Levitation of superconducting microrings for quantum magnetomechanics

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Levitation of superconductors is becoming an important building block in quantum technologies, particularly, in the rising field of magnetomechanics. In most of the theoretical proposals and experiments, solid geometries, such as spheres are considered for the levitator. Here we demonstrate that replacing them by superconducting rings brings two important advantages: First, the forces acting on the ring remain comparable to those expected for solid objects, whereas the mass of the superconductor is greatly reduced. In turn, this reduction increases the achievable trap frequency. Second, the flux trapped in the ring by in-field cooling yields an additional degree of control for the system. We construct a general theoretical framework with which we obtain analytical formulations for a superconducting ring levitating in an anti-Helmholtz quadrupole field and a dipole field for both zero-field and field cooling. The positions and the trapping frequencies of the levitated rings are analytically found as a function of the parameters of the system and the field applied during the cooling process. Unlike what is commonly observed in bulk superconductors, lateral and rotational stabilities are not granted for this idealized geometry. We, therefore, discuss the requirements for simple superconducting structures to achieve stability in all degrees of freedom.

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I. INTRODUCTION

Levitation of superconductors (SCs) has been an active topic of research in the past decades due to its significant scientific and technological potentials. Until recently, most of the research pertained to large-scale systems with applications including transportation and energy storage [1–5]. A typical levitation system consists of a source of the magnetic field (e.g., one or several permanent magnets) and one (or several) superconducting objects. The performance and optimization of such devices are based mainly on the choice and control of the magnetic-field sources on the optimization of the superconducting properties and on the selection of an advantageous geometry for the superconductor. Typically, a set of bulk high-T_c superconductors are used as levitators, each of them with sizes on the order of a few centimeters [6]. As a rule of thumb, levitation forces are enhanced by increasing the critical-current density of the SCs and the gradients of external fields; stability is increased by an adequate tuning of the field distribution and the cooling procedure [7]. At these scales, apart from the geometry, the key parameter of the superconductor is the critical-current density J_c, which is related to the internal vortex structure as a macroscopic, averaged quantity. The critical-state model is commonly used to describe the superconductor response [6]. The calculation of the current density distribution inside the superconducting materials is the basic step from which one can evaluate all the relevant parameters, such as force, stability, or energy.

The presence of an external magnetic field during the cooling below T_c [field-cooling (FC) process] can result in a substantially different current distribution in the SC compared to the case where the superconductor is cooled in the absence of an external magnetic-field [zero-field-cooling (ZFC) process], yielding different levitation forces and stabilities At macroscopic scales, the effects of cooling procedures have previously been considered in several works [8,9].

An important property of superconductors with nonsimply connected geometries, such as rings, is that the flux which threads a hole surrounded by a superconducting region must be conserved. At the macroscopic scale, one also needs to evaluate the current distribution inside the rings; the flux conservation through the hole is a consequence of the resulting flux distribution [10,11].

Here, we discuss superconducting rings with sizes on the order of hundreds of micrometers or smaller. At these scales, the basic principles of superconductivity still hold, but some assumptions generally taken for granted in macroscopic systems need to be addressed in more detail. In particular, macroscopic approximations, such as the critical-state model, are not applicable, and other approaches are required.

One of the key points, however, is that the details of the current distribution inside the ring become irrelevant, and flux conservation can be used as the condition for evaluating the total circulating current, which becomes the relevant quantity.

Recently, new applications of levitating superconductors have emerged in the context of quantum magnetomechanics where magnetically trapped superconducting objects can be used to create low-loss mechanical oscillators with long coherence times [12–17] or highly sensitive inertial sensors [18]. A typical system consists of a superconducting object

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levitated in a specifically designed magnetic field, which creates a trap for the superconductor. The objective of these proposed experiments is to perform ground-state cooling of these objects, allowing the creation of nonclassical states.

An example is the creation of a superposition in which the separation of the wave-function peaks is larger than the object itself. In the case of a sphere, this goal requires a separation of twice the radius, whereas for a ring the relevant dimension is the thickness, which can be far smaller. This geometry is, therefore, also advantageous for experiments seeking to probe gravitational interactions.

In this paper, we develop a comprehensive analytical theory for the levitation of superconducting microrings and demonstrate how their use can be highly advantageous for applications in quantum magnetomechanics. We use the flux conservation condition to analyze the motion of microrings in dipolar and quadrupolar magnetic-field geometries. For vertical displacements, we derive analytical expressions for the circulating current and the resulting oscillator dynamics.

We find that a ring with the same mass as a given sphere can achieve significantly higher trap frequencies by distributing the mass on a larger radius. This alleviates perturbations caused by low-frequency noise, and the larger forces that can, thus, be exerted are beneficial for active expansion of the wave function [12,15,19].

The idealized ring geometry does not lead to stable trapping in the lateral or angular degrees of freedom. We describe modifications by which stable levitation can nonetheless be achieved.

After defining key parameters of the levitator in Sec. II, we study different scenarios for cooling and trapping small superconducting rings in the presence of external fields (Fig. 1). The flux-conservation equation is used to derive analytical expressions for the circulating currents and to evaluate the interaction energy between the magnetic-field source and the levitating superconductor. We focus on two cases of interest. The first, a magnetic landscape created by a pair of anti-Helmholtz coils (AHC), is introduced in Sec. III. This geometry is of practical importance since it has been proposed for experiments in quantum magnetomechanics with spheres [12] and is discussed at length in Sec. III B. The second case studied is a superconducting ring coaxial with the field created by a point dipole. We find that for a given cooling position, a single parameter is sufficient to characterize the dynamics of the system along the rings' axis. We derive the equilibrium positions and study its vertical stability for the ZFC and FC cases in Secs. IV and V. In Sec. VI, we discuss the lateral and angular degrees of freedom of the ring and modifications to the architecture which enables stable trapping in all spatial directions. Conclusions are presented in Sec. VII.

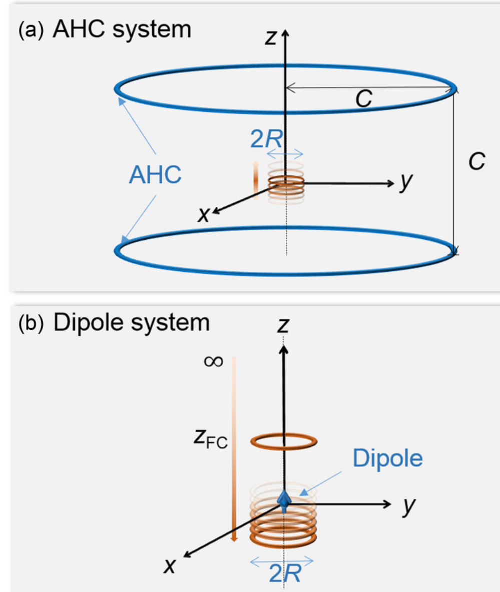

FIG. 1. Ring levitation scenarios. (a) SC ring of radius R in the field of anti-Helmholtz coils (AHC) of radius C. The colored arrows represent the movement of the ring. (b) A magnetic dipole generates the field, and the SC ring moves from infinity (zero-field-cooled case), or it is cooled down below its critical temperature at a given field-cooling position (z_{FC}) and moved afterwards.

II. MODEL FOR THE SUPERCONDUCTING RING

Consider a superconducting ring of mean radius R through which a total current I is circulating so that the cross section is a circle of radius $a \ll R$. The self-inductance of the ring can be regarded as [20]

$$L = \mu_0 R\left[\ln\left(\frac{8R}{a}\right) - b\right], \tag{1}$$

where μ_0 is the vacuum permeability and b is a dimensionless constant that accounts for the particular current distribution across the cross section of the superconductor: If the current is uniformly distributed over the cross section $b \simeq 1.75$, and if it flows over the surface $b \simeq 2$. Since $a \ll R$, the particular distribution of current inside the ring hardly affects the self-inductance. Thus, the total circulating current I is the relevant parameter. Note that the inductance L depends on the radius of the ring as $L \sim R \ln R$. In this paper, we assume that the superconducting ring is in the Meissner state with a London penetration depth λ smaller than all other relevant dimensions. This allows us to set $b = 2$.

While the ring is in the superconducting state, the magnetic flux threading it is maintained [21]. We will distinguish between the ZFC regime in which the ring has reached the superconducting state far from any field source so that the flux that threads it is zero and will be zero after any subsequent displacement; and the FC regime where the SC has been cooled when some flux was threading it, a flux that will remain after movements. Except otherwise indicated, the superconducting ring has the axis coincident with the z axis. The levitated superconductor has a total mass of M. Gravity has direction $-\hat{\mathbf{z}}$. We use standard cylindrical coordinates (ρ, ϕ, z).

In the limit $\lambda \ll a$ assumed throughout this paper, the flux threading the superconducting ring is quantized in multiples of the flux quantum Φ_0. If one is dealing with smaller dimensions such that λ is no longer negligible, instead of considering flux quantization, one should consider *fluxoid* quantization (which includes not only the flux threading the ring, but also a contribution of the currents in the superconductor volume) [22].

III. A SMALL RING IN THE FIELD OF AN ANTI-HELMHOLTZ COIL

A. Analytical description

The first case we analyze corresponds to a superconducting ring levitating in the field created by a pair of AHCs. Such a field configuration, using a levitated superconducting sphere, has been recently proposed for improving the performance in quantum systems with respect to optical counterparts (e.g., longer coherence times), enabling in this way access to a completely new parameter regime of macroscopic quantum physics [12,19].

We consider AHC coils with radii C (and, thus, separation between parallel coils C), coaxial with the z axis and centered at the origin, and with a circulating current I_0. The magnetic field created by the AHC $\mathbf{B}_{\mathrm{AHC}}$ and the vector potential $\mathbf{A}_{\mathrm{AHC}}$ at the central region (ρ, $|z| \ll C$) can be approximated by

$$\mathbf{B}_{\mathrm{AHC}}(\rho, z) = \mu_0 \frac{24}{25\sqrt{5}} \frac{I_0}{C^2}(-\rho\hat{\boldsymbol{\rho}} + 2z\hat{\mathbf{z}}), \tag{2}$$

$$\mathbf{A}_{\mathrm{AHC}}(\rho, z) = \mu_0 \frac{24}{25\sqrt{5}} \frac{I_0}{C^2} z\rho\hat{\boldsymbol{\phi}}. \tag{3}$$

We begin our discussion by assuming that the superconducting ring of radius $R \ll C$ is located at the origin in the absence of any applied field or current. The flux threading the SC ring is zero. We also consider that the ring can be moved in the vertical direction, maintaining the condition $|z| \ll C$. The current induced in the ring can be evaluated from the flux-conservation equation as

$$2\pi R A_\phi(R, z) + LI = 0. \tag{4}$$

Here $A_\phi(R, z)$ is the ϕ component of the magnetic vector potential evaluated at position (R, z). The first term in Eq. (4) is the flux threading the superconducting ring due to the field produced by the AHC coils, and the second one is the flux due to the current circulating in the ring.

From Eqs. (3) and (4), the current in the ring is

$$I = -\Theta \frac{z}{R}, \tag{5}$$

where we have defined the constant,

$$\Theta \equiv \mu_0 I_0 \frac{48\pi}{25\sqrt{5}} \frac{R^3}{C^2 L} = \partial_z B_{\mathrm{AHC},z} \frac{R^3}{L}, \tag{6}$$

that characterizes the system and can be identified as the current induced in the ring when the displacement is equal to the rings' radius. The field of the AHC is fully characterized by the gradient of the magnetic field along the axis $\partial_z B_{\mathrm{AHC},z}$.

The z component of the Lorentz force, F_z acting on a ring centered on the axis of the field is

$$F_z = -\Theta^2 \frac{L}{R^2} z\,\hat{\mathbf{z}}, \tag{7}$$

yielding a spring constant for vertical displacement,

$$\kappa_z \equiv -\frac{\partial F_z}{\partial z} = \Theta^2 \frac{L}{R^2}. \tag{8}$$

κ_z is always positive, which indicates vertically stable levitation. The vertical frequency of this trap is obtained from the spring constant as

$$\omega_z = \frac{\Theta}{R}\sqrt{\frac{L}{M}}. \tag{9}$$

Following Eqs. (1) and (6), ω_z depends on R as $\omega_z \sim R^{3/2}(\ln R)^{-1/2}$.

The potential-energy E of the system is the sum of its magnetic and gravitational potential energies. The magnetic energy of the system is given by the interaction of the AHC field with an induced dipole of magnetic moment $I\pi R^2\hat{\mathbf{z}}$. Setting the gravitational potential energy to zero at $z = 0$ gives

$$E = Mgz - \tfrac{1}{2} I\pi R^2 \hat{\mathbf{z}} \cdot \mathbf{B}_{\mathrm{AHC}}(0, z), \tag{10}$$

where g is the gravitational acceleration. By minimizing E with respect to the vertical position z we find the equilibrium position,

$$z_{eq} = -\frac{M\,g R^2}{\Theta^2 L}. \tag{11}$$

If the ring is cooled below the critical temperature at a vertical position $z_{FC} \neq 0$ where some flux is threading it before cooling (FC case), this flux is maintained after subsequent movements. The flux conservation equation for this case is

$$2\pi R A_\phi(R, z) + LI = 2\pi R A_\phi(R, z_{FC}). \tag{12}$$

Owing to the linear dependence of the potential vector on the vertical position (within the approximations considered here), the above expressions for the ZFC case remain valid with the change $z \rightarrow z - z_{FC}$ or $z_{eq} \rightarrow z_{eq} - z_{FC}$. Actually, this result permits selecting the desired value for the levitation position z_{eq} by adjusting the cooling position z_{FC} in an actual experimental setting.

B. Considerations for practical implementation

We now discuss the implementation of the system introduced above and apply the established theory. In Fig. 2, we compare the oscillation frequencies and resulting zero-point motion $\sqrt{\hbar/2M\omega}$ for levitated rings of varying inner radius to spheres of the same radius. These plots rest on several assumptions and choices, which we describe in what follows.

Let the field source be a macroscopic coil pair in the anti-Helmholtz configuration along the vertical axis, providing the idealized AHC field at its center with vertical B-field gradients of up to several 100 T/m. This assumption can easily be achieved, even for a relatively large volume around the center of the trap, hence, there is no need to consider deviations from the linear field dependence. We use a relatively modest vertical B-field gradient of 50 T/m throughout the calculations.

The levitating ring can, for example, be fabricated as a niobium ring on a silicon chip. Using silicon-on-insulator technology, the substrate can be made very thin, on the order of 100 nm, contributing negligibly to the mass of the levitator. The choice of the substrate thickness and shape still allows to add ballast to the levitator if required (see Sec. VI). For this example, we concentrate on the ease of fabrication and neglect the fact that niobium is a Type-II superconductor. A critical current density of $J_c = 5 \times 10^{10}$ A/m^2 is assumed, which is conservative for our parameters [23].

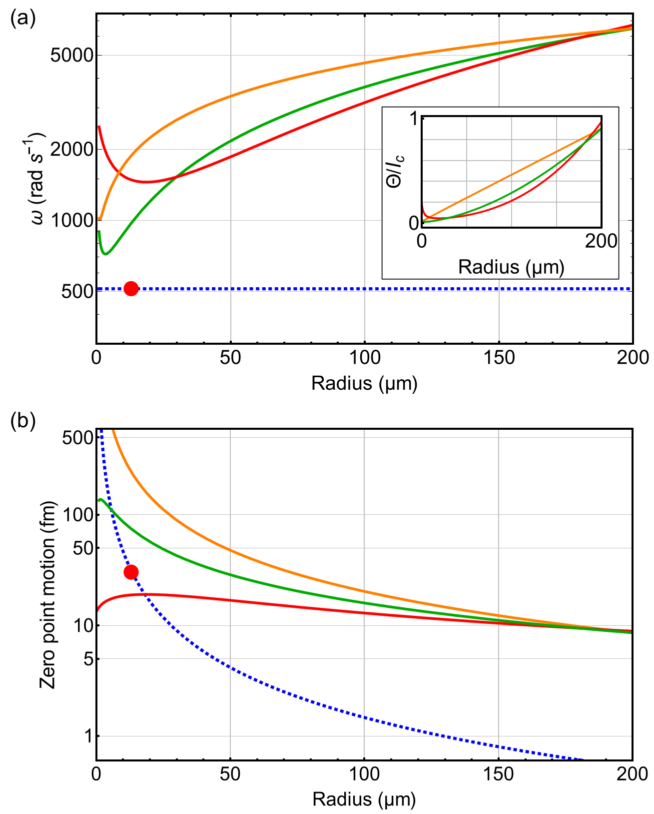

FIG. 2. (a) Trapping frequency of a sphere (dashed blue line) vs radius R compared to the trapping frequencies of superconducting rings with constant wire width w of 10 μm (orange), constant aspect ratio ($w = R/20$) (green) and constant system mass 10^{-10} kg (red). The red dot marks the size of a sphere of the same mass. The inset shows that the ratio $\Theta/I_c < 1$ for the considered range. See the text for further details. (b) Zero-point motion of the levitators. Line coloring as in panel (a).

An important technical restriction is the achievable thickness of the superconducting layer, which can be limited to a few micrometers due to mechanical instabilities of the deposited material. Depending on that layer thickness, the width of the superconductor has to be chosen appropriately to accommodate the induced current without exceeding the critical current I_c. It is, therefore, important to note that it might not always be possible to arbitrarily scale the rings' size whereas maintaining its ratio of radius, thickness, and width. Rather, the thickness of the layer will most likely be fixed at the highest attainable value. For the niobium layer in this example, we conservatively set this thickness to 1 μm.

The self-inductance of the ring is approximately given by Eq. (1), even though the cross section of the wire is now a rectangle and not a circle. For radius a in the equation, we take the mean of the wire width and the thickness of the layer. We set the current distribution parameter $b = 2$ [see Eq. (1)] as the current is always close to the surface in such a thin layer. We surmise that taking the rectangular shape into account for the derivation of the self-inductance will have a negligible influence on the calculated values for the aspect ratios of interest in this paper. For the extreme case of a conductor which is flat and wide (compared to the inner radius of the ring), an approximate expression of the self-inductance can be found in Ketchen *et al.* [24].

One of the main reasons behind the recent interest in superconducting levitation is the prospect of reaching significantly higher masses in an essentially dissipation-free environment. Using a ring instead of a sphere makes it possible to adjust the mass, radius, and trap frequency somewhat freely. The mass can be several orders of magnitude larger than typical optically levitated particles, whereas still attaining trap frequencies in the kilohertz range. This large trap frequency is beneficial as it brings the oscillator dynamics far above the regime of seismic noise, thereby enabling feedback cooling in a low-noise environment.

We show the dependence of the trap frequency [Fig. 2(a)] and zero-point motion [Fig. 2(b)], on the inner radius of Nb rings with a thickness of 1 μm for three different constraints. The red lines show the dependence for constant mass for which only the outer radius is adjusted. The green and orange lines display the dependence for a constant wire width of 10 μm and for a constant ratio of width to radius of 1/20, respectively.

As can be seen in Fig. 2(a), the trap frequency for rings is markedly higher than for comparable spheres, even when their masses are equal. The figure also illustrates a key result of this paper, namely, that higher trap frequencies can be achieved simply by increasing the radius of the ring. The reduction in mass, compared to a sphere of the same radius, leads to a zero-point motion amplitude which is approximately one order of magnitude larger than that of a sphere for ring radii greater than 50 μm [Fig. 2(b)]. Additionally, since the ring is only assumed to be 1-μm thick, the zero-point motion is three orders of magnitude greater, relative to the size of the object, compared to a sphere. Finally, the larger trap frequency for rings is indicative of the fact that forced expansion of the wave function by a repulsive potential will also occur on a significantly shorter timescale, reducing the adverse effects of decoherence [15].

For quantum magnetomechanical experiments, we are primarily interested in very small displacements of the levitator. However, the rings' starting position might be at a significant distance from the trap center. It is, therefore, important to make sure that the force needed to overcome gravity and surface adhesion at the starting position can be reached without exceeding the critical current of the ring. It might be necessary to gradually ramp up the magnetic-field gradient after lift-off. Furthermore, the equilibrium position of the trap shifts due to gravity, resulting in an induced current even at zero displacement. This current is, however, negligible compared to I_c in the parameter range considered here, even though we have chosen a superconductor thickness of only 1 μm: As shown in the inset of Fig. 2(a), the critical current is not reached, even for displacements on the order of a whole ring radius for the sizes chosen in these calculations.

IV. DIPOLE AND ZERO-FIELD-COOLED SUPERCONDUCTOR

We now apply the theoretical framework to the case of a ring levitating in the field of a magnetic dipole. We start here with the ZFC case and treat the FC case in the next section.

Consider a point dipole $\mathbf{m} = m\hat{\mathbf{z}}$ located at the origin of coordinates and the superconducting ring axially symmetric with the z axis, its center located at a position $z\hat{\mathbf{z}}$. In the ZFC case, the cooling distance is very long so that the ring becomes superconducting in the absence of any applied field and the flux initially threading the SC ring is zero. This zero-flux value is maintained when the SC is vertically descended towards the point dipole. The current I that flows in the superconducting ring in order to maintain this flux is found from the zero-flux threading condition as in Eq. (4),

$$2\pi R A_\phi(R, z) + LI = 0. \tag{13}$$

Here $A_\phi(R, z)$ is the ϕ component of the magnetic vector potential created by the dipole and evaluated at position $\mathbf{r} = R\,\hat{\boldsymbol{\rho}} + z\,\hat{\mathbf{z}}$.

The vector potential created by the point dipole is

$$\mathbf{A}(R, z) = A_\phi(R, z)\hat{\boldsymbol{\phi}} = \frac{\mu_0}{4\pi}\frac{mR}{(z^2 + R^2)^{3/2}}\hat{\boldsymbol{\phi}}. \tag{14}$$

Thus, the current flowing in the superconducting ring will depend on the distance z from the dipole as

$$I = -\frac{\mu_0 m R^2}{2L(z^2 + R^2)^{3/2}}. \tag{15}$$

The energy of the levitated ring can be evaluated from the Lorentz force acting over the current in the ring due to the field created by the dipole or, alternatively, from the interaction between the field created by the superconducting ring $\mathbf{B}_{SC}$ and the dipole. Including the gravitational potential (set as zero on the $z = 0$ plane), we obtain

$$E = Mgz + \frac{2\pi^2 R^2}{L}(\Delta A_\phi)^2 = Mgz - \frac{1}{2}\mathbf{m}\cdot\mathbf{B}_{SC}, \tag{16}$$

where ΔA_ϕ represents the variation (with respect to the cooling position—infinite in the present case) of the vector potential generated by the dipole and evaluated at the ring position. Note that the term $\frac{2\pi^2 R^2}{L}(\Delta A_\phi)^2$ equals $\frac{1}{2L}(\Delta\Phi)^2$, where $\Delta\Phi$ is the variation of the magnetic flux threading the ring due to the field created by the dipole. In the present case,

$$E = Mgz + \frac{\mu_0^2 m^2 R^4}{8L(z^2 + R^2)^3}. \tag{17}$$

Normalizing the positions to the radius of the ring $\zeta = z/R$ and the energy to $E_0 = \mu_0^2 m^2/8LR^2$, one can describe the above system with a normalized energy $e = E/E_0$ as

$$e = \alpha\zeta + \frac{1}{(1 + \zeta^2)^3}, \tag{18}$$

where the dimensionless constant,

$$\alpha = \frac{8LMgR^3}{\mu_0^2 m^2} \tag{19}$$

is a *single* positive parameter characterizing the system. E_0 is the energy of the superconducting ring when moved from infinity (ZFC) to the $z = 0$ position.

The energy e has a minimum as a function of ζ only if

$$\alpha < \frac{1029\sqrt{7}}{2028} \equiv \alpha_c \simeq 1.329. \tag{20}$$

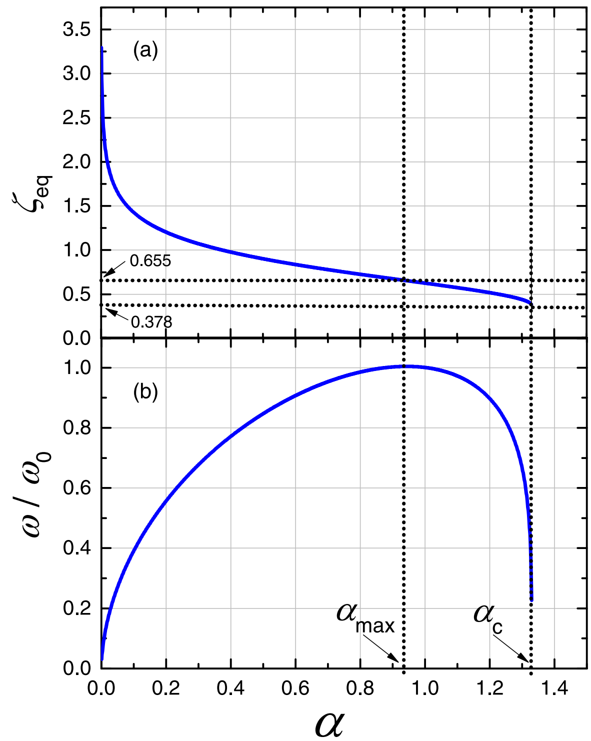

FIG. 3. (a) Equilibrium position and (b) frequency of the potential well for a superconducting ring levitating in the field of a magnetic dipole as a function of the α parameter characterizing the levitation (see the text). Normalization values are defined in the text.

This inequality sets a condition for the stable levitation of ZFC rings with the considered dipole. If the radius of the ring is fixed, the mass of the ring should be less than $\simeq 0.33 m^2\mu_0^2/(gLR^3)$. If the levitating material is fixed (with a given density), then, the above condition sets a maximum radius for the levitated ring.

For a given $\alpha < \alpha_c$, the normalized position of the stable levitation point ζ_{eq} can be found as the larger of the two solutions of the equation,

$$\alpha = \frac{6\zeta_{eq}}{(1 + \zeta_{eq}^2)^4}, \tag{21}$$

which has to be solved numerically; the smaller solution of Eq. (21) results in a maximum in energy [see Eq. (18) and note that α is a positive parameter]. The vertical oscillation spring constant in this potential well can be obtained from the second derivative of the energy, evaluated at the equilibrium position $\kappa_z = (1/2)\frac{\partial^2 E}{\partial z^2}|_{z=R\zeta_{eq}}$. With Eq. (17), the oscillation frequency can be expressed as

$$\omega = \omega_0\sqrt{\frac{3(7\zeta_{eq}^2 - 1)}{\left(1 + \zeta_{eq}^2\right)^5}}, \tag{22}$$

being $\omega_0 = \sqrt{E_0/2R^2M} = \frac{\mu_0 m}{4R^2}\sqrt{\frac{1}{ML}}$ [ζ_{eq} is the largest solution of Eq. (21)].

In Fig. 3 we plot the equilibrium positions ζ_{eq} and the frequency of the trap ω as a function of α. When $\alpha \to 0$, the equilibrium position tends to infinity as $\sim\alpha^{-1/7}$. $\alpha \to 0$ could be interpreted as a massless object or as a very large-radius ring. In addition, the frequency tends to zero as $\sim\alpha^{-4/7}$. When $\alpha \to \alpha_c$ the equilibrium position tends to $\zeta_{eq} \to 1/\sqrt{7} \simeq 0.378$, and the frequency of the trap goes to zero, corresponding to the limit in which the levitation becomes unstable. Interestingly, between these two limits there is a maximum in the stability of the force as a function of the parameter α. This maximum is achieved when the equilibrium position is $z_{eq} = \sqrt{3/7} \simeq 0.655$, corresponding to $\alpha_{\max} \simeq 0.935$ (after Fig. 3). The maximum value for the trapping frequency is $\omega_{\max} = \omega_0(49\sqrt{21})/(50\sqrt{20}) \simeq 1.004\omega_0$.

The fact that there exists an optimum frequency (maximum stability) means that for a given superconducting material (a given density), and for a given width of the superconductor, there is an optimum radius that can be found from the definition of α. Alternatively, if the geometry is also fixed, for a mass less than that needed for achieving the optimum α, it could be interesting in practice to apply a ballast to the ring in order to increase the stability.

V. DIPOLE AND FIELD-COOLED SUPERCONDUCTOR

We consider now that the superconducting ring has been cooled below T_c at a given position $z_{FC}\hat{\mathbf{z}}$. Now there is a flux threading the ring $\Phi_{FC} = 2\pi R A_\phi(z_{FC})$ that will be maintained after subsequent movement of the SC ring when already cooled. The flux conservation equation becomes

$$2\pi R A_\phi(z) + LI = \Phi_{FC}, \tag{23}$$

from which the current circulating in the ring is (defining $\zeta_{FC} = z_{FC}/R$ and $I_0 = \mu_0 m/2LR$)

$$I = I_0\left(\frac{1}{\left(1+\zeta_{FC}^2\right)^{3/2}} - \frac{1}{(1+\zeta^2)^{3/2}}\right). \tag{24}$$

Equation (16) also holds in this case. The energy of the levitated ring can now be written in terms of the α parameter and on the ζ_{FC} value. It is found that

$$e = \alpha\zeta + \frac{1}{(1+\zeta^2)^3} - \frac{2}{(1+\zeta^2)^{3/2}(1+\zeta_{FC}^2)^{3/2}}. \tag{25}$$

Without loss of generality, we consider only positive field-cooling positions ($\zeta_{FC} > 0$). Now, depending on the values of α and ζ_{FC}, the energy [Eq. (25)] can have one minimum at $\zeta < 0$, one minimum at $\zeta > 0$, two minima (one at $\zeta > 0$ and another at $\zeta < 0$), or no minima at all. Actually, for each ζ_{FC}, one can define a critical α for having positive equilibrium positions, α_{c+} and another critical α for achieving equilibrium positions at negative ζ's α_{c-}. These values can be found numerically. The functions $\alpha_{c+}(\zeta_{FC})$ and $\alpha_{c-}(\zeta_{FC})$ are plotted in Fig. 4. We observe that for $\zeta_{FC} \to \infty$ we recover the ZFC case where only equilibrium at positive ζ's is possible when $\alpha < \alpha_{c+}(\infty) = \alpha_c \simeq 1.329$. $\alpha_{c-} \to 0$ in this limit. However, when the cooling position is zero (that is, when the ring is cooled with the dipole in its center), there can be only one equilibrium position at negative ζ's. This is because the current in the ring creates an attractive force to the dipole after any movement. The equilibrium can be achieved only when the movement is performed to negative positions where the attraction to the dipole opposes gravity. Interestingly, for some values of ζ_{FC} and α (shadowed region in Fig. 4) one can have two equilibrium positions, one at positive and the other at negative ζ's. In Fig. 5 we show the numerically evaluated equilibrium positions as a function of the α parameter for different cooling distances ζ_{FC}. The shadowed area in Fig. 5 includes all possible equilibrium positions for all possible α's and ζ_{FC}'s.

To study the stability of such solutions one can follow the same procedure as above. To simplify, we split the treatment in two parts: for the positive and for the negative equilibrium positions. In Fig. 6 we present the results for the numerically evaluated frequency of the equilibrium position. For each ζ_{FC} there is an optimum α such that the equilibrium position is achieved with maximum trapping frequency. We

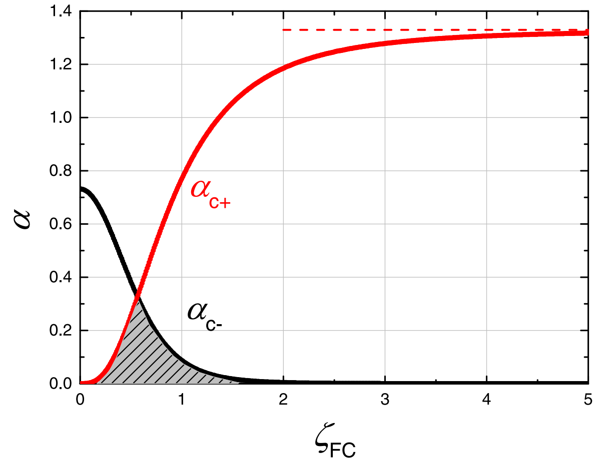

FIG. 4. For a superconducting ring levitating in the field of a magnetic dipole, critical values for the α parameter for achieving positive equilibrium position ($\zeta > 0$, red), and for a negative one ($\zeta < 0$, black). The shadowed area corresponds the (α, ζ_{FC}) values that results in two equilibrium positions.

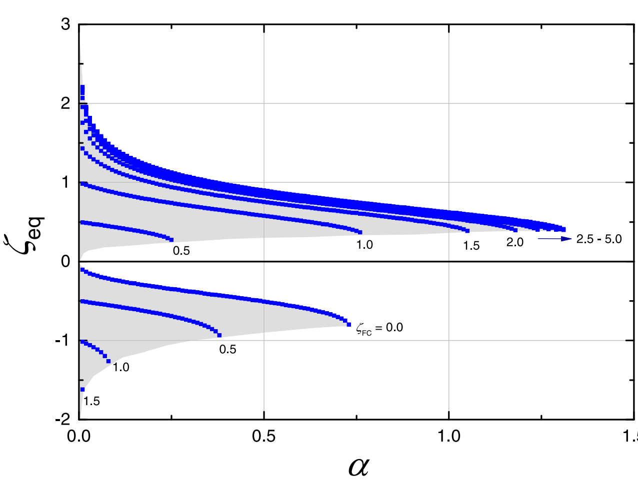

FIG. 5. For a superconducting ring levitating in the field of a magnetic dipole, equilibrium positions as a function of α for different cooling positions (indicated in the plot). The shadowed regions correspond to all the possible equilibrium positions for any ζ_{FC} value. Note that for some values of ζ_{FC} and α corresponding to the shadowed region in Fig. 4 there are two equilibrium positions.

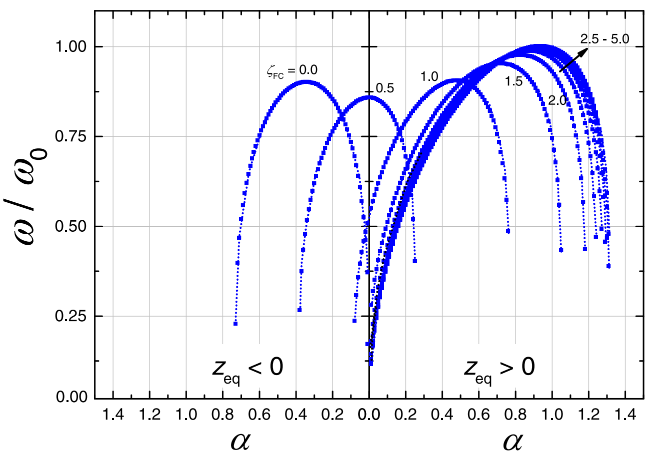

FIG. 6. For a superconducting ring levitating in the field of a magnetic dipole, frequency of the trap around the equilibrium position as a function of the α parameter for different cooling positions (indicated in the plot). For simplicity, we show on the left (right) side the frequencies for the negative (positive) equilibrium positions. Note the decreasing scale in the left part (α is always positive).

also find that the maximum achievable frequency is obtained in the ZFC case [$\omega_{\text{opt}}(\zeta_{FC} = \infty) \simeq 1.004\omega_0$], although for the other cooling positions, the optimum value of the frequency is above $0.85\omega_0$. In other words, the field cooling of the superconductor does not degrade substantially the stability of the equilibrium whereas it adds a new parameter to control the equilibrium position allowing a larger region for stabilizing the levitated ring.

A possible realization of this configuration can be constructed by using a small ring as an effective dipole or a micromagnet, around which a larger ring can be trapped. However, it should be pointed out that this configuration is not stable in the lateral direction, even for finite-sized superconducting rings [25]. Nonetheless, a ring on an additionally clamped structure, such as a cantilever, can be described by the relations given here. The symmetric nature of the dipole presents a particularly interesting feature in that, in general, two equilibrium positions $\pm z_{FC}$ can be found in the FC case, one above and one below the dipole, providing a double well for the ring in the z direction. The smallest separation between the wells is given by the quantized flux. Details of the quantization of the vertical levitation have bee studied by Haley [26].

VI. DISCUSSION

A. Lateral movements and rotational degrees of freedom

One of the main objectives of the present paper is to describe the forces resulting from the flux-conservation property of a levitated superconductor. In this sense, we have focused on the main characteristics and properties that can be derived from this property and have concentrated on the *vertical* movements and *vertical* stability, this being the direction of the rings' axis. Levitation is, however, only truly stable if it is so in all degrees of freedom, aside from rotation around the ring's axis. In a real ring with finite width and thickness, stability against rotation and lateral movements are provided by the current distribution induced on the ring's whole surface. This setting has recently been described and calculated numerically [27]. The effect of a finite size can be understood by returning closer to the picture of a macroscopic superconductor in that any field variation will induce a change in the surface currents of the ring, possibly in addition to a change in the flux threading the ring. In the quadrupole trap, this effect leads to a restoring force, giving positive stiffness in all degrees of freedom, apart from rotation about the ring's principal axis.

Even though the finite thickness of the ring in any actual implementation may lead to stable levitation, it is still worthwhile studying the stability of the idealized case and to explore further stabilizing mechanisms for levitation. This matter is discussed in the two following subsections.

B. Stability in the idealized thin ring case

For the (idealized) example of an infinitely thin ring, energy in a purely magnetic potential is minimized when the circulating current is zero. This is equivalent to preservation of the initial flux through the ring. In the ZFC case, the initial flux (zero) can be obtained at every position in space by tilting the ring until the local field is parallel to the surface. The zero-flux condition can similarly be achieved in regions of the trapping field with an arbitrary field gradient or curvature: Rotation of a ring by 180° about an axis on its plane will lead to a sign change in the flux in the ring's coordinate system. Since flux is a continuous quantity in free space, it follows immediately that an angle exists at which the flux must be zero. For field cooling, the entire volume in which the average field is, at least, $\Phi_{FC}/\pi R^2$ is accessible, again by adjusting the orientation accordingly. The volume where the field is smaller than this value cannot be reached without inducing a current. This means that in either case, an idealized thin ring cannot be trapped at the minimum of, for example, the field provided by anti-Helmholtz coils. By extension, the above arguments are equally valid for an infinitely thin SC forming an arbitrary closed curve, no matter how intricate its shape.

One might conversely consider seeking trapping configurations which rely on a maximum of the flux such that it decreases for any displacement or rotation. It follows from Earnshaw's theorem for the magnetic field that such a situation is impossible for static fields: Since there is no maximum of the magnetic field in free space, there can also be no maximum of the flux in a ring which is disconnected from all field sources.

C. Stabilizing mechanisms

Beyond finite-thickness effects, trapping of the levitator in the other degrees of freedom can be provided by other means, such as gyroscopic effects, time-orbiting potentials, or more complex topologies.

Spin-stabilized magnetic levitation is well known from the physics of spinning devices, such as the Levitron [28]. We expect that similar considerations apply to a rotating ring.

A further path towards stable levitation of thin rings is the use of time-averaged potentials. The combination of three nonorthogonal anti-Helmholtz coil pairs makes it possible to create a time-averaged trapping potential in which the flux through the ring is proportional to the distance from the trap center, regardless of its orientation. This configuration is equivalent to a ring rotating around both its planar axes in a static AHC field. The rich dynamics of such quasistatic potentials presents fertile ground for future work and offer an intriguing alternative outlook towards macroscopic superpositions of these quantum systems [29,30].

In general, the idea of flux conservation makes the levitation of a superconducting system "rigid," in the sense that any variation of the flux in any element would make the superconductor react to counteract this variation. This idea of rigid levitation has been exploited in several works of macroscopic levitation of superconductors where the rigidity comes from the hysteretic current penetration into the superconductor [31–33]. In a ring, it is the flux conservation which yields this rigidity, albeit only along one axis. Although a single ring cannot be trapped by the same mechanisms, the behavior of a macroscopic superconductor can be approached by a rigidly connected arrangement of rings. A planar arrangement of electrically isolated loops will be pinned to its initial vertical and lateral positions, albeit only in the FC case since the zero-current condition can only be fulfilled there for all loops. Lowering the symmetry of the trap potential by applying different gradients in all directions can, furthermore, provide rotational stability.

A minimal assembly of rings which allows stable levitation is found by using two nonparallel rigidly connected rings. In the ZFC case, the flux through both rings can only be minimized when both are parallel to the local magnetic field or both are on a plane in which the field components cancel over their areas. The first condition can be met, for example, by moving the rings along an axis until they reach the zero-current condition. As an example, if the rings are orthogonal to each other, their axes could lie along the lines $|x| = 2z$ in an AHC field. Once again, applying a different field gradient along y could provide rotational stability around the z axis, however, not against combinations of translation and rotation. These would be suppressed by the addition of gravity. The use of multiple rings in a similar arrangement, e.g., placed on the three angled surfaces of a tetrahedron or the four angled surfaces of a pyramid, would be fully stable in an AHC-like field with different gradients along all axes. Complex microscopic three-dimensional geometries made of superconducting materials pose a major challenge for fabrication, although recent work indicates that the creation of such structures is within reach of cutting-edge nanoassembly techniques [34].

Another simple way to provide stability against tilt is by adding a keel-like structure to the ring's supporting structure, which would provide restoring torque with respect to rotations on the x-y plane.

VII. CONCLUSION

We have introduced a theoretical framework for the levitation of superconducting rings over magnetic-field sources, considering the current flowing in the superconductor resulting from the flux-conservation conditions (assumed as a one-dimensional circuit) as the main parameter. These conditions provide a novel set of properties with respect to those arising from the induced currents in a single-connected superconductor.

Our approach has been applied to both a quadrupolar anti-Helmholtz field, a case recently proposed for quantum magnetomechanics experiments, as well as to the case of a superconducting ring levitating in the field of a magnetic dipole. A complete set of analytical formulas are derived for describing these cases. For the force acting on a ring in the z direction, we have found that, depending on the cooling position, a continuous range of equilibrium positions can be achieved. We have also found the conditions that the system should satisfy in order to have this equilibrium; an optimum for having the largest possible vertical stability is obtained. This condition is expressed in terms of a single parameter that combines information from the levitating object (mass, radius, and cross section) as well as from the field source (magnetic moment). We have seen that even in the simple dipole-SC ring system, there is the possibility of having one, two, or zero vertical equilibrium positions. The stability of the ring with respect to translation and rotation has been qualitatively discussed. Several strategies are proposed to achieve full stability. We have argued that although closed curved geometries are not trapped in DC fields, rigidly connected arrangements of several rings can be trapped, but only if they are three dimensional or their symmetry is sufficiently low. In actual implementations, the finite thickness of the superconducting rings will lead to current distributions that result in stable levitation.

In quantum magnetomechanical experiments, the relevant dynamics of the system are those along a single axis. Our model takes the flux conservation through the ring into account and provides an analytical toolkit for the design and optimization of levitating rings. The present paper will guide future experiments of such levitation systems by incorporating the degree of freedom provided by flux conservation conditions as a key element for characterizing and enhancing the magnetic levitation.

Finally, we note that a superconducting ring on a planar surface can be used to levitate additional systems, such as optical mirrors, nonlinear superconducting elements, crystals containing spin centers, or even living organisms [35]. This architecture, therefore, allows to combine the advantages of superconducting magnetic levitation with a wide range of physical mechanisms and will thereby constitute a veritable platform for the exploration of quantum-mechanical effects with massive objects of mesoscopic dimensions.

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