Low-energy effective field theories of fermion liquids and the mixed $U(1) \times \mathbb{R}^d$ anomaly

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In this paper, we study gapless fermionic and bosonic systems in d-dimensional continuum space with $U(1)$ particle-number-conservation and $\mathbb{R}^d$ translation symmetry. We present low-energy effective field theories for several gapless phases with the $U(1) \times \mathbb{R}^d$ symmetry. The $U(1) \times \mathbb{R}^d$ symmetry has a property that a $U(1)$ symmetry twist will induce a nonzero momentum proportional to the $U(1)$ charge density $\bar{\rho}$, which will be referred to as a mixed anomaly. The different effective field theories for different phases of the same system must have the same mixed anomaly. As a result, all the low-energy effective field theories must have fields with nonzero momenta of the order of $\bar{\rho}^{1/d}$. In particular, we present a low-energy effective field theory with infinite number of fields for Fermi liquid. We also present the Fermi-liquid effective field theory in the presence of a real space magnetic field and $\boldsymbol{k}$-space "magnetic" field, as well as in the presence of interaction described by Landau parameters. Our effective field theory correctly captures the mixed anomaly, which constrains the low-energy dynamics, such as determines the volume of the Fermi surface in terms of the mixed anomaly [i.e., in terms of the $U(1)$ charge density]. This is another formulation of the Luttinger-Ward-Oshikawa theorem.

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I. INTRODUCTION

A. Integrated Boltzmann equation as a theory for Fermi liquid and beyond

Fermi liquid is one of the most important states of matter since it describes most metals. Usually, a Fermi-liquid theory is based on the noninteracting fermionic quasiparticles. However, there is also a bosonized version of Fermi-liquid theory [1–4]. For example, a Fermi liquid for spinless fermions in d-dimensional (dD) space is described by the following *integrated Boltzmann equation* (if we ignore the collision term that is irrelevant under the renormalization group scaling) [5]:

$$\dot{u}(\boldsymbol{x}, \boldsymbol{k}_F, t) + \partial_{\boldsymbol{x}} \cdot [\boldsymbol{v}_F(\boldsymbol{k}_F) u(\boldsymbol{x}, \boldsymbol{k}_F, t)] + \partial_{\boldsymbol{k}_F} \cdot [\boldsymbol{f}(\boldsymbol{x}) u(\boldsymbol{x}, \boldsymbol{k}_F, t)] = 0, \tag{1}$$

where $u(\boldsymbol{x}, \boldsymbol{k}_F, t)$ is the dynamic field describing the Fermi surface fluctuations. Here, $\boldsymbol{k}_F$ parametrizes the Fermi surface, $\boldsymbol{v}_F$ is the Fermi velocity, $u(\boldsymbol{x}, \boldsymbol{k}_F, t)$ describes the Fermi surface displacement at Fermi momentum $\boldsymbol{k}_F$ and spatial location $\boldsymbol{x}$, and $\boldsymbol{f}$ is the force acting on a fermion. The Fermi surface displacement $u(\boldsymbol{x}, \boldsymbol{k}_F)$ describes the fluctuations of the integrated fermion occupation $g(\boldsymbol{k})$ along a line normal to the Fermi surface. This is why we call the above equation the integrated Boltzmann equation.

We can view the integrated Boltzmann equation as the equation of motion for the bosonized Fermi liquid [1–5]. Together with the total energy (assuming $\boldsymbol{f} = 0$)

$$E = \int d^d\boldsymbol{x} \frac{d^{d-1}\boldsymbol{k}_F}{(2\pi)^d} \frac{|\boldsymbol{v}_F(\boldsymbol{k}_F)|}{2} u^2(\boldsymbol{x}, \boldsymbol{k}_F), \tag{2}$$

we obtain a phase-space low-energy effective Lagrangian for the Fermi liquid [see Eq. (39)]. Such a low-energy effective field theory contains an infinite number of fields labeled by $\boldsymbol{k}_F$. Or, alternatively, the low-energy effective field theory can be viewed as having a single scaler field in $(d + d - 1)$D space, but the interaction in the $d - 1$ dimensions (parametrized by $\boldsymbol{k}_F$) is allowed to be nonlocal.

The bosonized description of the Fermi liquid is more general than Fermi-liquid theory. It may be used to describe a gapless state without a well defined quasiparticle (but with a well defined Fermi surface), as demonstrated in (1+1)D and in Refs. [5,6] for higher dimensions.

However, such a bosonized low-energy effective theory fails to capture one of the most important properties of a Fermi liquid: the volume enclosed by the Fermi surface is $(2\pi)^d \bar{\rho}$, where $\bar{\rho}$ is the density of the fermion in the ground state [7,8]. In fact, the fermion density $\bar{\rho}$ does not even appear in the above bosonized formulation.

In recent years, it was realized that the Lieb-Schultz-Mattis (LSM) theorem [9] and its higher-dimensional generalizations by Oshikawa [8] and Hastings [10] can be understood in term of a mixed anomaly between translation symmetry and an internal symmetry [11–17]. For a 1D system with $U(1)$ symmetry and translation symmetry, there is a similar theorem when the $U(1)$ charge per site is not an integer [18]. This suggests that such a 1D system also has a mixed anomaly when the $U(1)$ charge per site is not an integer. Similarly, for continuum systems with $U(1)$ and translation $\mathbb{R}^d$ symmetries, there should also be a mixed anomaly, whenever the $U(1)$ charge density is nonzero. The low-energy effective field theory for systems with $U(1) \times \mathbb{R}^d$ symmetry should capture this mixed $U(1) \times \mathbb{R}^d$ anomaly.

B. Emergent symmetry and mixed anomaly

To explain what a mixed anomaly is, we need to first explain emergent symmetry. Let us consider a 1D noninteracting spinless fermion $c(x)$ in continuum space with two Fermi points, one at $k = k_F^L$ and the other at $k = k_F^R$. The 1D system has a $U(1)$ fermion-number-conservation symmetry and $\mathbb{R}$ translation symmetry.

The low-energy effective field theory of the above model is described by the Lagrangian

$$\mathcal{L} = \psi_L^\dagger i(\partial_t - \partial_x)\psi_L + \psi_R^\dagger i(\partial_t + \partial_x)\psi_R, \tag{3}$$

for right-moving and left-moving fermions near the two Fermi points. The original exact symmetry of the model, $U(1)\times\mathbb{R}$, is enlarged to a bigger symmetry $U(1)\times\mathbb{R}\times\mathbb{R}_{\text{trn}}$ of the effective field theory:

$$\begin{aligned} U(1):&\ \ \psi_L \to e^{i\theta}\psi_L,\ \ \psi_R \to e^{i\theta}\psi_R;\\ \mathbb{R}:&\ \ \psi_L \to e^{i\delta x k_F^L}\psi_L,\ \ \psi_R \to e^{i\delta x k_F^R}\psi_R;\\ \mathbb{R}_{\text{trn}}:&\ \ \psi_L(x) \to \psi_L(x+\Delta x),\ \ \psi_R(x) \to \psi_R(x+\Delta x). \end{aligned} \tag{4}$$

$\mathbb{R}_{\text{trn}}$ is the translation symmetry in the effective field theory. $U(1)\times\mathbb{R}$ is regarded as the internal symmetry of the effective field theory, despite the fact that $\mathbb{R}$ comes from the translation symmetry of the original model.

We note that the "charges" of both $\mathbb{R}$ and $\mathbb{R}_{\text{trn}}$ symmetry are momenta. The charge of the $\mathbb{R}_{\text{trn}}$ symmetry is momenta of the order of $1/L$, while the charge of the $\mathbb{R}$ symmetry is momenta of the order of $\bar{\rho}$. In the limit where $L \to \infty$ and $\bar{\rho} \to$ constant $\neq 0$, the range of momenta is well separated, and they are conserved separately. This is why we have a larger emergent symmetry.

The emergent internal symmetry $U(1)\times\mathbb{R}$ has a property that the $U(1)$ symmetry twist around the 1D ring of the space induces a nonzero $\mathbb{R}$ charge (i.e., a nonzero momentum). Similarly, the $\mathbb{R}$ symmetry twist around the 1D ring induces a nonzero $U(1)$ charge (i.e., a nonzero fermion number).

This is similar to the 't Hooft anomaly [19]. Usually, the presence of the 't Hooft anomaly means that if we gauge the symmetry $U(1)\times\mathbb{R}$, the resulting gauge theory is not gauge invariant. Such a gauge noninvariance (i.e., mixed anomaly) for the $U(1)\times\mathbb{R}$ symmetry is related to the above phenomenon that a $U(1)$ symmetry twist can induce the $\mathbb{R}$ charge (i.e., the momentum).

Indeed, the $U(1)$ symmetry twist around the 1D ring of size L is given by a change of boundary condition of the fermion operator,

$$c(x+L) = e^{i\theta}c(x). \tag{5}$$

Such a symmetry twist shifts the momentum of each fermion by $\delta k = \theta/L$. Thus the total momentum (i.e., the $\mathbb{R}$ charge) is changed by $\Delta k = L\bar{\rho}\delta k = \theta\bar{\rho}$, where $\bar{\rho}$ is the fermion density.

The $\mathbb{R}$ symmetry twist around the 1D ring of size L is given by changing the ring size by δx. Such a symmetry twist shifts the total fermion number by $\delta x\bar{\rho}$.

We see that the so-called mixed anomaly of $U(1)\times\mathbb{R}$ is a just a fancy way to say that our fermion system has a nonzero density $\bar{\rho}$. However, here we stress two aspects that are hidden or unclear when we just say the system has a nonzero density: (1) the density is associated to a conserved quantity [i.e., a $U(1)$ symmetry] and (2) the uniform density implies a translation symmetry.

After realizing the presence of the mixed anomaly, we conclude that all the different low-energy effective field theories for various phases of the system must have the same mixed anomaly. This constrains the possible low-energy dynamics of the system, which will be discussed in more detail in the paper. In particular, a low-energy effective theory must have fields with nonzero $\mathbb{R}$ charges, i.e., with momenta of the order of $\bar{\rho}$.

In higher dimensions, the exact $U(1)\times\mathbb{R}^d$ symmetry of the original model may give rise to emergent $U(1)\times\mathbb{R}^d\times\mathbb{R}^d_{\text{trn}}$ symmetry in low-energy effective field theory. The emergent internal symmetry $U(1)\times\mathbb{R}^d$ has a mixed anomaly when the $U(1)$ charge density is nonzero. This is the so-called mixed anomaly discussed in this paper and in Refs. [11–17].

C. A summary of results

In this paper, we will carefully present the low-energy effective field theories for some gapless phases of bosons and fermions. The low-energy effective field theories contain a proper topological term that captures the mixed $U(1)\times\mathbb{R}^d$ anomaly. Such a mixed $U(1)\times\mathbb{R}^d$ anomaly ensures that the system must be gapless.

In particular, the low-energy effective field theory (44) for a Fermi liquid is obtained that contains the proper mixed $U(1)\times\mathbb{R}^d$ anomaly. Such a mixed $U(1)\times\mathbb{R}^d$ anomaly determines the volume enclosed by the Fermi surface in terms of $U(1)$ charge density, within the low-energy effective field theory. We also present the low-energy effective field theory (89) for a Fermi liquid with a real space magnetic field and $\boldsymbol{k}$-space "magnetic" field [20–22], as well as with interaction described by Fermi-liquid parameters. Those are the main results of this paper.

The equation of motion of our low-energy effective theory for a Fermi liquid is just the (integrated) quantum Boltzmann equation for transport after including the collision term. We present such quantum Boltzmann equation in the presence of both a real space magnetic field and a $\boldsymbol{k}$-space magnetic field, as well as in the presence of interaction described by Fermi-liquid parameters [see Eq. (97)]. Our quantum Boltzmann equation can be used to study the transport properties in such general situations.

We will also discuss the universal low-energy properties of the gapless phases for systems with $U(1)\times\mathbb{R}^d$ symmetry. Some of the features in the universal low-energy properties are determined by the mixed $U(1)\times\mathbb{R}^d$ anomaly, and we identify those features.

The results in this paper also apply to some lattice systems with $U(1)\times\mathbb{Z}^d_{\text{trn}}$ symmetry. If the $U(1)$ charge per unit cell is not a rational number, the low-energy emergent symmetry will be $U(1)\times\mathbb{R}^d\times\mathbb{Z}^d_{\text{trn}}$. The results in this paper are still valid, if we replace the $U(1)$ density $\bar{\rho}$ by $\bar{\rho} + l\rho_{\text{cell}}$, where ρ_{cell} is the density of the unit cells of the lattice and l is an integer. If the $U(1)$ charge per unit cell is a rational number, then the low-energy emergent symmetry may not be $U(1)\times\mathbb{R}^d\times\mathbb{Z}^d_{\text{trn}}$. The results in this paper may not apply.

In Sec. II, we will first discuss the low-energy effective field theory of 1D weakly interacting bosons. Then, in Sec. III, we will consider 1D weakly interacting fermions. In Sec. IV, we will obtain a low-energy effective field theory for a Fermi liquid in a general dimension. Section V discusses another gapless phase of fermions—a fermion-pair liquid—and its low-energy effective field theory. All those effective field theories capture the mixed $U(1) \times \mathbb{R}^d$ anomaly.

In this paper, we will use the natural unit where $\hbar = e = c = 1$.

D. Some remarks

We like to remark that a Fermi liquid at low energies also has many emergent symmetries. In particular, the $U(1)$ fermion-number-conservation symmetry is enlarged to $U^\infty(1)$ emergent symmetry [1–4]. Recently, it was pointed out that such an emergent $U^\infty(1)$ symmetry also has an anomaly in the presence of $U(1)$ flux [23]. Using such an $U^\infty(1)$ anomaly, one can also derive the relation between the volume enclosed by the Fermi surface and the density of the fermion.

We also like to remark that the mixed $U(1) \times \mathbb{R}^d$ anomaly discussed above is not the usual 't Hooft anomaly [19]. Usually, an anomaly corresponds to a symmetry protected trivial (SPT) order or a topological order in one higher dimension [24,25]. This kind of anomaly is labeled by a discrete index. However, for continuous symmetries, Ref. [24] pointed out that there can be a special kind of anomaly labeled by a continuous index, which will be referred to as *continuous anomaly*. Reference [24] gave an example of a continuous anomaly: a (2+1)D system with a $U(1)$ symmetry and an unquantized Hall conductance. (This example may be closely related to a recent work on the anomaly of emergent loop $U(1)$ symmetry [23]). The mixed $U(1) \times \mathbb{R}^d$ anomaly discussed in this paper is another example of a continuous anomaly that is labeled by the continuous particle density $\bar{\rho}$.

Continuous anomalies also correspond to gapped states in one higher dimension with topological terms. But now the topological terms have continuous coefficients [24]. For example, the (2+1)D continuous $U(1)$ anomaly of unquantized Hall conductance is characterized by a (3+1)D gapped state with a topological term $\theta \int \frac{F\wedge F}{8\pi^2}$. Had the coefficient been discrete, the bulk gapped state with the topological term would correspond to a SPT order or a topological order, and the anomaly would correspond to a SPT order or a topological order in one higher dimension. However, for continuous anomalies [such as the (2+1)D $U(1)$ continuous anomaly], the coefficient of the topological term is continuous and can be smoothly tuned to zero. So the bulk gapped state with this kind of topological term does not correspond to a nontrivial phase, and continuous anomalies do not correspond to a SPT phase in one higher dimension, but rather correspond to a bulk gapped state with an unquantized topological term, which can be viewed as a pseudo SPT state. We stress that a continuous anomaly for a symmetry is not robust against all symmetry preserving deformations of the Hamiltonian. For example, the mixed $U(1) \times \mathbb{R}^d$ anomaly is robust only against all symmetry preserving deformations that keep the particle density of the ground state unchanged [i.e., keep the $U(1)$ representation of the ground state unchanged]. This may be a general feature of continuous anomaly.

II. 1D BOSON LIQUID WITH $U(1) \times \mathbb{R}$ SYMMETRY

A. Gapless phase of weakly interacting bosons

In this section, we are going to consider 1D gapless systems in continuum space with $U(1)$ particle-number-conservation symmetry and $\mathbb{R}$ translation symmetry. The systems are formed by bosons with weak interaction, which gives rise to a gapless state: a "superfluid" state for bosons. We assume the system to have a size L with a periodic boundary condition. We will compute the distribution of the total momentum for many-body low-energy excitations, and how such a distribution depends on the $U(1)$ symmetry twist described by a constant $U(1)$ background vector potential a. We will see that such a dependence directly measures a mixed anomaly in $U(1) \times \mathbb{R}$ symmetry if we view $U(1) \times \mathbb{R}$ as an internal symmetry in the effective field theory [or, more precisely, if we view $U(1) \times \mathbb{R}$ as the internal symmetry in the emergent symmetry $U(1) \times \mathbb{R} \times \mathbb{R}_{\text{trn}}$ of the low-energy effective field theory].

Using the results from a careful calculation in the Appendix, we find the following low-energy effective field theory for the gapless phase of bosons:

$$\begin{aligned} L_{\text{ph}} &= \int dx \Big[\bar{\rho}\dot{\phi}(x,t) + \delta\rho(x,t)\dot{\phi}(x,t) \\ &\quad - \frac{\bar{\rho}}{2M_b}|\partial\phi|^2 - \frac{g}{2}\delta\rho^2 + \cdots\Big], \\ L_{\text{co}} &= \int dx \Big[\bar{\rho}\dot{\phi}(x,t) - \frac{\bar{\rho}}{2M_b}|\partial\phi|^2 + \frac{1}{2g}(\dot{\phi})^2 + \cdots\Big], \end{aligned} \tag{6}$$

where $\delta\rho$ is the boson density fluctuation, $\bar{\rho} = \bar{N}/L$ the boson density in the ground state, and ϕ an angular field $\phi(x,t) \sim \phi(x,t) + 2\pi$. L_{ph} is the phase-space Lagrangian and L_{co} is the coordinate-space Lagrangian. They both describe the system at low energies. People usually drop the total derivative term (also called topological term) $\bar{\rho}\dot{\phi}(x,t)$, since it does not affect the classical equation of motion of the fields. We will see that the topological term affects the dynamics in quantized theory and should not be dropped. In fact, the topological term captures the mixed anomaly in $U(1) \times \mathbb{R}$.

From the low-energy effective field theory, we find that the low-energy excitations are labeled by ($N \in \mathbb{N}, m \in \mathbb{Z}, n_k \in \mathbb{N}$). The total energy and total momentum of those excitations are given by [in the presence of a constant $U(1)$ connection a describing the $U(1)$ symmetry twist]

$$\begin{aligned} E &= \frac{N}{2M_b}\left(\frac{2\pi m}{L} + a\right)^2 + \frac{g}{2}(N - \bar{N})^2 + \sum_{k\neq 0}\left(n_k + \frac{1}{2}\right)v|k|, \\ k_{\text{tot}} &= \int dx\, \rho(\partial_x + a)\phi = N\left(\frac{2\pi m}{L} + a\right) + \sum_{k\neq 0} n_k k, \end{aligned} \tag{7}$$

where $N = \bar{N} + \delta N$ is the total number of bosons in the excited state. For $N = \bar{N}$ and $a = 0$, the possible values of (E, k_{tot}) are plotted in Fig. 1(a).

After quantization, we have the following operator algebra:

$$[\phi(x), \delta\rho(y)] = i\delta(x - y), \tag{8}$$

where $\delta\rho(x)$ is the boson density operator. Since

$$[\delta\rho(y), e^{i\phi(x)}] = \delta(x - y)e^{i\phi(x)}, \tag{9}$$

$e^{i\phi(x)}$ is the boson creation operator.

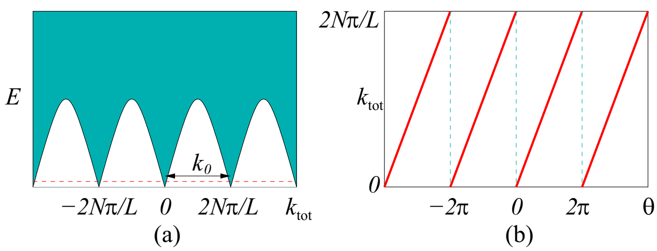

FIG. 1. (a) The distribution of the total energies and total momenta for low-energy states of 1D weakly interacting boson liquid. (b) The distribution of the total momenta $k_{\rm tot}$ for low-energy many-body states, and its dependence on the $U(1)$ symmetry twist θ. The red lines mark the total momenta $k_{\rm tot}$'s for many-body states near the ground state energy.

Let $\varphi(x) = 2\pi \int^x dx'\delta\rho(x')$. We find that

$$[\varphi(x), \phi(y)] = -2\pi\, i\,\Theta(x-y),$$
$$\Theta(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x < 0, \end{cases} \tag{10}$$

or

$$e^{i\alpha\varphi(x)} e^{i\beta\phi(y)} = e^{i\beta\phi(y)} e^{i\alpha\varphi(x)} e^{\alpha\beta[\phi(y),\varphi(x)]}$$
$$= e^{i\beta\phi(y)} e^{i\alpha\varphi(x)} e^{2\pi\, i\alpha\beta\Theta(x-y)}, \tag{11}$$

which can also be rewritten as

$$e^{-i\alpha\varphi(x)} e^{-i\beta\phi(y)} e^{i\alpha\varphi(x)} e^{i\beta\phi(y)} = e^{2\pi\, i\alpha\beta\Theta(x-y)},$$
$$e^{-i\beta\phi(y)} e^{-i\alpha\varphi(x)} e^{i\beta\phi(y)} e^{i\alpha\varphi(x)} = e^{-2\pi\, i\alpha\beta\Theta(x-y)}. \tag{12}$$

The above expression tells us that that the operator $e^{i\alpha\varphi(x)}$ causes an $e^{-2\pi\, i\alpha}$ phase shift for the operator $e^{i\phi(y)}$ for $y < x$, and keeps $e^{i\phi(y)}$ unchanged for $y > x$. So the operator $e^{i\varphi(x)}$ increases m by 1, i.e., increases the total momentum by $2\pi\bar\rho$. Similarly, the operator $e^{i\beta\phi(x)}$ causes a $e^{2\pi\, i\beta}$ phase shift for the operator $e^{i\varphi(y)}$ for $y < x$, and keeps $e^{i\varphi(y)}$ unchanged for $y > x$. So the operator $e^{i\phi(x)}$ increases $N = \frac{1}{2\pi}\int dx\,\partial_x\varphi = \int dx\,\delta\rho$ by 1.

From the above results, we see that under the $U(1)$ transformation θ,

$$\phi(x) \to \phi(x) + \theta, \quad \varphi(x) \to \varphi(x). \tag{13}$$

Under the $\mathbb{R}$ translation transformation, δx

$$\phi(x) \to \phi(x+\delta x), \quad \varphi(x) \to \varphi(x+\delta x) + 2\pi\bar\rho\delta x. \tag{14}$$

At low energies, we can rewrite the above translation transformation as two transformations: $\mathbb{R}_{\rm trn}$ translation transformation

$$\phi(x) \to \phi(x+\delta x), \quad \varphi(x) \to \varphi(x+\delta x), \tag{15}$$

and $\mathbb{R}$ transformation

$$\phi(x) \to \phi(x), \quad \varphi(x) \to \varphi(x) + 2\pi\bar\rho\delta x. \tag{16}$$

We see that the above $\mathbb{R}$ symmetry is an emergent internal symmetry in the low-energy effective field theory (6). We will see that $U(1)\times\mathbb{R}$, as an emergent internal symmetry, has a mixed anomaly.

We note that $e^{i\varphi(x)}$ is a local operator and $\int dx\,\rho = \frac{2\pi}{\int} dx\,\partial_x\varphi$ is an integer. Both imply that φ is also an angular variable, $\varphi(x,t) \sim \varphi(x,t) + 2\pi$. Using the two angular fields $\phi_1 := \phi$ and $\phi_2 := \varphi$, the low-energy effective theory can be written as

$$L_{\rm ph} = \int dx \left(\bar\rho\partial_t\phi_1 + \frac{K_{IJ}}{4\pi}\partial_x\phi_I\partial_t\phi_J - \frac{V_{IJ}}{2}\partial_x\phi_I\partial_x\phi_J\right), \tag{17}$$

where $I, J = 1, 2$ and

$$K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{18}$$

V is a positive definite symmetric matrix [from the $\frac{\bar\rho}{2M_b}|\partial\phi|^2 + \frac{g}{2}\delta\rho^2$ terms in Eq. (6)].

B. The universal properties of the gapless phase

From Eq. (7), we see that the total momentum of the ground state, $k_{\rm tot}$, depends on the $U(1)$ symmetry twist,

$$k_{\rm tot} = \rho\theta \quad \text{where } \theta = \int dx\, a = aL, \quad \rho = \frac{N}{L}, \tag{19}$$

where we have set $n_k = m = 0$ in Eq. (7). In other words, if we can follow a particular low-energy state as we change the $U(1)$ symmetry twist, we will see a change of the total momentum of the state. Such a relation between the $U(1)$ symmetry twist and total momentum, $k_{\rm tot} = \rho\theta$, is a universal property.

However, since the state is gapless, it is hard to following a particular low-energy state. So to make our statement meaningful, we consider the distribution of $k_{\rm tot}$'s. Such a distribution is plotted in Fig. 1(b). We consider how the distribution depends on the $U(1)$ symmetry twist θ.

From the distribution pattern in Fig. 1, we see two universal properties: the period in the distribution and the $\theta = aL$ dependence of the distribution,

$$k_0 = 2\pi\bar\rho, \quad \frac{dk_{\rm tot}}{d\theta} = \bar\rho, \tag{20}$$

which do not depend on the small changes in the interactions and the dispersion of the bosons, unless those changes cause a phase transition. Thus we say they are universal properties that characterize the gapless phase. The two universal properties are closely related, $(2\pi)^{-1}k_0 = \frac{dk_{\rm tot}}{d\theta} = \bar\rho$. We call $\bar\rho$ an index for the gapless phase. Physically, $\bar\rho$ is simply the density of the $U(1)$ charges in the ground state.

Let us give an argument why $\frac{dk_{\rm tot}}{d\theta}$ is universal. Let us assume the $U(1)$ symmetry twist is described by a boundary condition on a single-particle wave function at x_0: $\psi(x_0 + 0^+) = e^{i\theta}\psi(x_0 - 0^+)$. A usual translation $x \to x + \Delta x$ will shift the symmetry twist from x_0 to $x_0 + \Delta x$. So the symmetry twist breaks the translation symmetry. But we can redefine the translation operator to be the usual translation plus a $U(1)$ transformation $\psi(x) \to e^{i\theta}\psi(x)$ for $x \in [x_0, x_0 + \Delta x]$. The new translation operator generates the translation symmetry in the presence of the $U(1)$ symmetry twist. Due to the $U(1)$ transformation $\psi(x) \to e^{i\theta}\psi(x)$ for $x \in [x_0, x_0 + \Delta x]$, the eigenvalue of the new translation operator has a θ dependence given by $(e^{i\theta})^{\bar\rho\Delta x}$, where $\bar\rho\Delta x$ is the total $U(1)$ charges in the interval $[x_0, x_0 + \Delta x]$. In other words, the total momentum has a θ dependence given by $\theta\bar\rho$. This is the reason why $\frac{dk_{\rm tot}}{d\theta} = \bar\rho$. Since $\theta = 0$ and $\theta = 2\pi$ are equivalent, $\frac{dk_{\rm tot}}{d\theta} = \bar\rho$ implies the periodicity in Fig. 1, with the period $k_0 = 2\pi\bar\rho$.

The above discussion does not depend on interactions and boson dispersion. So the results (20) are universal properties.

C. Mixed anomaly for $U(1) \times \mathbb{R}$ symmetry

From the above argument, we also see that the shift of the low-energy momentum distribution by the $U(1)$ symmetry twist, $\frac{dk_{\text{tot}}}{d\theta} = \bar{\rho}$, is an invariant not only against small perturbations that preserve the $U(1) \times \mathbb{R}$ symmetry, but also against large symmetry preserving perturbations that can drive through a phase transition. The invariant for large perturbations is actually an anomaly [19]. This is because, under a different point of view [24,25], an anomaly corresponds to a symmetry protected trivial (SPT) order or a topological order in one higher dimension [24,25]. An anomalous theory can be viewed as a boundary theory of the corresponding SPT or topological order in one higher dimension. Any large perturbations and phase transitions on the boundary cannot change the SPT or topological order in one higher dimension, and thus cannot change the anomaly.

In our case, we can view $\frac{dk_{\text{tot}}}{d\theta} = \bar{\rho}$ as an anomaly in the low-energy effective field theories (6) and (17). We see that the topological term $\bar{\rho}(x,t)\dot{\phi}(x,t)$ in the low-energy effective field theory determines the anomaly. In fact, such an anomaly is a mixed anomaly between $U(1)$ symmetry and the translation $\mathbb{R}$ symmetry, which describe how a $U(1)$ symmetry twist can change the total momentum (i.e., $\frac{dk_{\text{tot}}}{d\theta} \neq 0$).

The presence of the anomaly implies that the ground state of the system must be either gapless or have a nontrivial topological order. Since there is no nontrivial topological order in 1D, the ground state must be gapless. In other words, *the field theories in Eq. (6) and Eq. (17) with $\bar{\rho} \neq 0$ must be gapless regardless of the interaction term described by $\cdots$, as long as the $U(1) \times \mathbb{R}$ symmetry is preserved. On the other hand, when $\bar{\rho} = 0$, the field theories in Eq. (6) and Eq. (17) allow a gapped phase with $U(1) \times \mathbb{R}$ symmetry.*

The mixed anomaly between the $U(1)$ symmetry and the $\mathbb{R}$ symmetry can also be detected via the patch symmetry transformations studied in Ref. [26]. The $U(1)$ patch symmetry transformations are given by

$$W_{U(1)}(x,y) = e^{i2\pi\alpha\int_x^y dx\rho} = e^{i\alpha[\varphi(y)-\varphi(x)]}e^{i2\pi\alpha\bar{\rho}(y-x)}, \quad (21)$$

which perform the $U(1)$ transformation, $\phi \to \phi + \alpha$, on the segment $[x,y]$. The $\mathbb{R}$ patch symmetry transformations are given by

$$W_{\mathbb{R}}(x,y) = e^{i\bar{\rho}\Delta x[\phi(y)-\phi(x)]}, \quad (22)$$

which perform the $\mathbb{R}$ transformation (the translation Δx) on the segment $[x,y]$. In the low-energy limit, $k \to 0$. So for a finite Δx, the translation is trivial for the phonon modes. The translation Δx has nontrivial actions only on the sector labeled by different N's and m's. For a translation Δx that acts on a segment $[x,y]$, its effect is to transfer $U(1)$ charge $\bar{\rho}\Delta x$ from x to y. This is why the $\mathbb{R}$ patch symmetry transformations are given by Eq. (22).

In the low-energy effective theories (6) and (17), the $U(1)$ transformation is given by $\phi \to \phi + \theta$. The term $\bar{\rho}\dot{\phi}$ implies $\bar{\rho}$ is the background $U(1)$ charge density. Therefore, the patch translation transformation has a form given by Eq. (22).

Assume $x_2 > x_1$. We have shown that $W_{U(1)}(x_1,x_2)$ shifts $e^{i\beta\phi(y)}$ by a phase $e^{-i2\pi\alpha\beta}$, if $x_1 < y < x_2$. Therefore,

$$\begin{aligned} &W_{U(1)}(x_1,x_2)W_{\mathbb{R}}(y_1,y_2) \\ &\quad = W_{\mathbb{R}}(y_1,y_2)W_{U(1)}(x_1,x_2)e^{i2\pi\alpha\bar{\rho}\Delta x}, \end{aligned} \quad (23)$$

for $x_1 < y_1 < x_2 < y_2$. The extra phase factor $e^{i2\pi\alpha\bar{\rho}\Delta x}$ indicates the appearance of the mixed $U(1) \times \mathbb{R}$ anomaly. Using the terminology of Ref. [26], we say that the $U(1)$ symmetry and the $\mathbb{R}$ symmetry have a "mutual statistics" between them, as a consequence of the mixed anomaly. So, according to Ref. [26], the $U(1)$ symmetry and the $\mathbb{R}$ are not independent, and we may denote the combined symmetry as $U(1) \vee \mathbb{R}$ to stress the mixed anomaly.

We like to remark that the ϕ and φ fields are canonical conjugate to each other. We see that the symmetries that shift ϕ and φ have a mixed anomaly, as captured by the nontrivial commutation relation between the patch operators for the symmetry transformations.

III. 1D FERMION LIQUID WITH $U(1) \times \mathbb{R}$ SYMMETRY

Weakly interacting 1D gapless fermionic systems

The 1D gapless fermionic systems with $U(1) \times \mathbb{R}$ symmetry and weak repulsive interaction are also in a gapless phase—a Tomonaga-Luttinger liquid for fermions. The low-energy effective theory also has a form

$$\begin{aligned} L_{\text{ph}} = \int dx\, \Big[&\bar{\rho}\dot{\phi}(x,t) + \delta\rho(x,t)\dot{\phi}(x,t) \\ &- \frac{\bar{\rho}}{2M_f}|\partial\phi|^2 + \frac{1}{2g}(\dot{\phi})^2 + \cdots\Big]. \end{aligned} \quad (24)$$

Considering noninteracting fermions in a system with a periodic boundary condition on a ring of size L, we find that the low-energy excitations are also labeled by $(N \in \mathbb{N}, m \in \mathbb{Z}, n_k \in \mathbb{N})$. However, the total energies and total momenta of those excitations are given by, for $N =$ odd,

$$\begin{aligned} E &= \frac{N}{2M_f}\left(2\pi\frac{m}{L} + a\right)^2 + \frac{g}{2}\delta N^2 + \sum_{k\neq 0}\left(n_k + \frac{1}{2}\right)v|k|, \\ k_{\text{tot}} &= N\left(2\pi\frac{m}{L} + a\right) + \sum_{k\neq 0} n_k k, \end{aligned} \quad (25)$$

and for $N =$ even,

$$\begin{aligned} E &= \frac{N}{2M_f}\left(2\pi\frac{m+\frac{1}{2}}{L} + a\right)^2 + \frac{g}{2}\delta N^2 + \sum_{k\neq 0}\left(n_k + \frac{1}{2}\right)v|k|, \\ k_{\text{tot}} &= N\left(2\pi\frac{m+\frac{1}{2}}{L} + a\right) + \sum_{k\neq 0} n_k k, \end{aligned} \quad (26)$$

where $N(2\pi\frac{m}{L} + a)$ or $N(2\pi\frac{m+\frac{1}{2}}{L} + a)$ are the momentum of the center of mass.

Again, we consider low-energy states $|\Psi_N\rangle$ of N fermions. Let k_{tot} be the total momenta of $|\Psi_N\rangle$. Since there are many different low-energy states $|\Psi_N\rangle$'s, we have a distribution of k_{tot}'s. Such a distribution is plotted in Fig. 2(a) for the $N =$ odd case, and in Fig. 2(b) for the $N =$ even case. We see that the

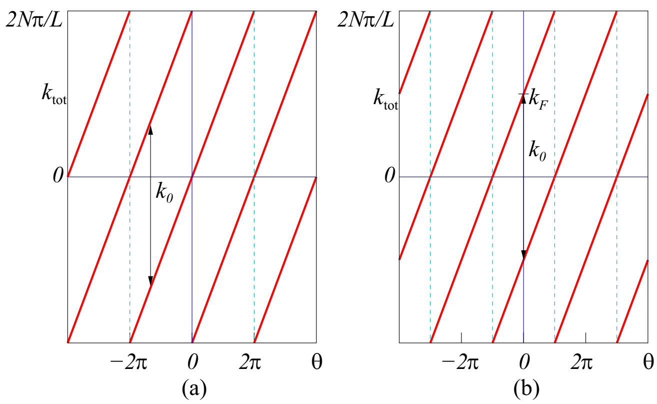

FIG. 2. The distribution of the total momenta $\Delta k_{\rm tot}$ and its dependence on the $U(1)$ symmetry twist, for a weakly interacting 1D fermion liquid.

$N =$ odd case and $N =$ even case have different distributions for $k_{\rm tot}$'s.

The shift of the distribution of $k_{\rm tot}$ by the $U(1)$ symmetry twist (see Fig. 2) can be interpreted as the $U(1)$ symmetry twist producing or pumping momentum. This directly measures the mixed $U(1)\times\mathbb{R}$ anomaly.

We also see that a local operator that creates a fermion (i.e., change N by 1) must also change m by $\frac{1}{2}$. Those operators have a form $e^{i(\phi\pm\frac{1}{2}\varphi)}$, where $\partial_x\varphi = 2\pi\rho$. Since the allowed operators are generated by $e^{i(\phi\pm\frac{1}{2}\varphi)}$, the fields

$$\phi_1 = \phi + \tfrac{1}{2}\varphi, \quad \phi_2 = \phi - \tfrac{1}{2}\varphi \tag{27}$$

are angular fields: $\phi_i \sim \phi_i + 2\pi$. Using

$$\phi = \tfrac{1}{2}(\phi_1+\phi_2), \quad \varphi = \phi_1 - \phi_2, \tag{28}$$

we find the low-energy effective theory to be

$$\begin{aligned} L &= \int dx \bigg[\frac{\bar\rho}{2}\partial_t(\phi_1+\phi_2) \\ &\quad + \frac{1}{4\pi}\partial_x(\phi_1-\phi_2)\partial_t(\phi_1+\phi_2) - \frac{V_{IJ}}{2}\partial_x\phi_I\partial_x\phi_J\bigg] \\ &= \int dx \bigg[\frac{\bar\rho}{2}\partial_t(\phi_1+\phi_2) + \frac{1}{4\pi}(\partial_x\phi_1\partial_t\phi_2 - \partial_x\phi_2\partial_t\phi_1) \\ &\quad + \frac{K_{IJ}}{4\pi}\partial_x\phi_I\partial_t\phi_J - \frac{V_{IJ}}{2}\partial_x\phi_I\partial_x\phi_J\bigg], \\ K &= \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \quad V = \text{positive definite.} \end{aligned} \tag{29}$$

Here, we have been careful to keep the total derivative terms $\frac{\bar\rho}{2}\partial_t(\phi_1+\phi_2) + \frac{1}{4\pi}(\partial_x\phi_1\partial_t\phi_2 - \partial_x\phi_2\partial_t\phi_1)$. Those are topological terms that do not affect the classical equation of motion, but have effects in quantum theory.

Effective theory similar to the above form has been obtained before for the edge state of fractional quantum Hall states [27,28]. But here we have to be more careful in keeping the topological term $\bar\rho\partial_t\phi_1$, which describes the mixed anomaly of $U(1)\times\mathbb{R}$ symmetry for the fermionic system.

We like to mention that the patch symmetry transformations are determined from the low-energy effective theories (24) or (29), and are still given by Eq. (21) and Eq. (22). So the mixed anomaly can still be detected via the commutation relation of the patch symmetry transformations (23).

In fact, Eq. (29) is the low-energy effective theory for a fermion system with Fermi momentum $k_F = \pi\bar\rho$. $\frac{1}{2\pi}\partial_x\phi_1$ describes the density of right-moving fermions and $\frac{1}{2\pi}\partial_x\phi_2$ describes the density of left-moving fermions. For example, the low-energy effective theory for right-moving fermions is given by

$$\begin{aligned} L &= \int dx \left(\frac{k_F}{2\pi}\partial_t\phi_1 + \frac{1}{4\pi}\partial_x\phi_1\partial_t\phi_1 - \frac{V_{11}}{2}\partial_x\phi_1\partial_x\phi_1\right) \\ &= \int dx \left(\bar\rho_1\partial_t\phi_1 + \frac{1}{4\pi}\partial_x\phi_1\partial_t\phi_1 - \frac{V_{11}}{2}\partial_x\phi_1\partial_x\phi_1\right). \end{aligned} \tag{30}$$

In the above expression, we stress the direction connection between the topological term and the Fermi momentum k_F, as well as the direction connection between the topological term and the density of the right-moving fermions: $\bar\rho_1 = \frac{k_F}{2\pi} = \frac{1}{2}\bar\rho$.

We see that the mixed anomaly of $U(1)\times\mathbb{R}$ symmetry is simply a nonzero Fermi momentum k_F. When $k_F = 0$, i.e., when the mixed anomaly vanishes, the fermion system can have a gapped ground state that does not break the $U(1)\times\mathbb{R}$ symmetry. But when $k_F \neq 0$, i.e., in the presence of a mixed anomaly, the fermion system cannot have a gapped ground state that does not break the $U(1)\times\mathbb{R}$ symmetry. This is a well known result, but restated in terms of the mixed anomaly of $U(1)\times\mathbb{R}$ symmetry.

IV. LOW-ENERGY EFFECTIVE THEORY OF d-DIMENSIONAL FERMI LIQUID AND THE MIXED $U(1)\times\mathbb{R}^d$ ANOMALY

A. Effective theory for the Fermi surface dynamics

In the last section, we discussed the low-energy effective theory of a 1D Fermi liquid, which contains a proper topological term that reflects the mixed $U(1)\times\mathbb{R}$ anomaly. In this section, we are going to generalize this result to higher dimensions. The generalization is possible since the higher dimensional Fermi liquid can be viewed as a collection of 1D Fermi liquids.

Let us use $\boldsymbol{k}_F$ to parametrize the Fermi surface. We introduce $u(\boldsymbol{x},\boldsymbol{k}_F)$ to describe the shift of the Fermi surface. Thus the total fermion number is given by

$$N = \bar N + \int d^d\boldsymbol{x}\frac{d^{d-1}\boldsymbol{k}_F}{(2\pi)^d}\,u(\boldsymbol{x},\boldsymbol{k}_F). \tag{31}$$

The total energy is

$$E = \bar E + \int d^d\boldsymbol{x}\frac{d^{d-1}\boldsymbol{k}_F}{(2\pi)^d}\frac{|\boldsymbol{v}_F(\boldsymbol{k}_F)|}{2}u^2(\boldsymbol{x},\boldsymbol{k}_F), \tag{32}$$

where

$$\boldsymbol{v}_F := \partial_{\boldsymbol{k}}H(\boldsymbol{k}) \tag{33}$$

is the Fermi velocity and $H(\boldsymbol{k})$ is the single fermion energy.

The equation of motion for the field $u(\boldsymbol{x},\boldsymbol{k}_F)$ is given by

$$(\partial_t + \boldsymbol{v}_F\cdot\partial_{\boldsymbol{x}})u(\boldsymbol{x},\boldsymbol{k}_F) = 0. \tag{34}$$

Let us introduce a field $\phi(\boldsymbol{x}, \boldsymbol{k}_F)$ via

$$-\boldsymbol{n}_F \cdot \partial_{\boldsymbol{x}} \phi(\boldsymbol{x}, \boldsymbol{k}_F) = u(\boldsymbol{x}, \boldsymbol{k}_F), \tag{35}$$

where

$$\boldsymbol{n}_F := \frac{\boldsymbol{v}_F}{|\boldsymbol{v}_F|}. \tag{36}$$

The equation of motion for ϕ becomes

$$(\partial_t + \boldsymbol{v}_F \cdot \partial_{\boldsymbol{x}})[\boldsymbol{n}_F \cdot \partial_{\boldsymbol{x}} \phi(\boldsymbol{x}, \boldsymbol{k}_F)] = 0 \tag{37}$$

and the total energy becomes

$$E = \bar{E} + \int d^d\boldsymbol{x} \frac{d^{d-1}\boldsymbol{k}_F}{(2\pi)^d} \frac{|\boldsymbol{v}_F(\boldsymbol{k}_F)|}{2} [\boldsymbol{n}_F \cdot \partial_{\boldsymbol{x}} \phi(\boldsymbol{x}, \boldsymbol{k}_F)]^2. \tag{38}$$

The phase-space Lagrangian that produces the above equation of motion and total energy is given by

$$L_{\text{ph}} = \int d^d\boldsymbol{x} \frac{d^{d-1}\boldsymbol{k}_F}{(2\pi)^d} \bigg\{ -\frac{1}{2} \partial_t \phi(\boldsymbol{x}, \boldsymbol{k}_F) \boldsymbol{n}_F \cdot \partial_{\boldsymbol{x}} \phi(\boldsymbol{x}, \boldsymbol{k}_F) - \frac{|\boldsymbol{v}_F(\boldsymbol{k}_F)|}{2} [\boldsymbol{n}_F \cdot \partial_{\boldsymbol{x}} \phi(\boldsymbol{x}, \boldsymbol{k}_F)]^2 \bigg\}. \tag{39}$$

Repeating a calculation similar to the 1D chiral Luttinger liquid [27,28], we find that after quantization, the operator $\phi(\boldsymbol{x}, \boldsymbol{k}_F)$ has the following commutation relation:

$$\begin{aligned} &[\phi(\boldsymbol{x}', \boldsymbol{k}'_F), \boldsymbol{n}_F \cdot \partial_{\boldsymbol{x}} \phi(\boldsymbol{x}, \boldsymbol{k}_F)] \\ &= -[\phi(\boldsymbol{x}', \boldsymbol{k}'_F), u(\boldsymbol{x}, \boldsymbol{k}_F)] \\ &= -i(2\pi)^d \delta^d(\boldsymbol{x} - \boldsymbol{x}') \delta^{d-1}(\boldsymbol{k}_F - \boldsymbol{k}'_F), \end{aligned} \tag{40}$$

which reproduces the equation of motion,

$$\begin{aligned} &\partial_t \phi(\boldsymbol{x}, \boldsymbol{k}_F, t) = i[H, \phi(\boldsymbol{x}, \boldsymbol{k}_F, t)] = -\boldsymbol{v}_F \cdot \partial_{\boldsymbol{x}} \phi(\boldsymbol{x}, \boldsymbol{k}_F, t), \\ &H = \int d^d\boldsymbol{x} \frac{d^{d-1}\boldsymbol{k}_F}{(2\pi)^d} \frac{|\boldsymbol{v}_F(\boldsymbol{k}_F)|}{2} [\boldsymbol{n}_F \cdot \partial_{\boldsymbol{x}} \phi(\boldsymbol{x}, \boldsymbol{k}_F)]^2. \end{aligned} \tag{41}$$

We also see that

$$[N, \phi(\boldsymbol{x}, \boldsymbol{k}_F)] = -i, \quad [N, e^{i\phi(\boldsymbol{x}, \boldsymbol{k}_F)}] = e^{i\phi(\boldsymbol{x}, \boldsymbol{k}_F)}. \tag{42}$$

Thus, $e^{i\phi(\boldsymbol{x}, \boldsymbol{k}_F, t)}$ is the operator that increases N by 1, and the $U(1)$ symmetry transformation is given by

$$e^{i\theta N} \phi(\boldsymbol{x}, \boldsymbol{k}_F) e^{-i\theta N} = \phi(\boldsymbol{x}, \boldsymbol{k}_F) + \theta. \tag{43}$$

We see that $\phi(\boldsymbol{x}, \boldsymbol{k}_F)$ is an angular field $\phi(\boldsymbol{x}, \boldsymbol{k}_F) \sim \phi(\boldsymbol{x}, \boldsymbol{k}_F) + 2\pi$.

However, in Eq. (39), we only have terms that $\phi(\boldsymbol{x}, \boldsymbol{k}_F)$ couples to δN. In a complete Lagrangian, $\phi(\boldsymbol{x}, \boldsymbol{k}_F)$ must also couple to $\bar{\rho}$—the density of the $U(1)$ charge in the ground state. This consideration motivates us to propose the complete phase-space Lagrangian to be

$$\begin{aligned} L_{\text{ph}} = \int d^d\boldsymbol{x} \frac{d^{d-1}\boldsymbol{k}_F}{(2\pi)^d} \bigg\{ &\frac{\bar{\rho}}{A_F} \dot{\phi}(\boldsymbol{x}, \boldsymbol{k}_F) \\ &- \frac{1}{2} \dot{\phi}(\boldsymbol{x}, -\boldsymbol{k}_F) \boldsymbol{n}_F \cdot \partial_{\boldsymbol{x}} \phi(\boldsymbol{x}, \boldsymbol{k}_F) \\ &- \frac{1}{2} \dot{\phi}(\boldsymbol{x}, \boldsymbol{k}_F) \boldsymbol{n}_F \cdot \partial_{\boldsymbol{x}} \phi(\boldsymbol{x}, \boldsymbol{k}_F) \\ &- \frac{|\boldsymbol{v}_F(\boldsymbol{k}_F)|}{2} [\boldsymbol{n}_F \cdot \partial_{\boldsymbol{x}} \phi(\boldsymbol{x}, \boldsymbol{k}_F)]^2 \bigg\}, \end{aligned} \tag{44}$$

where

$$A_F := \int \frac{d^{d-1}\boldsymbol{k}_F}{(2\pi)^d}, \tag{45}$$

and we have assumed a central reflection symmetry $\boldsymbol{k}_F \to -\boldsymbol{k}_F$, i.e., $\boldsymbol{v}_F(\boldsymbol{k}_F) = -\boldsymbol{v}_F(-\boldsymbol{k}_F)$ and $\boldsymbol{n}_F(\boldsymbol{k}_F) = -\boldsymbol{n}_F(-\boldsymbol{k}_F)$. The two terms, $\frac{\bar{\rho}}{A_F} \dot{\phi}(\boldsymbol{x}, \boldsymbol{k}_F)$ and

$$\int \frac{d^{d-1}\boldsymbol{k}_F}{(2\pi)^d} \frac{1}{2} \dot{\phi}(\boldsymbol{x}, -\boldsymbol{k}_F) \boldsymbol{n}_F(\boldsymbol{k}_F) \cdot \partial_{\boldsymbol{x}} \phi(\boldsymbol{x}, \boldsymbol{k}_F), \tag{46}$$

are total derivative topological terms.

We know that the volume enclosed by the Fermi surface is directly related to the fermion density $\bar{\rho}$ [7,8]. Naively, in our effective theory (44), the parameter $\bar{\rho}$ and Fermi surface $\boldsymbol{k}_F$ are not related. In the following, we like to show that in fact, $\bar{\rho}$ and the Fermi surface $\boldsymbol{k}_F$ are related, from within the effective field theory (44).

Consider a field configuration

$$\begin{aligned} &\phi(\boldsymbol{x}, \boldsymbol{k}_F) = \boldsymbol{a} \cdot \boldsymbol{x} \\ \text{or} \quad &\boldsymbol{n}_F \cdot \partial_{\boldsymbol{x}} \phi(\boldsymbol{x}, \boldsymbol{k}_F) = \boldsymbol{a} \cdot \boldsymbol{n}_F = u(\boldsymbol{x}, \boldsymbol{k}_F). \end{aligned} \tag{47}$$

There are two ways to compute the momentum for such a field configuration.

In the first way, the total momentum is computed via the deformation $u(\boldsymbol{x}, \boldsymbol{k}_F)$ of the Fermi surface (assuming the total momentum of the ground state to be zero),

$$\begin{aligned} \boldsymbol{k}_{\text{tot}} &= \int d^d\boldsymbol{x} \frac{d^{d-1}\boldsymbol{k}_F}{(2\pi)^d} \boldsymbol{k}_F u(\boldsymbol{x}, \boldsymbol{k}_F) \\ &= \int d^d\boldsymbol{x} \frac{d^{d-1}\boldsymbol{k}_F}{(2\pi)^d} \boldsymbol{k}_F(\boldsymbol{a} \cdot \boldsymbol{n}_F). \end{aligned} \tag{48}$$

Note that $\boldsymbol{n}_F$ is the normal direction of the Fermi surface. Therefore,

$$\begin{aligned} \boldsymbol{k}_{\text{tot}} &= \int d^d\boldsymbol{x} \frac{d^{d-1}\boldsymbol{k}_F}{(2\pi)^d} \boldsymbol{k}_F(\boldsymbol{a} \cdot \boldsymbol{n}_F) \\ &= \int d^d\boldsymbol{x} \int_{\boldsymbol{k} \in \boldsymbol{k}_F + \boldsymbol{a}} \frac{d^d\boldsymbol{k}_F}{(2\pi)^d} \boldsymbol{k} - \int d^d\boldsymbol{x} \int_{\boldsymbol{k} \in \boldsymbol{k}_F} \frac{d^d\boldsymbol{k}_F}{(2\pi)^d} \boldsymbol{k}, \end{aligned} \tag{49}$$

where $\int_{\boldsymbol{k} \in \boldsymbol{k}_F} d^d\boldsymbol{k}_F$ means integration over $\boldsymbol{k}$ inside the Fermi surface, and $\int_{\boldsymbol{k} \in \boldsymbol{k}_F + \boldsymbol{a}} d^d\boldsymbol{k}_F$ means integration over $\boldsymbol{k}$ inside the shifted Fermi surface (shifted by $\boldsymbol{a}$). Let

$$V_{\boldsymbol{k}_F} := \int_{\boldsymbol{k} \in \boldsymbol{k}_F} d^d\boldsymbol{k}_F \tag{50}$$

be the volume enclosed by the Fermi surface; we see that

$$\boldsymbol{k}_{\text{tot}} = V \frac{V_{\boldsymbol{k}_F}}{(2\pi)^d} \boldsymbol{a}, \tag{51}$$

where V is the volume of the system.

There is a second way to compute the total momentum $\boldsymbol{k}_{\text{tot}}$. We consider a time-dependent translation of the above configuration,

$$\phi(\boldsymbol{x}, \boldsymbol{k}_F) = \boldsymbol{a} \cdot [\boldsymbol{x} + \boldsymbol{x}_0(t)]. \tag{52}$$

The effective phase-space Lagrangian for $\boldsymbol{x}_0(t)$ is given by

$$L_{\text{ph}} = V \bar{\rho} \boldsymbol{a} \cdot \dot{\boldsymbol{x}}_0(t). \tag{53}$$

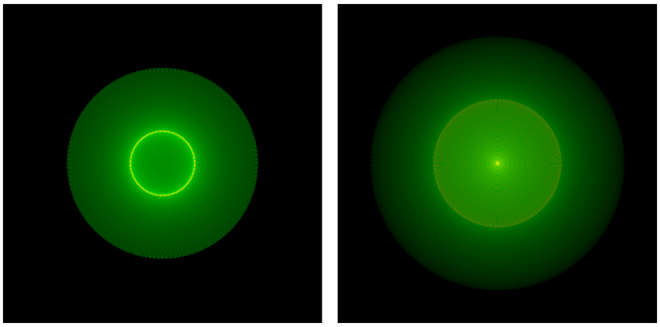

FIG. 3. The total momentum distributions for low-energy many-body states for a 2D Fermi liquid, with one-particle excitations (left, red channel), two-particle one-hole excitations (left, green channel), one-particle one-hole excitations (right, red channel), and two-particle two-hole excitations (right, green channel). The horizontal axis is k_x and the vertical axis is k_y. The shift of the momentum distributions by the $U(1)$ symmetry twist reflects the mixed anomaly in $U(1) \times \mathbb{R}^d$ symmetry.

We see that $V\bar{\rho}\boldsymbol{a}$ is the canonical momentum of the translation $\boldsymbol{x}_0$. Thus, the total momentum of the configuration is

$$\boldsymbol{k}_{\text{tot}} = V\bar{\rho}\boldsymbol{a}. \tag{54}$$

Comparing Eq. (51) and Eq. (54), we see that the volume included by the Fermi surface and the fermion density is related,

$$\frac{V_{\boldsymbol{k}_F}}{(2\pi)^d} = \bar{\rho}. \tag{55}$$

This is the Luttinger theorem.

B. The mixed anomaly in $U(1) \times \mathbb{R}^d$ symmetry

As we have pointed out, the topological term $\int d^d\boldsymbol{x} \frac{d^{d-1}\boldsymbol{k}_F}{(2\pi)^d} [\frac{\bar{\rho}}{A_F}\partial_t\phi(\boldsymbol{x},\boldsymbol{k}_F)]$ represents a mixed anomaly of $U(1) \times \mathbb{R}^d$ symmetry. To see this point, we note that $\boldsymbol{a}$ in Eq. (47) can be viewed as the $U(1)$ symmetry twist. The fact that the $U(1)$ symmetry twist can induce the $\mathbb{R}^d$ quantum number (i.e., the momentum) reflects the presence of the mixed anomaly of $U(1) \times \mathbb{R}^d$ symmetry. Equation (55) indicates that the mixed anomaly can constrain the low-energy dynamics, and in this case determines the volume enclosed by the Fermi surface.

In the above, we discussed how the $U(1)$ symmetry twist shifts the total momentum of a particular low-energy many-body state. However, in practice, we cannot pick a particular low-energy many-body state and see how its momentum is shifted by the $U(1)$ symmetry twist. What can be done is to examine all the low-energy many-body states and their total momentum distribution. The shift of the total momentum distribution by the $U(1)$ symmetry twist measures the mixed $U(1) \times \mathbb{R}^d$ anomaly. In Fig. 3, we plot the total momentum distributions for low-energy many-body states, with one-particle excitations, two-particle one-hole excitations, one-particle one-hole excitations, and two-particle two-hole excitations.

The mixed anomaly not only appears in Fermi-liquid phases of fermions, but also appears in any other phases of fermions. Thus the mixed anomaly constrains the low-energy dynamics in any of those phases. In the next section, we consider a fermion phase where the fermions pair up to form a boson liquid in d-dimensional space.

C. Effective theory of a Fermi liquid in most general setting

In Sec. IV A, we considered a Fermi liquid in free space. In this section, we like to include an electromagnetic field in real space, as well as a magnetic field in $\boldsymbol{k}$ space. We like to find the low-energy effective theory of a Fermi liquid for this more general situation.

First we consider the dynamics of a single particle in a very general setting. The classical state of the particle is described by a point in phase space parametrized by ξ^I. The single-particle dynamics is described by a single-particle phase-space Lagrangian,

$$L(\dot{\xi}^I, \xi^I) = \int dt\ [a_I(\xi^I)\dot{\xi}^I - H(\xi^I)], \tag{56}$$

which gives rise to the following single-particle equation of motion:

$$b_{IJ}\dot{\xi}^J = \frac{\partial H}{\partial \xi^I}, \quad b_{IJ} = \partial_{\xi^I}a_J - \partial_{\xi^J}a_I. \tag{57}$$

Here, $H(\xi^I)$ is the single-particle energy for the state ξ^I, and $a_I(\xi^I)$ is a phase-space vector potential that describes the phase "magnetic" field $b_{IJ}(\xi^I)$. The phase-space magnetic field includes both the real space magnetic field and the $\boldsymbol{k}$-space magnetic field [20–22].

For a particle in a d-dimensional free space described by coordinate-momentum pair $(\boldsymbol{x},\boldsymbol{k}) = (x^i, k_i)$, $i = 1,\dots,d$, the phase-space magnetic field $b_{IJ}(\xi^I)$ is a constant (i.e., independent of ξ^I), since $a_I(\xi^I)\dot{\xi}^I = \boldsymbol{k}\cdot\dot{\boldsymbol{x}}$ (i.e., $a_{x^i} = k_i,\ a_{k_i} = 0$). On the other hand, if there is a nonuniform real space and/or $\boldsymbol{k}$-space magnetic fields, the phase-space magnetic field $b_{IJ}(\xi^I)$ will not be uniform.

As an example, let us consider a particle in three-dimensional space. The phase space is six dimensional and is parametrized by $(\xi^I) = (\boldsymbol{x},\boldsymbol{k})$. The phase-space Lagrangian is given by

$$L = [\boldsymbol{k}\cdot\dot{\boldsymbol{x}} + \boldsymbol{A}(\boldsymbol{x})\cdot\dot{\boldsymbol{x}} + \widetilde{\boldsymbol{A}}(\boldsymbol{k})\cdot\dot{\boldsymbol{k}}] - H(\boldsymbol{k}) - V(\boldsymbol{x}). \tag{58}$$

Here, $\boldsymbol{A}(\boldsymbol{x})$ is the real space vector potential for the electromagnetic field that only depends on $\boldsymbol{x}$. $\widetilde{\boldsymbol{A}}(\boldsymbol{k})$ is the $\boldsymbol{k}$-space vector potential that is assumed to depend only on $\boldsymbol{k}$. Such a $\boldsymbol{k}$-space vector potential can appear for an electron in a crystal with spin orbital couplings. The corresponding equation of motion is given by

$$\dot{k}_i = -\frac{\partial V}{\partial x^i} + B_{ij}\dot{x}^j, \quad \dot{x}^i = \frac{\partial H}{\partial k_i} - \widetilde{B}^{ij}\dot{k}_j,$$
$$\text{or}\quad \dot{\boldsymbol{k}} = -\frac{\partial V}{\partial \boldsymbol{x}} + \dot{\boldsymbol{x}}\times\boldsymbol{B}, \quad \dot{\boldsymbol{x}} = \frac{\partial H}{\partial \boldsymbol{k}} - \dot{\boldsymbol{k}}\times\widetilde{\boldsymbol{B}}, \tag{59}$$

where

$$B_{ij} = \partial_{x^i}A_j - \partial_{x^j}A_i, \quad \widetilde{B}^{ij} = \partial_{k_i}\widetilde{A}^j - \partial_{k_j}\widetilde{A}^i,$$
$$\text{or}\quad \boldsymbol{B} = \partial_{\boldsymbol{x}}\times\boldsymbol{A}, \quad \widetilde{\boldsymbol{B}} = \partial_{\boldsymbol{k}}\times\widetilde{\boldsymbol{A}}. \tag{60}$$

Now consider a many-fermion system which is described by a particle-number distribution $g(\xi^I)$. The meaning of the distribution $g(\xi^I)$ is given by

$$dN = g(\xi^I)\text{Pf}[b(\xi^I)]\frac{d^{2d}\xi^I}{(2\pi)^d}, \tag{61}$$

where dN is the number of fermions in the phase-space volume $d^{2d}\xi^I$, and $\text{Pf}[b(\xi^I)]$ is the Pfaffian of the $2d \times 2d$ antisymmetric matrix,

$$[b(\xi^I)]_{IJ} = b_{IJ}(\xi^I). \tag{62}$$

In fact, $g(\xi^I)$ have a meaning as the occupation number per orbital since the number of orbitals (i.e., the single-particle quantum states) in the phase-space volume $d^{2d}\xi^I$ is given by $\text{Pf}[b(\xi^I)]\frac{d^{2d}\xi^I}{(2\pi)^d}$.

The above interpretation is correct since under the time evolution (57), the scaled phase-space volume $\text{Pf}[b(\xi^I)]\frac{d^{2d}\xi^I}{(2\pi)^d}$ is time independent, which corresponds to the unitary time evolution in quantum theory. To show such a result, we first choose a phase-space coordinate such that b_{IJ} is uniform in the phase space. In this case, the time evolution $\dot{\xi}^I$ is described by a divergentless vector field, $\frac{\partial H}{\partial \xi^I}$, in the phase space, and the phase-space volume $d^{2d}\xi^I$ is time independent. We note that the phase-space volume given by the combination $\text{Pf}[b(\xi^I)]\frac{d^{2d}\xi^I}{(2\pi)^b}$ is invariant under the coordinate transformation. Such an invariant combination is invariant under the time evolution (57) for a general coordinate since the equation of motion is covariant under the coordinate transformation. We see that the phase space has a symplectic geometry.

For our example (58), the 6×6 matrix b_{IJ} is given by

$$(b_{IJ}) = \begin{pmatrix} B_{ij} & \delta_{ij} \\ -\delta_{ij} & \widetilde{B}^{ij} \end{pmatrix}. \tag{63}$$

We find that

$$\begin{aligned}\text{Pf}(b) &= \text{Pf}\begin{pmatrix} B_{ij} & \delta_{ij} \\ -\delta_{ij} & \widetilde{B}^{ij} \end{pmatrix} \\ &= \text{Pf}(B, \widetilde{B}) = 1 + B_{ij}\widetilde{B}^{ji} + O(B_{ik}\widetilde{B}^{kj})^2. \end{aligned} \tag{64}$$

The effective theory for the Fermi liquid in such a general setting is simply a hydrodynamical theory for an incompressible fluid in the phase space. In the following, we will present such a theory for small fluctuations near the ground state. First, the ground state of the Fermi liquid is described by the following distribution (or phase-space density):

$$\bar{g}(\xi^I) = \begin{cases} 1 & \text{for } H(\xi^I) < 0 \\ 0 & \text{for } H(\xi^I) > 0. \end{cases} \tag{65}$$

The generalized Fermi surface is the $(2d-1)$-dimensional submanifold in the phase space where $\bar{g}(\xi^I)$ has a jump. A many-body collective excitation is described by another incompressible distribution $g(\xi^I) = 0, 1$. For low-energy collective excitations near the ground state, we may describe such an incompressible distribution via the displacement of the generalized Fermi surface,

$$u(\xi_F^I) = \sqrt{\sum_I (\Delta\xi^I)^2}, \tag{66}$$

where ξ_F^I parametrize the $(2d-1)$-dimensional generalized Fermi surface, and $\Delta\xi^I$ describe the shift of the generalized Fermi surface in the normal direction.

Let us introduce an integration over the generalized Fermi surface,

$$\int \text{Pf}[b(\xi^I)]\frac{d^{2d-1}\xi_F}{(2\pi)^d} := \int \text{Pf}[b(\xi^I)]\frac{d^{2d}\xi}{(2\pi)^d}\,|\partial_{\xi^I}\bar{g}|. \tag{67}$$

The number of fermions in the collective excited state described by $u(\xi^I)$ is given by

$$\begin{aligned} N &= \int \text{Pf}(b)\frac{d^{2d}\xi}{(2\pi)^d}\, g(\xi^I) \\ &= \bar{N} + \int \text{Pf}(b)\frac{d^{2d-1}\xi_F}{(2\pi)^d}\, u(\xi_F^I). \end{aligned} \tag{68}$$

The energy of the collective excited state is given by

$$E = \bar{E} + \int \text{Pf}(b)\frac{d^{2d-1}\xi_F}{(2\pi)^d}\,\frac{1}{2}|h_.|u^2(\xi_F^I), \tag{69}$$

where

$$h_I := \partial_{\xi^I} H, \qquad |h_.| := \sqrt{\sum_I h_I^2}. \tag{70}$$

The equation of motion of $u(\xi_F^I, t)$ can be obtained in two ways. First, we note that $|h_I|u$ is the single-particle energy, which is invariant under the single-particle time evolution $\xi_F^I(t)$ that satisfies the single-particle equation of motion (57). Thus,

$$\frac{d}{dt}|h_.|u(\xi_F^I(t), t) = 0. \tag{71}$$

This allows us to obtain the equation of motion for the $u(\xi_F^I, t)$ field using the single-particle equation of motion (57),

$$\big(\partial_t + h^I(\xi_F^I)\partial_{\xi^I}\big)|h_.|(\xi_F^I)u(\xi_F^I, t) = 0, \tag{72}$$

where

$$h^I = b^{IJ}h_J, \tag{73}$$

and the repeated index J is summed. Here, b^{IJ} is the matrix inversion of of b_{IJ}:

$$b_{IJ}b^{JK} = \delta_{IK}. \tag{74}$$

Second, we note that $\text{Pf}(b)u$ is the density of fermions on the generalized Fermi surface [see Eq. (68)]. The corresponding current density is given by $h^I\text{Pf}(b)u$ since $\dot{\xi}^I = h^I$ [see Eq. (57)]. The fermion conservation gives us another equation of motion for $u(\xi_F^I, t)$,

$$\partial_t \text{Pf}\big[b(\xi_F^I)\big]u(\xi_F^I, t) + \partial_{\xi^I}\big\{h^I(\xi_F^I)\text{Pf}\big[b(\xi_F^I)\big]u(\xi_F^I, t)\big\} = 0. \tag{75}$$

Since the single-particle dynamics leads to the two equations, they must be consistent. This requires that

$$|h_.|(\partial_t + \partial_{\xi^I}h^I)\text{Pf}(b)u = \text{Pf}(b)(\partial_t + h^I\partial_{\xi^I})|h_.|u. \tag{76}$$

In other words, h^I, $|h_.|$, and $\text{Pf}(b)$ are related and they satisfy

$$\frac{|h_.|}{\text{Pf}(b)}(\partial_t + \partial_{\xi^I}h^I)\frac{\text{Pf}(b)}{|h_.|} = \partial_t + h^I\partial_{\xi^I}. \tag{77}$$

Let us introduce a scalar field $\phi(\xi_F^I,t)$ via

$$-|h_.|^{-1}h^I\partial_{\xi_F^I}\phi(\xi_F^I,t)=u(\xi_F^I,t). \tag{78}$$

The equation of motion for ϕ is given by

$$(\partial_t+h^I\partial_{\xi^I})h^J\partial_{\xi_F^J}\phi=0, \tag{79}$$

which can be simplified further as

$$(\partial_t+h^I\partial_{\xi^I})\phi=0 \tag{80}$$

since $h^I(\xi_F^I)$ does not depend on time.

The above equation of motion and the expression of total energy (69) allow us to determine the phase-space Lagrangian,

$$L_{\text{ph}}=-\int\frac{d^{2d-1}\xi_F}{(2\pi)^d}\frac{\text{Pf}(b)}{2|h_.|}\big(\dot\phi h^I\partial_{\xi_F^I}\phi+\big[h^I\partial_{\xi_F^I}\phi\big]^2\big), \tag{81}$$

up to total derivative topological terms.

To include topological terms, we assume a symmetry described by a map in phase space,

$$\xi^I\to\bar\xi^I,\quad h^I(\xi^I)=-h^I(\bar\xi^I), \tag{82}$$

which generalize the $\boldsymbol{k}_F\to-\boldsymbol{k}_F$ symmetry used before. The phase-space Lagrangian can now be written as

$$\begin{aligned}L_{\text{ph}}=\int\frac{d^{2d-1}\xi_F}{(2\pi)^d}\bigg(&\frac{\bar N}{VA_F}\dot\phi(\xi^I,t)-\frac{\text{Pf}[b(\xi^I)]}{2|h_.(\xi^I)|}\\&\times\big\{\dot\phi(\bar\xi^I,t)h^I(\xi^I)\partial_{\xi_F^I}\phi(\xi^I,t)+\dot\phi(\xi^I,t)h^I(\xi^I)\\&\times\partial_{\xi_F^I}\phi(\xi^I,t)+\big[h^I(\xi^I)\partial_{\xi_F^I}\phi(\xi^I,t)\big]^2\big\}\bigg),\end{aligned} \tag{83}$$

where $\bar N$ is the number of fermions in the ground state, and

$$VA_F=\int\frac{d^{2d-1}\xi_F}{(2\pi)^d}1 \tag{84}$$

is the total volume of the generalized Fermi surface.

From the first three terms in Eq. (83), we see that $\dot\phi(\xi^I,t)$ directly couples to the total density of the fermions,

$$\begin{aligned}N=\int\frac{d^{2d-1}\xi_F}{(2\pi)^d}\bigg\{&\frac{\bar N}{VA_F}-\frac{\text{Pf}[b(\xi^I)]}{2|h_.(\xi^I)|}\big[h^I(\xi^I)\partial_{\xi_F^I}\phi(\xi^I,t)\\&-h^I(\xi^I)\partial_{\xi_F^I}\phi(\bar\xi^I,t)\big]\bigg\}.\end{aligned} \tag{85}$$

In particular, the uniform part of $\dot\phi(\xi^I,t)$ couples to total number of fermions,

$$L_{\text{ph}}=N\dot\phi_{\text{uniform}}(t)+\cdots. \tag{86}$$

This indicates that $\phi\sim\phi+2\pi$ is an angular field, and the $U(1)$ transformation is given by

$$\phi(\xi^I,t)\to\phi(\xi^I,t)+\theta. \tag{87}$$

D. Effective theory for a Fermi liquid with real space and k-space magnetic fields

Now, let us apply the above formalism to develop the low-energy effective theory of a Fermi liquid, for three-dimensional fermions with real space magnetic field $\boldsymbol{A}(\boldsymbol{x})$ and $\boldsymbol{k}$-space magnetic field $\widetilde{\boldsymbol{A}}(\boldsymbol{k})$. The dynamics of a single fermion is described by Eq. (58) (with $V=0$). We have

$$\begin{aligned}&(h_I)=(0,\boldsymbol{v}_F),\qquad |h_.|=|\boldsymbol{v}_F|,\quad \boldsymbol{v}_F:=\partial_{\boldsymbol{k}}H,\\&(h^I)=(\widetilde{\boldsymbol{v}}_F,\boldsymbol{f}),\qquad \text{Pf}(b)=1-2\boldsymbol{B}\cdot\widetilde{\boldsymbol{B}}+\cdots,\\&\widetilde{\boldsymbol{v}}_F=\boldsymbol{v}_F-(\boldsymbol{v}_F\times\boldsymbol{B})\times\widetilde{\boldsymbol{B}}+\cdots,\quad \boldsymbol{f}:=\boldsymbol{v}_F\times\boldsymbol{B}+\cdots.\end{aligned} \tag{88}$$

Here, $\boldsymbol{f}=\dot{\boldsymbol{k}}$ has a physical meaning as the force acting on each fermion. $\widetilde{\boldsymbol{v}}_F=\dot{\boldsymbol{x}}$ has a physical meaning as the velocity of each fermion. When $\widetilde{\boldsymbol{A}}\neq0$, the velocity of a fermion at the Fermi surface is not given by $\boldsymbol{v}_F=\partial_{\boldsymbol{k}}H$. $\widetilde{\boldsymbol{v}}_F$ is also called the anomalous velocity [21].

By substituting the above into Eq. (83), we obtain

$$\begin{aligned}L_{\text{ph}}=\int d^d\boldsymbol{x}\frac{d^{d-1}\boldsymbol{k}_F}{(2\pi)^d}\bigg\{&\frac{\bar\rho}{A_F}\dot\phi(\boldsymbol{x},\boldsymbol{k}_F)-\frac{\text{Pf}(b)\dot\phi(\boldsymbol{x},-\boldsymbol{k}_F)}{2|\boldsymbol{v}_F|}\big(\widetilde{\boldsymbol{v}}_F\cdot\partial_{\boldsymbol{x}}+\boldsymbol{f}\cdot\partial_{\boldsymbol{k}_F}\big)\phi(\boldsymbol{x},\boldsymbol{k}_F,t)-\frac{\text{Pf}(b)\dot\phi(\boldsymbol{x},\boldsymbol{k}_F)}{2|\boldsymbol{v}_F|}\big(\widetilde{\boldsymbol{v}}_F\cdot\partial_{\boldsymbol{x}}+\boldsymbol{f}\cdot\partial_{\boldsymbol{k}_F}\big)\\&\times\phi(\boldsymbol{x},\boldsymbol{k}_F,t)-\frac{\text{Pf}(b)}{2|\boldsymbol{v}_F|}\big[\big(\widetilde{\boldsymbol{v}}_F\cdot\partial_{\boldsymbol{x}}+\boldsymbol{f}\cdot\partial_{\boldsymbol{k}_F}\big)\phi(\boldsymbol{x},\boldsymbol{k}_F,t)\big]^2\bigg\}-\int d^d\boldsymbol{x}\frac{d^{d-1}\boldsymbol{k}_F}{(2\pi)^d}\frac{d^{d-1}\boldsymbol{k}'_F}{(2\pi)^d}\frac{\text{Pf}(b)\text{Pf}(b')V(\boldsymbol{k}_F,\boldsymbol{k}'_F)}{2|\boldsymbol{v}_F||\boldsymbol{v}'_F|}\\&\times\big[\big(\widetilde{\boldsymbol{v}}_F\cdot\partial_{\boldsymbol{x}}+\boldsymbol{f}\cdot\partial_{\boldsymbol{k}_F}\big)\phi(\boldsymbol{x},\boldsymbol{k}_F,t)\big]\big[\big(\widetilde{\boldsymbol{v}}'_F\cdot\partial_{\boldsymbol{x}}+\boldsymbol{f}'\cdot\partial_{\boldsymbol{k}_F}\big)\phi(\boldsymbol{x},\boldsymbol{k}'_F,t)\big].\end{aligned} \tag{89}$$

Up to first order in $\boldsymbol{B}$ and $\widetilde{\boldsymbol{B}}$, the above can be simplified as

$$\begin{aligned}L_{\text{ph}}=\int d^d\boldsymbol{x}\frac{d^{d-1}\boldsymbol{k}_F}{(2\pi)^d}\bigg\{&\frac{\bar\rho}{A_F}\dot\phi(\boldsymbol{x},\boldsymbol{k}_F)-\frac{\dot\phi(\boldsymbol{x},-\boldsymbol{k}_F)}{2|\boldsymbol{v}_F|}\big(\boldsymbol{v}_F\cdot\partial_{\boldsymbol{x}}+\boldsymbol{f}\cdot\partial_{\boldsymbol{k}_F}\big)\phi(\boldsymbol{x},\boldsymbol{k}_F,t)-\frac{\dot\phi(\boldsymbol{x},\boldsymbol{k}_F)}{2|\boldsymbol{v}_F|}\big(\boldsymbol{v}_F\cdot\partial_{\boldsymbol{x}}+\boldsymbol{f}\cdot\partial_{\boldsymbol{k}_F}\big)\phi(\boldsymbol{x},\boldsymbol{k}_F,t)\\&-\frac{1}{2|\boldsymbol{v}_F|}\big[\big(\boldsymbol{v}_F\cdot\partial_{\boldsymbol{x}}+\boldsymbol{f}\cdot\partial_{\boldsymbol{k}_F}\big)\phi(\boldsymbol{x},\boldsymbol{k}_F,t)\big]^2\bigg\}-\int d^d\boldsymbol{x}\frac{d^{d-1}\boldsymbol{k}_F}{(2\pi)^d}\frac{d^{d-1}\boldsymbol{k}'_F}{(2\pi)^d}\frac{V(\boldsymbol{k}_F,\boldsymbol{k}'_F)}{2|\boldsymbol{v}_F||\boldsymbol{v}'_F|}\\&\times\big[\big(\boldsymbol{v}_F\cdot\partial_{\boldsymbol{x}}+\boldsymbol{f}\cdot\partial_{\boldsymbol{k}_F}\big)\phi(\boldsymbol{x},\boldsymbol{k}_F,t)\big]\big[\big(\boldsymbol{v}'_F\cdot\partial_{\boldsymbol{x}}+\boldsymbol{f}'\cdot\partial_{\boldsymbol{k}_F}\big)\phi(\boldsymbol{x},\boldsymbol{k}'_F,t)\big].\end{aligned} \tag{90}$$

Here, we have assumed a central reflection symmetry $\boldsymbol{k} \to -\boldsymbol{k}$, and the mapping $\xi^I \to \bar{\xi}^I$ is given by $(\boldsymbol{x}, \boldsymbol{k}) \to (\boldsymbol{x}, -\boldsymbol{k})$. We also included the interaction term for the Fermi surface fluctuations, the $V(\boldsymbol{k}_F, \boldsymbol{k}'_F)$ term, where $\boldsymbol{v}_F, \boldsymbol{v}'_F$ are the Fermi velocities at $\boldsymbol{k}_F, \boldsymbol{k}'_F$.

Note that the fermion density at the Fermi surface $\boldsymbol{k}_F$ is given by [see Eq. (78)]

$$u(\boldsymbol{x}, \boldsymbol{k}_F) = -\frac{\boldsymbol{v}_F}{|\boldsymbol{v}_F|} \cdot \partial_{\boldsymbol{x}}\phi(\boldsymbol{x}, \boldsymbol{k}_F, t) - \frac{\boldsymbol{f}}{|\boldsymbol{v}_F|} \cdot \partial_{\boldsymbol{k}_F}\phi(\boldsymbol{x}, \boldsymbol{k}_F, t), \tag{91}$$

and thus the total fermion number density is given by

$$\rho = \bar{\rho} - \int \frac{d^{d-1}\boldsymbol{k}_F}{(2\pi)^d}\left(\frac{\boldsymbol{v}_F}{|\boldsymbol{v}_F|} \cdot \partial_{\boldsymbol{x}}\phi + \frac{\boldsymbol{f}}{|\boldsymbol{v}_F|} \cdot \partial_{\boldsymbol{k}_F}\phi\right). \tag{92}$$

This expression helps us to understand why the interaction term for the Fermi surface fluctuations has a form given in Eq. (90).

The above expression also allows us to see that $\dot{\phi}(\boldsymbol{x}, \boldsymbol{k}_F)$ couples to the total fermion density [see the first three terms in Eq. (90)]. Thus, $\phi(\boldsymbol{x}, \boldsymbol{k}_F) \sim \phi(\boldsymbol{x}, \boldsymbol{k}_F) + 2\pi$ is an angular field and the $U(1)$ symmetry transformation is given by

$$\phi(\boldsymbol{x}, \boldsymbol{k}_F) \to \phi(\boldsymbol{x}, \boldsymbol{k}_F) + \theta. \tag{93}$$

Equation (77) now becomes (to the first order in $\boldsymbol{B}$ and $\widetilde{\boldsymbol{B}}$)

$$\partial_{\boldsymbol{x}} \cdot \boldsymbol{v}_F + \partial_{\boldsymbol{k}_F} \cdot \boldsymbol{f} = |\boldsymbol{v}_F|^{-1}\big(\boldsymbol{v}_F \cdot \partial_{\boldsymbol{x}} + \boldsymbol{f} \cdot \partial_{\boldsymbol{k}_F}\big)|\boldsymbol{v}_F|. \tag{94}$$

This will help us to compute the equation of motion for the ϕ field. The resulting equation of motion is given by [written in terms of $u(\boldsymbol{x}, \boldsymbol{k}_F, t)$]

$$\big(\partial_t + \boldsymbol{v}_F \cdot \partial_{\boldsymbol{x}} + \boldsymbol{f} \cdot \partial_{\boldsymbol{k}_F}\big)|\boldsymbol{v}_F|u(\boldsymbol{x}, \boldsymbol{k}_F, t) + \big(\boldsymbol{v}_F \cdot \partial_{\boldsymbol{x}} + \boldsymbol{f} \cdot \partial_{\boldsymbol{k}_F}\big)\int \frac{d^{d-1}\boldsymbol{k}'_F}{(2\pi)^d} V(\boldsymbol{k}_F, \boldsymbol{k}'_F)u(\boldsymbol{x}, \boldsymbol{k}'_F, t) = 0. \tag{95}$$

The above is the Boltzmann equation, which can be used to compute the transport properties after adding the collision terms. In terms of $\phi(\boldsymbol{x}, \boldsymbol{k}_F, t)$, we have

$$\big(\partial_t + \boldsymbol{v}_F \cdot \partial_{\boldsymbol{x}} + \boldsymbol{f} \cdot \partial_{\boldsymbol{k}_F}\big)\phi(\boldsymbol{x}, \boldsymbol{k}_F, t) + \int \frac{d^{d-1}\boldsymbol{k}'_F}{(2\pi)^d}\frac{V(\boldsymbol{k}_F, \boldsymbol{k}'_F)}{|\boldsymbol{v}'_F|}\big[\boldsymbol{v}'_F \cdot \partial_{\boldsymbol{x}}\phi(\boldsymbol{x}, \boldsymbol{k}'_F, t) + \boldsymbol{f}' \cdot \partial_{\boldsymbol{k}_F}\phi(\boldsymbol{x}, \boldsymbol{k}'_F, t)\big] = 0. \tag{96}$$

The above equations of motion are valid only to the first order in $\boldsymbol{B}$ and $\widetilde{\boldsymbol{B}}$. The exact equations of motion, in several different forms, are given by

$$\begin{aligned}
&\big(\partial_t + \partial_{\boldsymbol{x}} \cdot \widetilde{\boldsymbol{v}}_F + \partial_{\boldsymbol{k}_F} \cdot \boldsymbol{f}\big)\mathrm{Pf}(b)u(\boldsymbol{x}, \boldsymbol{k}_F, t) + \mathrm{Pf}(b)\left(\frac{\widetilde{\boldsymbol{v}}_F}{|\boldsymbol{v}_F|} \cdot \partial_{\boldsymbol{x}} + \frac{\boldsymbol{f}}{|\boldsymbol{v}_F|} \cdot \partial_{\boldsymbol{k}_F}\right)\int \frac{d^{d-1}\boldsymbol{k}'_F}{(2\pi)^d}\mathrm{Pf}(b')V(\boldsymbol{k}_F, \boldsymbol{k}'_F)u(\boldsymbol{x}, \boldsymbol{k}'_F, t) = 0,\\
&\big(\partial_t + \widetilde{\boldsymbol{v}}_F \cdot \partial_{\boldsymbol{x}} + \boldsymbol{f} \cdot \partial_{\boldsymbol{k}_F}\big)|\boldsymbol{v}_F|u(\boldsymbol{x}, \boldsymbol{k}_F, t) + \big(\widetilde{\boldsymbol{v}}_F \cdot \partial_{\boldsymbol{x}} + \boldsymbol{f} \cdot \partial_{\boldsymbol{k}_F}\big)\int \frac{d^{d-1}\boldsymbol{k}'_F}{(2\pi)^d}\mathrm{Pf}(b')V(\boldsymbol{k}_F, \boldsymbol{k}'_F)u(\boldsymbol{x}, \boldsymbol{k}'_F, t) = 0,\\
&\big(\partial_t + \widetilde{\boldsymbol{v}}_F \cdot \partial_{\boldsymbol{x}} + \boldsymbol{f} \cdot \partial_{\boldsymbol{k}_F}\big)\phi(\boldsymbol{x}, \boldsymbol{k}_F, t) + \int \frac{d^{d-1}\boldsymbol{k}'_F}{(2\pi)^d}\frac{\mathrm{Pf}(b')}{|\boldsymbol{v}'_F|}V(\boldsymbol{k}_F, \boldsymbol{k}'_F)\big[\boldsymbol{v}'_F \cdot \partial_{\boldsymbol{x}}\phi(\boldsymbol{x}, \boldsymbol{k}'_F, t) + \boldsymbol{f}' \cdot \partial_{\boldsymbol{k}_F}\phi(\boldsymbol{x}, \boldsymbol{k}'_F, t)\big] = 0.
\end{aligned} \tag{97}$$

E. Emergent $U^\infty(1)$ symmetry

From the effective theory (89), we see that when there is no real space magnetic field $\boldsymbol{B} = 0$, we have $\boldsymbol{f} = 0$ and the effective theory has a $U^\infty(1)$ symmetry generated by

$$\phi(\boldsymbol{x}, \boldsymbol{k}_F) \to \phi(\boldsymbol{x}, \boldsymbol{k}_F) + \theta(\boldsymbol{k}_F), \tag{98}$$

where $\theta(\boldsymbol{k}_F)$ can be any function of $\boldsymbol{k}_F$. This is the so-called emergent $U^\infty(1)$ symmetry, which is a key character of Fermi liquid [1–4,23]. We also see that the above transformation is no longer a symmetry in the presence of real space magnetic field $\boldsymbol{B} \neq 0$. This may be related to the anomaly in the emergent $U^\infty(1)$ symmetry discussed in Ref. [23].

V. FERMION-PAIR LIQUID AND THE MIXED $U(1) \times \mathbb{R}^d$ ANOMALY

In this section, we are going to consider a fermion system in d-dimensional continuous space, with $U(1)$ particle-number-conservation symmetry and $\mathbb{R}^d$ translation symmetry. We assume the space to have a size $L_1 \times L_2 \times \cdots \times L_d$ and a periodic boundary condition. We will compute the distribution of the total momentum for many-body low-energy excitations $\boldsymbol{k}_{\text{tot}}$, and how such a distribution depends on the $U(1)$ symmetry twist described by a constant vector potential $\boldsymbol{a}$.

Using the results from the Appendix, we find that the low-energy effective theory is described by the following phase-space Lagrangian [see Eq. (A14)]:

$$L = \int d^d\boldsymbol{x}\left[\bar{\rho}_p\dot{\phi}_p(\boldsymbol{x}, t) + \delta\rho_p(\boldsymbol{x}, t)\dot{\phi}_p(\boldsymbol{x}, t) - \frac{\bar{\rho}}{2M_p}|\boldsymbol{\partial}\phi_p|^2 - \frac{g}{2}\delta\rho_p^2 + \cdots\right], \tag{99}$$

where $\bar{\rho}_p = \frac{1}{2}\bar{\rho}$ is the fermion-pair density in the ground state, and $\phi_p \sim \phi_p + 2\pi$ is the angular field for the fermion pair. The total energy and total crystal momentum of those excitations are given by

$$E = \frac{\bar{N}_p}{2M_p} \sum_\mu \left(\frac{m_\mu}{L_\mu} + 2a_\mu\right)^2 + \sum_{\boldsymbol{k}\neq 0} \left(n_{\boldsymbol{k}} + \frac{1}{2}\right) v|\boldsymbol{k}|,$$

$$\boldsymbol{k}_{\text{tot}} = N_p \sum_\mu \left(\frac{m_\mu}{L_\mu} + 2a_\mu\right)\hat{\boldsymbol{x}}_\mu + \sum_{\boldsymbol{k}\neq 0} n_{\boldsymbol{k}} \boldsymbol{k}. \quad (100)$$

We see that the $U(1)$ symmetry twist $\boldsymbol{a}$ induces a change in the total momentum,

$$\boldsymbol{k}_{\text{tot}} = 2N_p \boldsymbol{a} = 2\bar{\rho}_p \boldsymbol{a} V = \bar{\rho} \boldsymbol{a} V. \quad (101)$$

Such momentum dependence of the $U(1)$ symmetry twist $\boldsymbol{a}$ reflects the mixed $U(1) \times \mathbb{R}^d$ anomaly. Equations (101) and (54) are identical, implying the identical mixed anomaly, which is captured by the topological term $\int d^d\boldsymbol{x}\, [\bar{\rho}_p \dot{\phi}_p(\boldsymbol{x}, t)]$.

The mixed anomaly can also be measured by the periodicity $k_{0\mu}$ in the distribution of $\boldsymbol{k}_{\text{tot}}$, and the periodicity $\Delta N = 2$ in the distribution of N, for the low-energy excitations. The period in the μ direction times ΔN is

$$\Delta N k_{0\mu} = 2\frac{N_p}{L_\mu} = \frac{N}{L_\mu}. \quad (102)$$

The periodicities $k_{0\mu}$ and ΔN are universal low-energy properties of the fermion-pair gapless phase. Their product $\Delta N k_{0\mu}$ is even more robust since it is invariant even across any phase transitions, and thus corresponds to an anomaly.

For the gapless state formed by four-fermion bound states, the periodicities will be

$$k_{0\mu} = \frac{1}{4}\frac{N}{L_\mu}, \quad \Delta N = 4. \quad (103)$$

Their product is still $\Delta N k_{0\mu} = \frac{N}{L_\mu}$.

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APPENDIX: DYNAMICAL VARIATIONAL APPROACH AND LOW-ENERGY EFFECTIVE THEORY

1. Coherent state approach

A quantum state is described by a complex vector,

$$|\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \end{pmatrix},$$

in a Hilbert state, with inner product

$$\langle\phi|\psi\rangle = \sum_n \phi_n^* \psi_n. \quad (A1)$$

The motion of a quantum state is described by the time dependent vector $|\psi(t)\rangle$, which satisfies an equation of motion (called the Schrödinger equation) with only a first order time derivative,

$$i\frac{d}{dt}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle, \quad (A2)$$

where the Hermitian operator $\hat{H}$ is the Hamiltonian.

The Schrödinger equation (A2) also has a phase-space Lagrangian description. If we choose the Lagrangian L to be

$$L\left(\frac{d}{dt}|\psi\rangle, |\psi\rangle\right) = \langle\psi(t)|\, i\frac{d}{dt} - \hat{H}|\psi(t)\rangle$$
$$= i\psi_n^*(t)\dot{\psi}_n(t) - \psi_m^*(t) H_{mn} \psi_n(t), \quad (A3)$$

then the action $S = \int dt\, L(\frac{d}{dt}|\psi\rangle, |\psi\rangle)$ will be a functional for the paths $|\psi(t)\rangle$ in the Hilbert space. The stationary paths $|\psi_{\text{sta}}(t)\rangle$ of the action will correspond to the solutions of the Schrödinger equation. Since the Schrödinger equation can be derived from the Lagrangian, we can say that the Lagrangian $L(\frac{d}{dt}|\psi\rangle, |\psi\rangle)$ provides a complete description of a quantum system.

In the variational approach to the ground state, we consider a variational state $|\psi_{\xi^i}\rangle$ that depends on variational parameters ξ^i. We then found an approximation of the ground state $|\psi_{\bar{\xi}^i}\rangle$ by choosing $\bar{\xi}^i$ that minimize the average energy,

$$\bar{H}(\xi^i) = \langle\psi_{\xi^i}|\hat{H}|\psi_{\xi^i}\rangle. \quad (A4)$$

If we choose the variational parameters ξ^i properly, the low-energy excitations are also described by the fluctuations of variational parameters. In other words, the dynamics of the variational parameters ξ^i describe the low-energy excitations. This leads to a dynamical variational approach (or coherent state approach) that gives us a description of both ground state and low-energy excitations.

The dynamics of the full quantum system is described by the phase-space Lagrangian $L = \langle\psi|\, i\frac{d}{dt} - \hat{H}|\psi\rangle$. The dynamics of the variational parameters is described by the evolution of the quantum states in a submanifold of the total Hilbert space, given by the variational states $|\psi_{\xi^i}\rangle$. Here we want to obtain the dynamics of the quantum states, restricted to the submanifold parametrized by ξ^i. Such a dynamics is described by the same phase-space Lagrangian restricted in the submanifold,

$$L(\dot{\xi}^i, \xi^i) = \langle\psi_{\xi^i(t)}|\, i\frac{d}{dt} - \hat{H}|\psi_{\xi^i(t)}\rangle = a_i(\xi^i)\dot{\xi}^i - \bar{H}(\xi^i), \quad (A5)$$

where

$$a_i = i\langle\psi_{\xi^i}|\frac{\partial}{\partial\xi^i}|\psi_{\xi^i}\rangle, \quad \bar{H}(\xi^i) = \langle\psi_{\xi^i}|\hat{H}|\psi_{\xi^i}\rangle. \quad (A6)$$

The resulting equation of motion is given by

$$b_{ij}\dot{\xi}^j = \frac{\partial\bar{H}}{\partial\xi^i}, \quad b_{ij} = \partial_i a_j - \partial_j a_i, \quad (A7)$$

which describes the classical motion of ξ^i.

The above phase-space Lagrangian actually only describes the classical dynamics of the variables ξ^i. To obtain the low-energy effective theory for the quantum dynamics of the variables ξ^i, we need to quantize the phase-space Lagrangian (A5) to obtain the low-energy effective Hilbert space $\mathcal{H}_{\text{eff}}$ and the low-energy effective Hamiltonian H_{eff} acting with

$\mathcal{H}_{\text{eff}}$. Roughly, the low-energy effective Hilbert space $\mathcal{H}_{\text{eff}}$ is a representation of the operator's algebra,

$$i[\hat{\xi}^i, \hat{\xi}^j] = b^{ij}(\hat{\xi}^i), \tag{A8}$$

where b^{ij} is the inverse of b_{ij}: $b_{ij}b^{jk} = \delta_{ik}$. The low-energy effective Hamiltonian is given by

$$H_{\text{eff}} = \bar{H}(\hat{\xi}^i). \tag{A9}$$

Let us use the above approach to describe a hardcore boson on a single site. The total Hilbert space is two dimensional, spanned by $|0\rangle$ (no boson) and $|1\rangle$ (one boson). The coherent state is described by a unit vector $\boldsymbol{n} = (n_x, n_y, n_z) \in S^2$. We may also use θ, ϕ to describe $\boldsymbol{n}$:

$$\begin{aligned} n_x(\theta, \phi) &= \cos\phi \sin\theta, \\ n_x(\theta, \phi) &= \sin\phi \sin\theta, \\ n_z(\theta, \phi) &= \cos\theta. \end{aligned} \tag{A10}$$

We can choose the coherent state to be

$$|\boldsymbol{n}(\theta, \phi)\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{-i\phi}\sin\frac{\theta}{2} \end{pmatrix}. \tag{A11}$$

The phase-space Lagrangian to describe the classical dynamics of θ, ϕ is given by

$$\begin{aligned} L &= \sin^2\frac{\theta}{2}\dot{\phi} - \bar{H}(\theta, \phi) \\ &= \rho\dot{\phi} - \bar{H}(\rho, \phi), \quad \rho \equiv \frac{1 - n_z}{2} = \sin^2\frac{\theta}{2}. \end{aligned} \tag{A12}$$

We will use (ρ, ϕ) to parametrize the phase space, where $\rho \in [0, 1]$ has a physical meaning, being the average number of bosons on the site. Here we stress that (ρ, ϕ) parametrize S^2. Quantizing the above classical phase-space Lagrangian, we suppose to obtain a quantum system with Hilbert space $\mathcal{H} = \text{span}(|0\rangle, |1\rangle)$.

2. Low-energy effective theory of bosonic superfluid phase

Using the above result, we obtain the following low-energy effective theory for interacting bosons in a d-dimensional cubic lattice with periodic boundary condition, whose sites are labeled by $\boldsymbol{i}$:

$$L = \sum \rho_{\boldsymbol{i}}\dot{\phi}_{\boldsymbol{i}} - \bar{H}(\rho_{\boldsymbol{i}}, \phi_{\boldsymbol{i}}), \tag{A13}$$

where $\phi_{\boldsymbol{i}} \sim \phi_{\boldsymbol{i}} + 2\pi$ is an angular variable. If we assume $\rho_{\boldsymbol{i}}, \phi_{\boldsymbol{i}}$ to have a smooth dependence on the space coordinate $\boldsymbol{x} \sim \boldsymbol{i}$, the above can be rewritten as a field theory,

$$\begin{aligned} L = \int d^d\boldsymbol{x} \bigg[&\bar{\rho}\dot{\phi}(\boldsymbol{x}, t) + \delta\rho(\boldsymbol{x}, t)\dot{\phi}(\boldsymbol{x}, t) \\ &- \frac{\bar{\rho}}{2m}|\boldsymbol{\partial}\phi|^2 - \frac{g}{2}\delta\rho^2 + \cdots \bigg], \end{aligned} \tag{A14}$$

where we have assumed that $\bar{H}(\rho, \phi)$ is minimized as $\rho = \bar{\rho}$, which corresponds to the average number of bosons per site.

To quantize the above low-energy effective theory, we expand to

$$\begin{aligned} \delta\rho &= \rho_0 + \sum_{\boldsymbol{k}\neq 0} \rho_{\boldsymbol{k}} \frac{e^{i\boldsymbol{k}\cdot\boldsymbol{x}}}{L^{d/2}}, \\ \phi &= \phi_0 + 2\pi\frac{\boldsymbol{m}\cdot\boldsymbol{x}}{L} + \sum_{\boldsymbol{k}\neq 0} \phi_{\boldsymbol{k}} \frac{e^{i\boldsymbol{k}\cdot\boldsymbol{x}}}{L^{d/2}}, \end{aligned} \tag{A15}$$

where L is the size of the cubic lattice. The $\boldsymbol{k} \neq 0$ modes give rise to a collection of quantum oscillators after quantization. $\rho_0, \phi_0, \boldsymbol{m}$ describe the $\boldsymbol{k} = 0$ mode, where the integer vector $\boldsymbol{m} = (m_1, \dots, m_d)$ describes the winding numbers of the phase ϕ. The effective Lagrangian for the $\boldsymbol{k} = 0$ modes is given by

$$L_0 = \left[L^d\bar{\rho}\dot{\phi}_0 + L^d\rho_0\dot{\phi}_0 - \frac{\bar{\rho}L^{d-2}}{2m}(2\pi\boldsymbol{m})^2 - L^d\frac{g}{2}\rho_0^2 \right]. \tag{A16}$$

After quantization, the $\boldsymbol{k} = 0$ mode describes a particle on a ring with $L^d\bar{\rho}$ flux through the ring. Let $\hat{L}_z \sim L^d(\bar{\rho} + \rho_0)$ be the angular operator of the quantized particle. After quantization, the Hamiltonian is given by

$$\hat{H}_0 = \frac{\bar{\rho}L^{d-2}}{2m}(2\pi\boldsymbol{m})^2 + \frac{g}{2L^d}(\hat{L}_z - L^d\bar{\rho})^2. \tag{A17}$$

The many-body low-energy excitations are labeled by $(\boldsymbol{m}, N, n_{\boldsymbol{k}\neq 0})$, where integer N is the eigenvalues of $\hat{L}_z$ (the total number of bosons) and integer $n_{\boldsymbol{k}\neq 0}$ is the number of excited phonons for the $\boldsymbol{k}$ mode. The total energy and the total crystal momentum are given by

$$\begin{aligned} E &= \bar{N}\frac{(2\pi\boldsymbol{m})^2/L^2}{2m} + \frac{g}{2}\frac{(N - \bar{N})^2}{L^d} + \sum_{\boldsymbol{k}\neq 0}\left(n_{\boldsymbol{k}} + \frac{1}{2}\right)v|\boldsymbol{k}|^2, \\ \boldsymbol{k}_{\text{tot}} &= \int d^d\boldsymbol{x}\, (\bar{\rho} + \delta\rho)\boldsymbol{\partial}\phi = N\frac{2\pi\boldsymbol{m}}{L} + \sum_{\boldsymbol{k}\neq 0} n_{\boldsymbol{k}}\boldsymbol{k}, \end{aligned} \tag{A18}$$

where the phonon velocity $v = \sqrt{\frac{g\bar{\rho}}{m}}$, and $\bar{N} \equiv L^d\bar{\rho}_0$.

The above results are very standard, except that we carefully keep the topological term $\bar{\rho}(\boldsymbol{x}, t)\dot{\phi}(\boldsymbol{x}, t)$ in the Lagrangian. Our quantum system has $U(1)$ particle-number-conservation symmetry and $\mathbb{Z}^d$ lattice translation symmetry. In the continuum field theory (A14), we take the limit of zero lattice spacing. In this case, both the $U(1)$ and $\mathbb{Z}^d$ symmetries are internal symmetries of the field theory. It turns out that the $U(1) \times \mathbb{Z}^d$ symmetry has a mixed 't Hooft anomaly when $\bar{\rho}$ is not an integer, which constrains the low-energy dynamics of the interacting bosons.

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