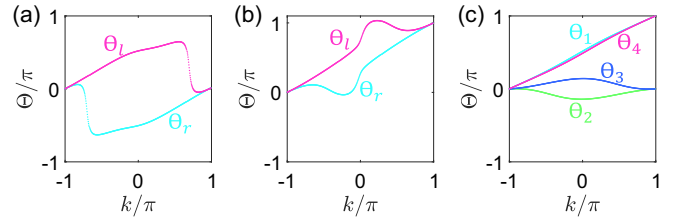


FIG. 5. Zak phase and *partial* global Zak phase. The parameters are $t_1 = 1$, $t_2 = 1/2$, and $\gamma = 1/2$ for (a), $\gamma = 9/10$ for (b), and $\gamma = 3/2$ for (c).

$(h_x, h_y) = (1, 0)$; at $\gamma = 1$, the two EPs merge into single EP at $(h_x, h_y) = (1, 0)$; and the EP vanishes for $\gamma > 1$. The edge DP or the EPs remain on the unit circle, and the topology is fully determined by the central DP (black cross). From the unit circle winding around the central DP, we can observe how the nontrivial topology is created by the non-Hermiticity γ . For $\gamma = 0$ in Fig. 4(a), the central DP is outside the unit circle; thus, the system is in the topologically trivial phase. The nonzero non-Hermiticity γ moves the central DP along $h_y = 0$ towards the negative h_x direction in the parameter space. For $\gamma = 1/2$ in Fig. 4(b), the central DP moves to $(h_x, h_y) = (3.284, 0)$, and the system enters the white region of the phase diagram as shown in Fig. 2(a). For $\gamma = \sqrt{3}/2$ in Fig. 4(c), the central DP moves to $(h_x, h_y) = (1, 0)$ and locates on the unit circle; the system is at the boundary of the white and cyan regions. For $\gamma = 9/10$ in Fig. 4(d), the central DP is enclosed in the unit circle, the topology of the system changes, and the system enters the cyan region. For $\gamma = 1$ in Fig. 4(e), the central DP keeps inside the unit circle; the system is in the nontrivial phase for the nonzero t_2 , but the energy bands are not completely separated. For $\gamma = 3/2$ in Fig. 4(f), the central DP is still inside the unit circle; in this situation, the four bands are completely separated and the system is in the nontrivial phase.

VI. ZAK PHASE AND PARTIAL GLOBAL ZAK PHASE

When the four complex energy bands are separated at $\gamma^2 > t_1^2$, each energy band is associated with a Zak phase,

$$\Theta_n = i \oint dk \langle \varphi_n | \partial_k | \psi_n \rangle. \quad (8)$$

In the definition of Θ_n , $|\varphi_n\rangle$ is the left eigenstate, and $|\psi_n\rangle$ is the right eigenstate, $H_k|\psi_n\rangle = E_n|\psi_n\rangle$ and $H_k^\dagger|\varphi_n\rangle = E_n^*|\varphi_n\rangle$, where the subscript n is the band index. E_1 denotes the band with positive real and imaginary energies, E_2 denotes the band with positive real and negative imaginary energies, E_3 denotes the band with negative real and imaginary energies, and E_4 denotes the band with negative real and positive imaginary energies as shown in Figs. 3(f) and Fig. 1(g). Their wave functions are $|\psi_1\rangle$, $|\psi_2\rangle$, $|\psi_3\rangle$, and $|\psi_4\rangle$, respectively. The system has the inversion symmetry, which ensures that the Zak phase for each separated energy band is an integer of π . Thus, the Zak phase is used for topological characterization. In Fig. 5(c), the Zak phases for the bands E_1 and E_4 are π , and the Zak phases for the bands E_2 and E_3 are 0; which

are consistent with the geometric picture in Fig. 4(f), the central DP belongs to energy bands E_1 and E_4 and predicts the existence of one pair of the topological zero modes with gain for the system under the OBC.

For the energy bands embedded with EPs ($\gamma^2 \leq t_1^2$ and $\gamma^2 + t_2^2 \neq t_1^2$), there are only two energy bands E_r and E_l . The two-state coalescence EP₂₁₂ exists only in the energy bands E_1 and E_2 , and the two-state coalescence EP₂₃₄ exists only in the energy bands E_3 and E_4 . In this sense, we define two *partial* global Zak phases,

$$\Theta_r = \Theta_1 + \Theta_2; \Theta_l = \Theta_3 + \Theta_4. \quad (9)$$

In the calculation of the *partial* global Zak phase, the momentum ranges $[k_{EP} - \Delta k, k_{EP} + \Delta k]$ are removed because that the coalesced wave functions are self-orthogonal at the EPs [137], where Δk is an infinite small positive real number. The *partial* global Zak phase is valid for the topological characterization.

For $\gamma^2 \leq t_1^2$ and $\gamma^2 + t_2^2 < t_1^2$, both *partial* global Zak phases Θ_r and Θ_l are 0 as shown in Fig. 5(a). This indicates the phase is topologically trivial without any edge state under the OBC. However, for $\gamma^2 \leq t_1^2$ and $\gamma^2 + t_2^2 > t_1^2$, both *partial* global Zak phases Θ_r and Θ_l are π as shown in Fig. 5(b). This indicates the phase is topologically nontrivial and one pair of topological edge states appear under the OBC.

VII. CONCLUSION

We propose the anti- \mathcal{PT} -symmetric non-Hermitian SSH model as a prototypical anti- \mathcal{PT} -symmetric topological lattice. The gain and loss are alternatively introduced in pairs in the standard SSH model through holding the inversion symmetry. The inversion symmetric gain and loss result in the thresholdless breaking of anti- \mathcal{PT} symmetry, and the energy spectrum is partially or fully complex. We provide novel insights on the roles played by the anti- \mathcal{PT} symmetry and non-Hermiticity in the topological phases. The large non-Hermiticity constructively creates the nontrivial topology and greatly expands the topologically nontrivial region of the SSH model. The topological edge states localized at two boundaries of the lattice are degenerate and suitable for topological lasing. Besides, the dissipation can solely induce the nontrivial topology. In comparison to the \mathcal{PT} -symmetric non-Hermitian SSH model, only the arrangement of gain and loss in the anti- \mathcal{PT} -symmetric non-Hermitian SSH model is different; the proposed anti- \mathcal{PT} -symmetric non-Hermitian SSH model can be easily implemented in the microring resonator arrays, coupled optical waveguides, photonic crystals, electronic circuits, and acoustic lattices [69–87,89,90].

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