Hidden wave function of twisted bilayer graphene: The flat band as a Landau level

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We study zero-energy states of the chirally symmetric continuum model (CS-CM) of twisted bilayer graphene. The zero-energy state obeys the Dirac equation on a torus in the external non-Abelian magnetic field. These zero-energy states could form a flat band—a band where the energy is constant across the Brillouin zone. We prove that the existence of the flat band implies that the wave function of any state from the flat band has a zero and vice versa. We found a hidden flat band of unphysical states in the CS-CM that has a pole instead of a zero. Our main result is that in the basis of the flat band and hidden wave functions the flat band could be interpreted as a Landau level in the external magnetic field. From that interpretation we show the existence of extra flat bands in the magnetic field.

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Introduction. Twisted bilayer graphene (TBG) has recently drawn a lot of attention from the physics community due to its interesting properties and applications [1–42]. One of the most prominent features is the recent discovery of correlated insulators and superconductivity, which are observed in a narrow range of twist angles near $\theta = 1.05^\circ$, which is usually referred to as the magic angle. At this angle the system develops a nearly flat band near charge neutrality. Recently, the flat band was explored analytically in a chiral model of TBG that neglects the hoppings within the same sublattices of different TBG sheets [43]. In this paper we continue the exploration of the mathematical structures of the flat band and demonstrate the connection with the vector bundles over the Riemann surfaces of higher genus that could provide some deeper understanding of the physics behind the chirally symmetric continuum model (CS-CM) of twisted bilayer graphene. From a physical standpoint this will allow us to study the behavior of flat bands in an external magnetic field.

TBG consists of two graphene sheets placed on top of each other at small angle $\theta \ll 1$ that form a long-period pattern (moiré pattern). One can estimate that the period of the resulting superlattice is of the order $L(\theta) \sim \frac{a}{\theta} \gg a$, where a is the graphene lattice constant. That allows us to consider a continuum model for the Hamiltonian instead of a lattice one. This approach was used by Bistritzer and MacDonald [44,45] and by Lopes dos Santos *et al.* [46]. Thus we can write an effective Hamiltonian for this model [43] as

$$H_0 = \begin{pmatrix} iv_0\vec{\sigma}_{\theta/2}\vec{\nabla} & T(r) \\ T^\dagger(r) & iv_0\vec{\sigma}_{-\theta/2}\vec{\nabla} \end{pmatrix}, \quad T(r) = \begin{pmatrix} t_{aa}(r) & t_{ab}(r) \\ t_{ba}(r) & t_{bb}(r) \end{pmatrix},$$

where we already used the fact that the superlattice is much bigger than the interatomic lattice of separate sheets of graphene. Therefore we can use the Dirac equation to describe excitations in each individual graphene sheet. The off-diagonal term T is responsible for hopping between sheets of the TBG and sublattices a and b of the individual graphene sheets. The H_0 acts on the four-dimensional wave function $\Psi = (\psi_{a1}, \psi_{b1}, \psi_{a2}, \psi_{b2})^T$, where the second index relates to the individual graphene sheets of the TBG and the first index relates to the sublattice of the given graphene sheet.

The numerical study of that model confirmed the existence of the flat band at magic angle $\theta_1^* \approx 1.05^\circ$. The modification studied here neglects the coupling between the sublattices of the graphene $t_{aa} = t_{bb} = 0$. In this case, the system acquires an additional chiral symmetry and is usually referred to as a *chirally symmetric continuum model* (CS-CM). After an appropriate change of basis the Hamiltonian of the CS-CM can be cast in the following form:

$$H = UH_0U^{-1} = \begin{pmatrix} 0 & \mathcal{D} \\ \mathcal{D}^* & 0, \end{pmatrix}, \quad \text{where}$$
$$\mathcal{D} = \begin{pmatrix} 2i\bar{\partial} + W(\vec{r}) & V(\vec{r}) \\ U(\vec{r}) & 2i\bar{\partial} - W(\vec{r}) \end{pmatrix}, \tag{1}$$

where $\vec{r}$ is the vector in the two-dimensional (2D) graphene sheet, $\bar{\partial} = \frac{1}{2}(\partial_x - i\partial_y)$ is the antiholomorphic derivative along the sheet, and $V(r), U(r)$ are the hopping potentials between the two sheets of the TBG that could be expressed linearly through $t_{ab}(r)$ and $t_{ba}(r)$. This Hamiltonian acts on the rotated wave functions $\Psi_U = U\Psi = (\phi, \psi)^T = (\phi_1, \phi_2, \psi_1, \psi_2)^T$.

The spectrum of the model is governed by the following eigenvalue problem:

$$H\Psi_U = E\Psi_U \Leftrightarrow \begin{cases} \mathcal{D}\psi = E\phi \\ \mathcal{D}^*\phi = E\psi. \end{cases} \tag{2}$$

Since we are interested in the existence of a flat band near charge neutrality, these equations simplify, and we should

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study only the following equation:

$$-\frac{i}{2}\mathcal{D}\psi = (\bar{\partial} + \bar{A})\psi = 0, \quad \mathcal{D}^*\phi = 0,$$
$$\text{where } \bar{A} = -\frac{i}{2}\begin{pmatrix} -W(\vec{r}) & V(\vec{r}) \\ U(\vec{r}) & W(\vec{r}) \end{pmatrix}. \tag{3}$$

Due to the enhanced chiral symmetry the equations on ψ and ϕ are decoupled, which allows for a deeper analytical investigation of the properties of the CS-CM. These equations could be interpreted as a Dirac equation in a non-Abelian magnetic field $\bar{A} \in \mathbf{su}(2)$ on a Riemann surface [47,48]. We will use this interpretation to bolster our intuition and draw interesting conclusions. Since we study a periodic system, we should impose Bloch boundary [49] conditions

$$\psi_{\vec{k}}(\vec{r} + \vec{a}_{1,2}) = e^{i\vec{k}\vec{a}_{1,2}}\psi_{\vec{k}}(\vec{r}), \tag{4}$$

where $\vec{a}_{1,2}$ are the periods of the moiré superlattice and the vector $\vec{k}$ defines the location in the moiré Brillouin zone (MBZ). If the solution exists for any point k in the MBZ, then the system has a *flat band*. One can show that such potentials exist [43]. However, for a general chosen potential $\bar{A}$, the system of Eqs. (3) and (4) has a smooth finite solution *only* at finite numbers of points k in the MBZ.

The purpose of this paper is to consider a *generic potential* $\bar{A}$ in Eq. (3) and get general properties independent of the concrete form of $\bar{A}$ that could shed light on the physics behind the CS-CM. Our main result is that once a system possesses a flat band we can separate TBG into a system of two individual sheets with positive and negative effective magnetic fields, which support Landau levels of different chirality [50]. The negative magnetic field could be canceled by an external magnetic field, resulting in additional flat bands. Therefore the number of flat bands increases in the presence of the magnetic field. *This is the main physical result of this paper.*

The paper is organized as follows. First, we will build an integral of motion of Eq. (3), which, as was shown in Ref. [43], is related to the Fermi velocity. Hence we will refer to it as a Fermi integral of motion I_F. We prove that the flat band appears if and only if this invariant is equal to zero, $I_F = 0$. Then we demonstrate that the system of Eqs. (3) and (4) admits an additional solution, which is singular and therefore unphysical. This second solution will allow us to rewrite the system of equations in the form of two Dirac equations on a torus with effective magnetic fields. This shows the direct connection of the flat band to the Landau levels. Finally, we will show that introducing an external magnetic field can make the second solution nonsingular. Hence this leads to additional flat bands which, in principle, could be seen in experiment. The mathematical details are delegated to Appendixes A and B.

Fermi integral, zeros of wave functions, and the flat band. For simplicity, let us consider Eq. (3) alone without taking into account boundary conditions (4):

$$\mathcal{D}\psi = (\bar{\partial} + \bar{A})\psi = 0, \quad \psi = (\psi_1, \psi_2)^T \in \mathbb{C}^2, \quad \operatorname{tr}\bar{A} = 0. \tag{5}$$

We start by studying the properties of the vector-valued function ψ that satisfy the equation $\mathcal{D}\psi = 0$. Such equations have been broadly studied in some fields of mathematics. Hence, to simplify further computation and exploit the results, we adopt some mathematical terminology. We assume that our TBG is separated into geometric domains U_α such that when we jump from one domain to another we should appropriately change the vector-valued function ψ:

$$\begin{pmatrix} \psi_{1\alpha} \\ \psi_{2\alpha} \end{pmatrix} = g_{\alpha\beta}\begin{pmatrix} \psi_{1\beta} \\ \psi_{2\beta} \end{pmatrix}.$$

The collection $\psi = \{\psi_\alpha\}$ is said to be a section of a vector bundle E, which is a collection of domains $\{U_\alpha\}$ with translation functions $g_{\alpha\beta}$. If there is only one domain, the bundle is said to be trivial. An example of a nontrivial vector bundle is provided by separating the TBG into a set of fundamental domains by acting with translations $\vec{a}_{1,2}$. Translation functions in this case are boundary conditions (4).

From the mathematical point of view, the holomorphic equation $\bar{\partial}\psi = 0$ is similar to Eq. (5): $\mathcal{D}\psi = 0$. Then mathematicians say that if $\mathcal{D}\psi = 0$, then the wave function ψ is a *meromorphic* function and $\bar{A}$ is a holomorphic connection. If ψ is also finite everywhere, we would call such a function *holomorphic*. The convenience of such terminology is that such ψ share a lot of properties with usual holomorphic functions studied in the complex analysis.

From the physical point of view, any wave function must be finite. So we must assume that ψ is also *a holomorphic* function in the above sense of vector bundle [51] E.

Let us consider two finite solutions ψ_1, ψ_2 of Eq. (5). One can compute the Wronskian of these solutions

$$I_F(\psi_1, \psi_2) = \det(\psi_1, \psi_2) = I_F(r), \quad \text{then}$$
$$\bar{\partial} I_F(r) = -\operatorname{tr}\bar{A}\cdot I_F(r), \quad \operatorname{tr}\bar{A} = 0 \Rightarrow I_F(r) = I_F(z), \tag{6}$$

where we have used the fact that $\bar{A} \in \mathbf{su}(2)$ and hence is traceless. We come to the conclusion that the Wronskian $I_F(z)$ must be an analytic function. If $\psi_{1,2}$ are finite everywhere, $I_F(z)$ is holomorphic and therefore must be constant (because of the Liouville theorem) across the plane of TBG. Because of this property we can consider I_F as an integral of motion of Eq. (5). This property could be generalized to other systems and will provide a necessary and sufficient condition for the existence of a flat band in the system.

From flat band to zero Wronskian $I_F = 0$. Here, we prove that we cannot have a flat band unless $I_F = 0$. Therefore, applying negation, a flat-band wave function has a zero, and hence $I_F = 0$. We will say in a minute which two solutions we need to pick up. For the application to TBG we should study Eq. (3) on a torus as was explained in the previous section. Namely, we can consider TBG as a torus $\mathbb{C}/\Lambda$, where $\Lambda = ma_1 + na_2$, $m, n \in \mathbb{Z}$, $a_{1,2} = a^x_{1,2} + ia^y_{1,2}$. We must impose boundary conditions (4) to glue the wave function as we shift along lattice Λ. These boundary conditions are the gluing functions of the vector bundle over the torus. Without loss of generality we will set $a_1 = 1$ and $a_2 = \tau$.

We define $\mathbb{C}^2_K$ to be the vector bundle with boundary conditions (gluing functions) (4) with quasimomentum K. Again, Eq. (5) with connection $\bar{A}$ defines a meromorphic section of this vector bundle.

For the sake of argument, we assume that there are at least two points K_1, K_2 in the MBZ where the solution exists. We would like to stress that $K_{1,2}$ are different from special points K, K' usually considered in the study of TBG, where due to discrete symmetry C_3 the band must have zero energy $E = 0$. Of course, in the TBG the potential must respect the C_3 symmetry, and therefore the Dirac points must exist at points K, K'. However, in the general twisted bilayer material, such symmetry could be absent, and to make the discussion more general, we make an assumption that there are at least two points $K_{1,2}$ where the gap closes. Therefore, to keep the discussion general, we just assume that due to some lucky choice of $\bar{A}$ in Eq. (3) a system has zero energy at some points $K_{1,2}$ of the MBZ.

Relation (6) still holds true as it is not sensitive to boundary conditions. If we have two holomorphic solutions $\psi_{K_{1,2}}$ at two different points of the Brillouin zone $K_{1,2}$, we can compute the Wronskian

$$I_{F,K_1+K_2}(z) = I_F\left(\psi_{K_1}, \psi_{K_2}\right) = \det\left(\psi_{K_1}, \psi_{K_2}\right), \quad (7)$$

but because the $I_F(z)$ is holomorphic and bounded in the complex plane of TBG [due to periodicity conditions (4) and the fact that ψ_{K_1}, ψ_{K_2} are finite], we must conclude that $I_F(z) = \text{const}$. Moreover, using boundary conditions (4), we have

$$I_{F,K_1+K_2}(z + a_{1,2}) = I_{F,K_1+K_2}(z)e^{i(K_1+K_2)a_{1,2}}. \quad (8)$$

However, if $K_1 + K_2 \neq 0$ and $I_F(z)$ is constant, the boundary conditions are satisfied only if $I_F(z) = 0$. Hence there are only *two possibilities*:

(1) $I_F(z) = 0$, and K_1, K_2 are arbitrary.

(2) $I_F(z) \neq 0$, but $K_1 = -K_2$.

We start with the second possibility. We normalize the solutions such that $I_F = 1$. Then we immediately get that ψ_{K_1}, ψ_{-K_1} are nowhere zero, because otherwise the Wronskian would be equal to zero at points where $\psi_{\pm K_1} = \vec{0}$. Since $I_F(z)$ is nonzero, the solutions ψ_{K_1} and ψ_{-K_1} are linearly independent at each point of the TBG. If we consider now matrix $M = (\psi_{K_1}, \psi_{-K_1})$, it satisfies the following equation:

$$(\bar{\partial} + \bar{A})M = 0, \quad \bar{A} = -\bar{\partial}M \cdot M^{-1}, \quad (9)$$

where we used the fact that if $\det M \neq 0$, the matrix M is invertible. We would like to point out that Eq. (9) does not mean $\bar{A}$ is a pure gauge (and hence a flat connection), since $M \in SL(2, \mathbb{C})$ rather than $SU(2)$ and therefore is not a genuine gauge transformation.

Let us consider another solution ψ of Eq. (5). Since $\psi_{\pm K_1}$ are linearly independent, we can always represent ψ as a linear combination of these solutions:

$$\psi = v_1(r)\psi_{K_1} + v_2(r)\psi_{-K_1}.$$

Applying the operator $D = \bar{\partial} + \bar{A}$, we get

$$D\psi = \bar{\partial}v_1\, \psi_{K_1} + \bar{\partial}v_2\, \psi_{-K_1} = 0. \quad (10)$$

Since $\psi_{\pm K_1}$ are linearly independent at each point of the torus $\mathbb{C}/\Lambda$, it follows that the coefficients v_i must be holomorphic, $\bar{\partial}v_i = 0$. The functions ψ and $\psi_{\pm K_1}$ are finite and nonzero everywhere; hence v_i are bounded. From the maximum principle for analytic functions on a complex plane, v_i are constant. Therefore, if we have an arbitrary solution of Eq. (5) at point k of the Brillouin zone, we must have

$$\psi_k = v_{k,K_1}\psi_{K_1} + v_{k,-K_1}\psi_{-K_1}, \quad v_{k,\pm K_1} \in \mathbb{C}, \quad (11)$$

but it is easy to see that with any choice of numbers $v_{\pm K_1}$ we are not able to satisfy boundary conditions (4) in the MBZ. Therefore we cannot have a flat band if $I_F \neq 0$.

From zero Wronskian to a flat band. Let us prove the converse. Namely, if $I_F = 0$ for some points K_1, K_2 in the MBZ, the system develops a flat band. In other words, Eq. (5) has a solution at any point k in the MBZ.

We start by noticing that since $I_F(z) = 0$ and ψ_{K_1,K_2} satisfy Eq. (5), then wave function ψ_{K_1} has a zero. Let us prove this statement by contradiction. Assume the opposite: that $\psi_{K_1}(r) \neq 0$ at any point of the torus, $\mathbb{C}/\Lambda$, or fundamental domain of TBG. Because the torus is compact, the minimum $\min_{r\in\mathbb{C}/\Lambda} |\psi_{K_1}(r)| = m > 0$ is reachable. If $I_F(z) = 0$, the wave functions ψ_{K_1,K_2} are proportional to each other:

$$\psi_{K_2}(r) = \gamma(r)\psi_{K_1}(r), \quad D\psi_{K_2} = \bar{\partial}\gamma(r)\psi_{K_1}(r) = 0,$$

where $\gamma(r)$ is bounded as $|\gamma(r)| < \frac{|\psi_{K_2}(r)|}{m}$ and holomorphic, $\bar{\partial}\gamma(r) = 0$. Then the function $\gamma(r) = \gamma(z)$ must be constant by the maximum principle. However, this is impossible since ψ_{K_1,K_2} satisfy different boundary conditions. Hence we must conclude that ψ_K has at least one simple zero [52].

We can now follow the procedure described in Ref. [43] and construct a solution at any point k of the Brillouin zone. The specific potential studied in Ref. [43] had an extra property: $I_F \propto v_F$, and hence such solution implied the existence of a flat band. Our reasoning managed to generalize this condition to an arbitrary potential $\bar{A}$.

Hidden wave function. We can draw some additional conclusions from the existence of the holomorphic section with a zero at any point K of the Brillouin zone. For the sake of argument we will assume that ψ_K has one simple zero, but this could be easily generalized to the case of multiple zeros.

Let us notice that at general K there can be only one holomorphic section ψ_K. Indeed, if there are two holomorphic linearly independent sections $\psi_K^{1,2}$, their Wronskian must be a holomorphic nonzero double-periodic function with specific boundary conditions. However, this implies that $2K = 0 \mod \Lambda$ and the Wronskian is constant. We come to the conclusion that if the other solution exists, it must be meromorphic.

Let us spell out the motivation as to why a singular wave function satisfying Eqs. (3) and (4) actually exists. The holomorphic section ψ_K (flat-band wave function) of the bundle $\mathbb{C}^2_K$ forms a sub-bundle, which we denote as γ. Then one can consider an exact short sequence:

$$0 \to \gamma \to \mathbb{C}^2_K \to (\gamma)^\perp \to 0, \quad (12)$$

where $(\gamma)^\perp = \mathbb{C}^2_K/\gamma$. Roughly speaking, we split the two-dimensional Hilbert space into two one-dimensional ones. The first one is defined to be along the flat-band function ψ_K at each point of the TBG. The second one is chosen to be alongside any other linearly independent wave function. For the sake of argument, one can think of the orthogonal wave function $\psi_\perp$.

The bundles γ and $(\gamma)^{\perp}$ or wave functions ψ and $\psi_{\perp}$ are one-dimensional wave functions on periodic TBG and therefore could be assigned (first) Chern numbers c_1. These numbers could be computed in a fashion similar to the case of the usual Chern numbers in the topological insulators, but where the computations are performed in real space rather than in a momentum space. From the mathematical point of view [53] the Chern number c_1 is just the number of zeros minus the number of poles. Since the $\mathbb{C}_K$ bundle is in some sense trivial, the Chern numbers for the sub-bundles γ and $(\gamma)^{\perp}$ must satisfy the following relation:

$$c_1((\gamma)^{\perp}) = -c_1(\gamma), \tag{13}$$

suggesting that if γ has a holomorphic section, $(\gamma)^{\perp}$ has a section, but instead of a zero it has a pole. This reasoning is not enough for the proof of its existence, because (12) may not split. In other words, the wave function $\psi_{\perp}$ is ill defined: Going around a torus cycle will not only produce a phase but also add a multiple of ψ. From the physical point of view it means that we cannot simply represent this 2D system as a stack of two topological materials with opposite Chern numbers.

Luckily, the theory of vector bundles over the Riemann surfaces was actively studied by Donaldson [54]. One can show that the short sequence (12) is split over the torus. Below we present a physical construction that can be used to find the solution explicitly. In Appendix B we prove the existence using algebro-geometric methods.

Let us start from a holomorphic section ψ_K that is a solution of Eq. (5), satisfies boundary conditions (4), and has a zero at some point z_0. Let us assume for a moment that we somehow managed to find another solution ϕ_K that is linearly independent of ψ_K. If such a solution exists, the Wronskian $\widetilde{I}(\psi_K, \phi_K)$ should be a meromorphic function

$$\det(\psi_K, \phi_K) = \widetilde{I}(z) \tag{14}$$

that satisfies double-periodic boundary conditions $\widetilde{I}(z + a_{1,2}) = e^{2i\vec{K}\vec{a}_{1,2}}\widetilde{I}(z)$ [see Eq. (4)]. Unlike the previous section, ϕ_K might have poles, so we cannot conclude that $\widetilde{I}$ is constant. However, an analytic function with these properties exists and is unique up to a normalization factor [53]. Namely, this function is represented as

$$\widetilde{I}(z) = e^{2i\vec{K}\vec{a}_1 z}\frac{\vartheta(z - z_0;\tau)}{\vartheta(z - z_\infty;\tau)}, \quad z_0 - z_\infty = \vec{K}\frac{\vec{a}_2 - \tau\vec{a}_1}{\pi},$$

where $\vartheta(z;\tau)$ is a Jacobi theta function. Since ψ_K is finite everywhere, ϕ_K must have a pole at the point $z = z_\infty$. From this we get a simple linear equation that ϕ_K should satisfy

$$\phi_K^1\psi_K^2 - \phi_K^2\psi_K^1 = \widetilde{I}(z). \tag{15}$$

Having determined $\widetilde{I}$, let us now construct ϕ_K. At any point $z \in \mathbb{C}/\Lambda$ this equation has at least one solution. Since at point $z = z_0$ both sides of Eq. (15) have a simple zero, we can analytically continue the solution at this point. Let us pick an arbitrary solution to Eq. (15) and denote it as $\zeta_K(r)$. Any other solution of Eq. (15) is

$$\zeta_K^{\lambda}(r) = \zeta_K(r) + \lambda(r)\psi_K, \tag{16}$$

where $\lambda(r)$ is an arbitrary function.

We can derive a relation for the function $\zeta_K(r)$. Namely, we apply an operator $\bar{\partial}$ to the Wronskian to get

$$\begin{aligned}\bar{\partial}\widetilde{I}(z) &= \bar{\partial}\det(\psi_K, \zeta_K)\\ &= \det(\mathcal{D}\psi_K, \zeta_K) + \det(\psi_K, \mathcal{D}\zeta_K)\\ &= \det(\psi_K, \mathcal{D}\zeta_K) = 0.\end{aligned} \tag{17}$$

This means that in general $\mathcal{D}\zeta_K$ is proportional to the wave function ψ_K

$$\mathcal{D}\zeta_K = \eta(r)\psi_K, \tag{18}$$

for some function $\eta(r)$ which may have singularities. To clarify what we have done, the solution ζ_K is just an arbitrary solution to Eq. (15) and does not satisfy Eq. (5). In Eq. (15) we can arbitrarily choose ζ_K^1. It could have some singularities. To avoid this problem, we set $\zeta_K^1 = 1$ on the torus. Then the singularities of ζ_K^2 come only from the function $\widetilde{I}(z)$.

As we discussed before the function ζ_K is not unique, so we can consider ζ_K^{λ} from Eq. (16). This freedom allows us to set the right-hand side of Eq. (18) to zero. Indeed,

$$\begin{aligned}\zeta_K^{\lambda} &= \zeta_K + \lambda(r)\psi_K, \quad \mathcal{D}\zeta_K^{\lambda}(r) = \mathcal{D}\zeta_K(r) + \bar{\partial}\lambda\psi_K(r)\\ &= [\eta(r) + \bar{\partial}\lambda(r)]\psi_K.\end{aligned} \tag{19}$$

Therefore we just need to solve the following equation on a torus:

$$\bar{\partial}\lambda = -\eta + C\delta^{(2)}(z - z_0), \tag{20}$$

with periodic boundary conditions $\lambda(r + a_{1,2}) = \lambda(r)$. The term proportional to the δ function is allowed since ψ_K has a zero at point $z = z_0$. To solve (20), we make a two-dimensional Fourier transform over the torus

$$\lambda(k) = \int d^2\vec{r}\lambda(\vec{r})e^{ik_x x + ik_y y}. \tag{21}$$

Then Eq. (20) could be cast as

$$\bar{k}\lambda(k) = -\eta(k) + Ce^{ikz_0}, \quad k = k_x + ik_y. \tag{22}$$

This equation has a solution for any k if we tune $C = \eta(0)$. This way the right-hand side is zero at $k = 0$, so dividing by $\bar{k}$ we get

$$\lambda(k) = -\frac{k\eta(k)}{|k|^2}, \quad \lambda(0) = 0. \tag{23}$$

Therefore we managed to find a second solution to Eq. (5) with boundary conditions (4) that is singular but linearly independent of the holomorphic solution.

One can check that $\lambda(r)\psi_K$ is finite everywhere and therefore the pole of ζ_K could not be removed. We have checked numerically that if one follows the above procedure the resulting wave function has a simple pole and satisfies the system of Eqs. (5) and (4).

Hidden Landau levels. In this section we use units such that the fundamental magnetic flux $\Phi_0 = \frac{h}{e} = 1$. We have two solutions at the Brillouin point K: ψ_K^0 with a zero at a point z_0 and ψ_K^{∞} with a pole at a point z_∞. We wish to change the basis to these functions because the original operator $\mathcal{D}$ in (3) would look very simple in this basis. Unfortunately, we cannot do this with original ψ_K^{∞}, ψ_K^0 because they have a pole and a zero.

We can introduce finite everywhere wave functions $\hat{\psi}^{\infty}$, $\hat{\psi}^{0}$

$$\hat{\psi}^{\infty} = e^{i\vec{K}\vec{a}_1 z - \frac{1}{2}B_1 z\bar{z}}\vartheta(z - z_{\infty};\tau)\psi_K^{\infty},$$
$$\hat{\psi}^{0} = \frac{e^{i\vec{K}\vec{a}_1 z + \frac{1}{2}B_1 z\bar{z}}}{\vartheta(z - z_0;\tau)}\psi_K^{0}, \tag{24}$$

where B_1 is a constant magnetic field corresponding to flux 1 in the moiré lattice. The Jacobi theta function cancels the corresponding zero and pole.

One can introduce the matrix S which changes the basis:

$$S = (\hat{\psi}^{0}, \hat{\psi}^{\infty}),$$
$$\det S = \det(\hat{\psi}^{0}, \hat{\psi}^{\infty}) = 1. \tag{25}$$

Then since $\det S = 1$, we can invert this matrix at each point of the lattice $\mathbb{C}/\Lambda$. This matrix allows us to rewrite the Dirac operator as

$$\hat{\mathcal{D}} = S^{-1}\mathcal{D}S = \begin{pmatrix} \bar{\partial} - \frac{1}{2}B_1 z & 0 \\ 0 & \bar{\partial} + \frac{1}{2}B_1 z \end{pmatrix}. \tag{26}$$

With the use of transformation S we managed to remove the potential $\bar{A}$ from the original Dirac operator $\mathcal{D}$ defined in Eq. (3) but at the cost of introducing two effective magnetic fields. We would like to point out that the same consideration could be repeated for a holomorphic part of the Hamiltonian $\mathcal{D}^*$ with the same type of arguments and results.

This shows that in this basis we have just effectively split TBG into two sheets with effective magnetic field B_1. The magnitude of this field is the same in both sheets but differs in sign. The form of the equations is exactly the same as for the Landau level problem on a torus (see Appendix A and Ref. [50] for a detailed discussion). Since the matrix S is nonsingular, the physical solutions for this auxiliary problem must be finite too. In one layer the effective magnetic field supports a wave function with a zero, while in the other the solution has a pole and is therefore unphysical. An analogous conclusion was derived from the different arguments in Ref. [55], but in our case we managed to show that our system does split into a sum of two systems with nonzero Chern numbers for a generic potential.

What is the advantage of representation (26)? The key feature of this representation is that it allows us to easily study the system in an external magnetic field. Such an external field corresponds to adding an identity matrix to the antiholomorphic connection in Eq. (5). It will not be sensitive to the transformation in Eq. (26). Therefore the equation for the zero mode has the following form:

$$\hat{D}_B f = \begin{pmatrix} \bar{\partial} - \frac{1}{2}B_1 z + \bar{A}_{U(1)} & 0 \\ 0 & \bar{\partial} + \frac{1}{2}B_1 z + \bar{A}_{U(1)} \end{pmatrix} f = 0,$$
$$f = (f_-, f_+), \tag{27}$$

where $\bar{A}_{U(1)}$ is a gauge potential for the external magnetic field that creates a magnetic field in the direction perpendicular to the plane of the TBG. Again, physical solutions of these equations are the ones with no singularities. We see that we got a simple Landau problem again! We dedicate Appendix A to a detailed description of this well-known problem.

The most important consequence of this is the emergence of extra flat bands. For simplicity we assume that $\bar{A}_{U(1)}$ has a flux $\Phi_{\text{ext}} = \int d^2x[\partial\bar{A} - \bar{\partial}A] \in \mathbb{Z}$ through the moiré superlattice. Equation (27) shows us that the system decouples into two noninteracting layers with fluxes $\Phi_{\text{tot}} \equiv \Phi = \Phi_{\text{ext}} \pm 1$. The result of Appendix A is that for $\Phi_{\text{ext}} \geqslant 1$ there are no flat bands, for $\Phi_{\text{ext}} = 0$ there is exactly one, and for $\Phi_{\text{ext}} \leqslant -1$ there are $2|\Phi_{\text{ext}}|$ flat bands.

Note that we have studied only the antiholomorphic part of the Hamiltonian (1). The holomorphic part (the other chirality) exhibits the same properties but for $\Phi_{\text{ext}} \to -\Phi_{\text{ext}}$. This means that in total there are $2|\Phi_{\text{ext}}|$ flat bands for $|\Phi_{\text{ext}}| > 0$ and for $\Phi_{\text{ext}} = 0$ there are only two flat bands. We would like to point out that if we did not take into account the hidden wave function, we would expect to have $|\Phi_{\text{ext}}| + 2$ flat bands in the presence of the external magnetic field.

Physical consequences and conclusion. Let us summarize our key findings. We started from the CS-CM-type Hamiltonian (1) with a *generic* $\bar{A}$ and assumed that it has a flat band. We proceeded by deriving an extra (nonphysical) zero-mode ψ_K^{∞} of the Hamiltonian (1). This solution let us define the transformation S [Eq. (25)] and represent the original Dirac operator in the form (26). This new form is very simple, and it allowed us to explicitly demonstrate the emergence of extra flat bands in the presence of an external transverse magnetic field.

TBG is believed to be approximately described by the CS-CM [44,45]. As was shown in Ref. [43], such a system possesses a flat-band solution at $\theta = 1.05^\circ$. Since we derived some general properties of the solutions of the CS-CM, we can argue that our results are applicable to TBG. At the first magic angle $\theta = 1.05^\circ$, unity fundamental flux through a moiré lattice corresponds to a magnetic field of about 13 T. Such magnetic fields are accessible; hence our prediction of extra flat bands can be, in principle, verified in experiment.

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APPENDIX A: LANDAU LEVELS ON A TORUS

In this Appendix we briefly review the wave functions on a torus; we will mostly follow Haldane and Rezayi [50]. We consider a complex torus $\mathbb{C}/\Lambda$, $\Lambda = \{n + m\tau; n, m \in \mathbb{Z}\}$ and want to find solutions to the following equation:

$$\hat{D}_B f = \left(\bar{\partial} + \tfrac{1}{2}eBz\right)f = 0, \quad F = \bar{\partial}A - \partial\bar{A} = B, \tag{A1}$$

which is either of the two equations in (27). To establish the boundary conditions, we consider a shift of z by a lattice vector $a_i = 1, \tau$ to get

$$\left(\bar{\partial} + \tfrac{1}{2}eBz + \tfrac{1}{2}eBa_i\right)f = 0. \tag{A2}$$

To remove the change in the gauge potential, we should make a gauge transformation

$$f \to f e^{-\frac{1}{2}eBa_i\bar{z}+\frac{1}{2}eB\bar{a}_i z}; \quad \text{(A3)}$$

this can be used to define the boundary conditions. Namely,

$$\begin{aligned} T_1 &: f(z+1) = \psi(z)e^{-\frac{1}{2}eB\bar{z}+\frac{1}{2}eBz}, \\ T_\tau &: f(z+\tau) = f(z)e^{-\frac{1}{2}eB\tau\bar{z}+\frac{1}{2}eB\bar{\tau}z}. \end{aligned} \quad \text{(A4)}$$

We should check the consistency of these boundary conditions, that $T_1T_\tau = T_\tau T_1$. One can check that

$$T_1T_\tau f(z) = e^{eB(\tau-\bar{\tau})}T_\tau T_1 f(z). \quad \text{(A5)}$$

The difference between these phases is $e^{eB(\tau-\bar{\tau})} = 1$. That gives a condition for the consistent boundary conditions (A4)

$$\pi\Phi = eB\,\mathrm{Im}\,\tau, \quad eB = \frac{\pi\Phi}{\mathrm{Im}\,\tau}, \quad \Phi \in \mathbb{Z}. \quad \text{(A6)}$$

These conditions give that the integral over a fundamental period is equal to $\Phi = \frac{1}{2\pi}\int F d^2z = \frac{eB}{\pi}\,\mathrm{Im}\,\tau$.

Then we can consider a general k from the Brillouin zone to get

$$\begin{aligned} &\left(\bar{\partial} + \tfrac{1}{2}eBz\right)f_k(z) = 0, \\ &f_k(z+a_i) = f_k(z)e^{-\frac{1}{2}eBa_i\bar{z}+\frac{1}{2}eB\bar{a}_i z + i(k,a_i)}, \end{aligned} \quad \text{(A7)}$$

where $(k,z) = k_x x + k_y y = \mathrm{Im}\,k\bar{z}$, $k = k_x + ik_y$, $z = x + iy$.

For $\Phi = -1$, Eq. (A7) is easy to solve; we get

$$f_{k,-1}(z) = \vartheta\left(z + \frac{ik}{2eB};\tau\right)e^{\frac{1}{2}eB(z+\frac{ik}{2eB})^2 + i\frac{1}{2}\bar{k}z - \frac{1}{2}eBz\bar{z}}.$$

The zero of this function is located at

$$z_0 = \frac{1}{2} + \frac{1}{2}\tau - \frac{ik}{2eB}, \quad \text{(A8)}$$

whereas for $\Phi = +1$ we get a solution with a pole:

$$f_{k,+1}(z) = \frac{1}{\vartheta\left(z + \frac{ik}{2eB};\tau\right)}e^{\frac{1}{2}eB(z+\frac{ik}{2eB})^2 + i\frac{1}{2}\bar{k}z - \frac{1}{2}eBz\bar{z}}.$$

If $\Phi \neq -1$, the solution is just

$$f_{K,\Phi}(z) = \prod_{k_1+\ldots+k_\Phi = K} f_{k_i,-1}(z). \quad \text{(A9)}$$

It seems that there are now an infinite number of wave functions at given K. One can show that there are only a finite number of linearly independent solutions (A9).

To compute this dimension, we can use the Riemann-Roch formula [53] for the operator $\hat{D}_B$ in Eq. (A1). This gives that $\dim\ker\hat{D}_B = \Phi$, so when Φ is negative we do not have any finite solutions for the Landau levels at any point of the Brillouin zone. The case $\Phi = 0$ is special—there are some zero modes but only at special points of the Brillouin zone.

APPENDIX B: THE SPLITTING OF A SHORT EXACT SEQUENCE AND ČECH COHOMOLOGY

In this Appendix we would like to clarify the existence of the second solution from the cohomology point of view. Although this approach is a bit involved, it is mathematically rigorous and could be generalized to higher-genus Riemann surfaces [56].

We start with the rigorous formulation of the problem. Assume that we have a Riemann surface $\mathcal{M}$ with some holomorphic vector bundle π of rank 2, $\pi : E \to \mathcal{M}$ and some connection $\bar{A}$. Namely, we have a covering of the Riemann surface with open subsets $\{U_\alpha\}$ where the vector bundle could be trivialized

$$\mathcal{M} = \bigcup_\alpha U_\alpha, \quad E|_{U_\alpha} \approx U_\alpha \times \mathbb{C}^2. \quad \text{(B1)}$$

When we move from one covering U_α to another U_β, we need to glue the section with holomorphic gluing functions $g^0_{\alpha\beta}$, $\bar{\partial}g^0_{\alpha\beta} = 0$. The connection $\bar{A}_\beta$ transforms as

$$\bar{A}_\beta = \hat{g}^{-1}_{\alpha\beta}\bar{A}_\alpha\hat{g}_{\alpha\beta} + \hat{g}^{-1}_{\alpha\beta}\bar{\partial}\hat{g}_{\alpha\beta}. \quad \text{(B2)}$$

We want to find holomorphic sections of these vector bundles—a collection of functions $\{\psi_\alpha\}$, such that the following conditions are satisfied:

$$\psi^0_\alpha = g^0_{\alpha\beta}\psi^0_\beta, \quad (\bar{\partial} + \bar{A}_\alpha)\psi^0_\alpha = 0. \quad \text{(B3)}$$

We can get rid of the connection $\bar{A}_\alpha$ by solving Eq. (B3) at each covering and performing the gauge transformation.

Then we can generally study the following problem:

$$\psi_\alpha = g_{\alpha\beta}\psi_\beta, \quad \bar{\partial}\psi_\alpha = 0. \quad \text{(B4)}$$

So we just need to find *meromorphic* sections of the vector bundle E defined by cocycles $g_{\beta\alpha}$ in the assumption that we have *holomorphic* sections of the bundle (B4) ψ^h. Namely, we have a collection of holomorphic functions $\psi^h_\alpha(z)$ that are defined at each of the coverings and satisfy boundary conditions

$$\psi^h_\alpha = g_{\alpha\beta}\psi^h_\beta. \quad \text{(B5)}$$

We want these functions ψ^h_α to be nonzero at any coverings of $\mathcal{M}$. Whenever we encounter a zero in some covering U_α, $\psi^h_\alpha(z^\alpha_0) = 0$, we redefine holomorphic section and gluing functions $\hat{\psi}^h_\alpha = \frac{1}{z - z^\alpha_0}\psi^h_\alpha$ and $\hat{g}_{\alpha\beta} = \gamma^1_{\alpha\beta}g_{\alpha\beta}$, $\gamma^1_{\alpha\beta} = \frac{z - z^\alpha_0}{z - z^\beta_0}$. This new function is nowhere zero and changes as

$$\hat{\psi}^h_\alpha = \gamma^1_{\alpha\beta}g_{\alpha\beta}\hat{\psi}^0_\beta, \quad \text{where } \gamma^1_{\alpha\beta} \in \mathbb{C}, \quad \gamma^1_{\alpha\beta}\gamma^1_{\beta\gamma}\gamma^1_{\gamma\alpha} = 1. \quad \text{(B6)}$$

Since this section is nowhere zero we can find another set of holomorphic functions that is linearly independent of $\hat{\psi}_\alpha$ at each point. We refer to this set of functions as $\hat{\psi}^\infty_\alpha$ and with an analogous procedure introduce $\gamma^2_{\alpha\beta}$ to remove all the zeros it can possibly have. Because of this, at each covering we can change basis to $\hat{\psi}_\alpha$ and $\hat{\psi}^\infty_\alpha$. One can check that gluing functions in this new basis of the vector bundle become

$$\hat{g}_{\alpha\beta} = \begin{pmatrix} \gamma^2_{\alpha\beta} & h_{\alpha\beta} \\ 0 & \gamma^1_{\alpha\beta} \end{pmatrix}, \quad \text{where}$$

$$\gamma^2_{\alpha\beta}\gamma^2_{\beta\gamma}\gamma^2_{\gamma\alpha} = 1 \quad \text{and} \quad h_{\alpha\gamma} = \gamma^2_{\alpha\beta}h_{\beta\gamma} + h_{\alpha\beta}\gamma^1_{\beta\gamma}. \quad \text{(B7)}$$

If we got that $h_{\alpha\beta} = 0$, then the functions $\hat{\psi}^\infty_\alpha$ would change through each other as

$$\hat{\psi}^\infty_\alpha = \gamma^2_{\alpha\beta}g_{\alpha\beta}\hat{\psi}^\infty_\beta \quad \text{(B8)}$$

and define a legitimate section of the vector bundle E. Since the set of functions $\gamma^2_{\alpha\beta}$ represents a line bundle, it has a meromorphic section: a set of meromorphic functions $f_\alpha(z)$ with property $\gamma^2_{\alpha\beta} = \frac{f_\beta}{f_\alpha}$. Then new functions

$$\psi^\infty_\alpha = f_\alpha \hat{\psi}^\infty_\alpha \tag{B9}$$

are holomorphic everywhere and transform as

$$\psi^\infty_\alpha = g_{\alpha\beta}\psi^\infty_\beta, \tag{B10}$$

therefore representing a legitimate section of the original vector bundle E but containing a pole at some point.

Let us show that we can get rid of $h_{\alpha\beta}$ by a proper redefinition of the arbitrarily chosen $\hat{\psi}^\infty_\alpha$. Namely, we notice that the choice of $\hat{\psi}^\infty_\alpha$ is not unique. At each covering we can make a change

$$\hat{\psi}^\infty_\alpha \to \hat{\psi}^\infty_\alpha + h_\alpha(z)\hat{\psi}^0_\alpha. \tag{B11}$$

It changes gluing functions as

$$h_{\alpha\beta} \to h_{\alpha\beta} + \gamma^2_{\alpha\beta}h_\beta(z) - h_\alpha(z)\gamma^1_{\alpha\beta}. \tag{B12}$$

This gives that $h_{\alpha\beta}$ belong to $H^1(\mathcal{O}(\gamma_1\gamma_2^{-1}))$ and by Serre duality are dual to $H^0(\mathcal{O}(\kappa\gamma_1^{-1}\gamma_2))$, where κ is a tangent line bundle. This line bundle does not have any holomorphic section if its Chern class is negative. We get

$$\begin{aligned} c_1(\kappa\gamma_1^{-1}\gamma_2) &= 2g - 2 - c(\gamma_1) + c(\gamma_2) \\ &= 2g - 2 - 2c(\gamma_1) = -2c(\gamma_1) < 0, \end{aligned} \tag{B13}$$

and we used $g = 1$ (torus), $c_1(\gamma_1) \geqslant 1$ ($\hat{\psi}_\alpha$ has at least one simple zero), and $c_1(\gamma_1) + c_1(\gamma_2) = 0$ [consequence of Eq. (13)].

Since $H^1(\mathcal{O}(\gamma_1\gamma_2^{-1})) = 0$, the cohomology class represented by $h_{\alpha\beta}$ is trivial, meaning that we can always pick h_α such that $h_{\alpha\beta} = 0$ in Eq. (B12). Then, as we discussed above, $\hat{\psi}^\infty_\alpha$ will represent a meromorphic section of the vector bundle E. This procedure could be generalized to higher-genus Riemann surfaces and other vector bundles.

[1] Y. Cao, V. Fatemi, S. Fang, K. Watanabe, T. Taniguchi, E. Kaxiras, and P. Jarillo-Herrero, Unconventional superconductivity in magic-angle graphene superlattices, Nature (London) **556**, 43 (2018).

[2] Y. Cao, V. Fatemi, A. Demir, S. Fang, S. L. Tomarken, J. Y. Luo, J. D. Sanchez-Yamagishi, K. Watanabe, T. Taniguchi, E. Kaxiras, R. C. Ashoori, and P. Jarillo-Herrero, Correlated insulator behaviour at half-filling in magic angle graphene superlattices, Nature (London) **556**, 80 (2018).

[3] H. C. Po, L. Zou, A. Vishwanath, and T. Senthil, Origin of Mott Insulating Behavior and Superconductivity in Twisted Bilayer Graphene, Phys. Rev. X **8**, 031089 (2018).

[4] A. Thomson, S. Chatterjee, S. Sachdev, and M. S. Scheurer, Triangular antiferromagnetism on the honeycomb lattice of twisted bilayer graphene, Phys. Rev. B **98**, 075109 (2018).

[5] L. Zou, H. C. Po, A. Vishwanath, and T. Senthil, Band structure of twisted bilayer graphene: Emergent symmetries, commensurate approximants and Wannier obstructions, Phys. Rev. B **98**, 085435 (2018).

[6] F. Guinea and N. R. Walet, Electrostatic effects and band distortions in twisted graphene bilayers, Proc. Natl. Acad. Sci. U. S. A. **115**, 13174 (2018).

[7] S. Carr, S. Fang, P. Jarillo-Herrero, and E. Kaxiras, Pressure dependence of the magic twist angle in graphene superlattices, Phys. Rev. B **98**, 085144 (2018).

[8] Y. Su and S.-Z. Lin, Spontaneous vortex-antivortex lattice in superconducting twisted bilayer graphene, Phys. Rev. B **98**, 195101 (2018).

[9] J. González and T. Stauber, Kohn-Luttinger Superconductivity in Twisted Bilayer Graphene, Phys. Rev. Lett. **122**, 026801 (2019).

[10] F. Wu, T. Lovorn, E. Tutuc, I. Martin, and A. H. MacDonald, Layer Pseudospin Skyrmions and Topological Moiré Bands in Twisted Homobilayers, Phys. Rev. Lett. **122**, 086402 (2019).

[11] D. K. Efimkin and A. H. MacDonald, Helical network model for twisted bilayer graphene, Phys. Rev. B **98**, 035404 (2018).

[12] J. Peng and J.-S. Wang, Current-induced heat transfer in double-layer graphene, arXiv:1805.09493.

[13] N. F. Q. Yuan and L. Fu, A model for metal-insulator transition in graphene superlattices and beyond, Phys. Rev. B **98**, 045103 (2018).

[14] C. Xu and L. Balents, Topological Superconductivity in Twisted Multilayer Graphene, Phys. Rev. Lett. **121**, 087001 (2018).

[15] E. Riccardi, O. Kashuba, M. Cazayous, A. Sacuto, and Y. Gallais, Raman scattering as a selective probe of chiral electronic excitations in bilayer graphene, Phys. Rev. Materials **3**, 014002 (2018).

[16] M. Ochi, M. Koshino, and K. Kuroki, Possible correlated insulating states in magic-angle twisted bilayer graphene under strongly competing interactions, Phys. Rev. B **98**, 081102(R) (2018).

[17] H. K. Pal, On magic angles and band flattening in twisted bilayer graphene, arXiv:1805.08803.

[18] F. Wu, A. H. MacDonald, and I. Martin, Theory of Phonon-Mediated Superconductivity in Twisted Bilayer Graphene, Phys. Rev. Lett. **121**, 257001 (2018).

[19] Y.-H. Zhang, D. Mao, Y. Cao, P. Jarillo-Herrero, and T. Senthil, Nearly flat Chern bands in moiré superlattices, Phys. Rev. B **99**, 075127 (2019).

[20] X.-C. Wu, K. A. Pawlak, C.-M. Jian, and C. Xu, Emergent superconductivity in the weak Mott insulator phase of bilayer graphene moiré superlattice, arXiv:1805.06906.

[21] J. Kang and O. Vafek, Symmetry, Maximally Localized Wannier States, and a Low-Energy Model for Twisted Bilayer Graphene Narrow Bands, Phys. Rev. X **8**, 031088 (2018).

[22] B. Andrei Bernevig, Z. Song, N. Regnault, and B. Lian TBG I: Matrix elements, approximations, perturbation theory and a $k \cdot p$ 2-band model for twisted bilayer graphene, arXiv:2009.11301.

[23] Z.-D. Song, B. Lian, N. Regnault, and A. B. Bernevig, TBG II: Stable symmetry anomaly in twisted bilayer graphene, arXiv:2009.11872.

[24] A. Bernevig, Z.-D. Song, N. Regnault, and B. Lian, TBG III: Interacting Hamiltonian and exact symmetries of twisted bilayer graphene, arXiv:2009.12376.

[25] B. Lian, Z.-D. Song, N. Regnault, D. K. Efetov, A. Yazdani, and B. A. Bernevig, TBG IV: Exact insulator ground states and phase diagram of twisted bilayer graphene, arXiv:2009.13530.

[26] B. A. Bernevig, B. Lian, A. Cowsik, F. Xie, N. Regnault, and Z.-D. Song, TBG V: Exact analytic many-body excitations in twisted bilayer graphene Coulomb Hamiltonians: Charge gap, Goldstone modes and absence of Cooper pairing, arXiv:2009.14200.

[27] F. Xie, A. Cowsik, Z. Son, B. Lian, B. A. Bernevig, and N. Regnault, TBG VI: An exact diagonalization study of twisted bilayer graphene at non-zero integer fillings, arXiv:2010.00588.

[28] J. M. Pizarro, M. J. Calderón, and E. Bascones, The nature of correlations in the insulating states of twisted bilayer graphene, J. Phys. Commun. **3**, 035024 (2018).

[29] M. Koshino, N. F. Q. Yuan, T. Koretsune, M. Ochi, K. Kuroki, and L. Fu, Maximally Localized Wannier Orbitals and the Extended Hubbard Model for Twisted Bilayer Graphene, Phys. Rev. X **8**, 031087 (2018).

[30] D. M. Kennes, J. Lischner, and C. Karrasch, Strong correlations and $d + id$ superconductivity in twisted bilayer graphene, Phys. Rev. B **98**, 241407(R) (2018).

[31] H. Isobe, N. F. Q. Yuan, and L. Fu, Unconventional Superconductivity and Density Waves in Twisted Bilayer Graphene, Phys. Rev. X **8**, 041041 (2018).

[32] L. Rademaker and P. Mellado, Charge-transfer insulation in twisted bilayer graphene, Phys. Rev. B **98**, 235158 (2018).

[33] J.-B. Qiao and L. He, Heterostrain engineering on twisted graphene bilayer around the first magic angle, Phys. Rev. B **98**, 235402 (2018).

[34] M. Zarenia, A. R. Hamilton, F. M. Peeters, and D. Neilson, Sign Reversal of Electron-Hole Coulomb Drag in Double Bilayer Graphene Heterostructures, Phys. Rev. Lett. **121**, 036601 (2018).

[35] T.-f. Chung, Y. Xu, and Y. P. Chen, Transport measurements in twisted bilayer graphene: Studies of electron-phonon coupling and Landau level crossing, Phys. Rev. B **98**, 035425 (2018).

[36] M. Fidrysiak, M. Zegrodnik, and J. Spałek, Unconventional topological superconductivity and phase diagram for a two-orbital model of twisted bilayer graphene, Phys. Rev. B **98**, 085436 (2018).

[37] T. J. Peltonen, R. Ojajärvi, and T. T. Heikkilä, Mean-field theory for superconductivity in twisted bilayer graphene, Phys. Rev. B **98**, 220504 (2018).

[38] P. Cazeaux, M. Luskin, and D. Massatt, Energy minimization of 2D incommensurate heterostructures, Arch. Ration. Mech. Anal. **235**, 1289 (2020).

[39] K. Wang, T. Hou, Y. Ren, and Z. Qiao, Enhanced robustness of zero-line modes in graphene via a magnetic field, Front. Phys. **14**, 23501 (2019).

[40] T. M. R. Wolf, J. L. Lado, G. Blatter, and O. Zilberberg, Electrically Tunable Flat Bands and Magnetism in Twisted Bilayer Graphene, Phys. Rev. Lett. **123**, 096802 (2019).

[41] V. Kozii, M. P. Zaletel, and N. Bultinck, Superconductivity in a doped valley coherent insulator in magic angle graphene: Goldstone-mediated pairing and Kohn-Luttinger mechanism, arXiv:2005.12961.

[42] V. Kozii, H. Isobe, J. W. F. Venderbos, and L. Fu, Nematic superconductivity stabilized by density wave fluctuations: Possible application to twisted bilayer graphene, Phys. Rev. B **99**, 144507 (2019).

[43] G. Tarnopolsky, A. J. Kruchkov, and A. Vishwanath, Origin of Magic Angles in Twisted Bilayer Graphene, Phys. Rev. Lett. **122**, 106405 (2019).

[44] R. Bistritzer and A. H. MacDonald, Moiré bands in twisted double-layer graphene, Proc. Natl. Acad. Sci. U. S. A. **108**, 12233 (2011).

[45] R. Bistritzer and A. H. MacDonald, Moiré butterflies in twisted bilayer graphene, Phys. Rev. B **84**, 035440 (2011).

[46] J. M. B. Lopes dos Santos, N. M. R. Peres, and A. H. Castro Neto, Continuum model of the twisted graphene bilayer, Phys. Rev. B **86**, 155449 (2012).

[47] P. San-Jose, J. González, and F. Guinea, Non-Abelian Gauge Potentials in Graphene Bilayers, Phys. Rev. Lett. **108**, 216802 (2012).

[48] N. Berline, E. Getzler, and M. Vergne, *Heat Kernels and Dirac Operators* (Springer, New York, 1996).

[49] Please note that in Ref. [43] the authors introduced additional matrix twist to the boundary conditions. One can show that the results of our paper are not affected by this change; therefore for simplicity of the argument we will consider only boundary conditions of the form (4).

[50] F. D. M. Haldane and E. H. Rezayi, Periodic Laughlin-Jastrow wave functions for the fractional quantized Hall effect, Phys. Rev. B **31**, 2529 (1985).

[51] Strictly speaking, we can allow integrable singularities like $1/z^{1/4}$, but they are not consistent with Eq. (5) because we assume that $\bar{A}$ does not have poles.

[52] Holomorphicity of $\gamma(z)$ concludes that the wave function has a simple zero $\psi_K(z) \sim f(\bar{z})(z - z_0) + \mathcal{O}(z - z_0)$ rather than some nonanalytical behavior.

[53] P. Griffiths and J. Harris, *Principles of Algebraic Geometry* (Wiley, New York, 1978).

[54] S. K. Donaldson, An application of gauge theory to four-dimensional topology, J. Differ. Geom. **18**, 279 (1983).

[55] J. Liu, J. Liu, and Xi. Dai, Pseudo Landau level representation of twisted bilayer graphene: Band topology and implications on the correlated insulating phase, Phys. Rev. B **99**, 155415 (2019).

[56] R. C. Gunning, *Lectures on Vector Bundles on Riemann surfaces* (Princeton University Press, Princeton, 1967).