Gaffnian and Haffnian: Physical relevance of nonunitary conformal field theory for the incompressible fractional quantum Hall effect

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We motivate a close look on the usefulness of the Gaffnian and Haffnian quasihole manifold (null spaces of the respective model Hamiltonians) for well-known gapped fractional quantum Hall phases. The conformal invariance of these subspaces is derived explicitly from microscopic many-body states. The resultant conformal field theory (CFT) description leads to an intriguing emergent primary field with $h = 2,\ c = 0$, and we argue the quasihole manifolds are quantum mechanically well defined and well behaved. Focusing on the incompressible phases at $\nu = \frac{1}{3}$ and $\frac{2}{5}$, we show the low-lying excitations of the Laughlin phase are quantum fluids of Gaffnian and Haffnian quasiholes, and give a microscopic argument showing that the Haffnian model Hamiltonian is gapless against Laughlin quasielectrons. We discuss the thermal Hall conductance and shot-noise measurements at $\nu = \frac{2}{5}$, and argue that the experimental observations can be understood from the dynamics within the Gaffnian quasihole manifold. A number of detailed predictions on these experimental measurements are proposed, and we discuss their relationships to the conventional CFT arguments and the composite fermion descriptions.

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I. INTRODUCTION

The most important aspects of the fractional quantum Hall effect (FQHE) are the universal topological properties that manifest in experimental measurements [1]. Unlike symmetry-protected topological phases, topological properties such as the Hall conductivity in FQHE are robust against any types of small perturbations. These properties arise from strong interaction between electrons confined not only in a two-dimensional manifold, but also in a single Landau level due to the strong magnetic field. The truncation of the Hilbert space (due to the smallness of the sample thickness and the magnetic length) at low temperature plays the crucial role here, leading to ground states with long-range topological entanglement, fractionalization of quasiparticle charges (i.e., anyons), nontrivial degeneracy of the quasihole manifold (i.e., non-Abelions), and chiral edge theories at the boundary of the FQH fluids [2–6].

Many exotic topological phases have been theoretically proposed, and their experimental realizations can potentially lead to robust storage and manipulations of quantum information [7,8]. The greatest challenge, however, is the existence of various different energy scales in realistic systems. Strictly speaking, in an effective description of a topological system, all energy scales are set to either infinity (e.g., the incompressibility gap) or zero (e.g., the quasihole degeneracy), and we can denote the associated "model Hamiltonian" as $\hat{H}_{\text{topo}}$. All the topological aspects are thus coming from a (possibly infinitely dimensional) sub-Hilbert space $\mathcal{H}_{\text{topo}}$ in which all states have *zero energy*, while all states in the complementary sub-Hilbert space $\bar{\mathcal{H}}_{\text{topo}}$ have *infinite energy*. For example, in Affleck-Kennedy-Lieb-Tasaki (AKLT) models with open boundary conditions, $\mathcal{H}_{\text{topo}}$ consists of the degenerate ground-state manifold with different edge configurations [9]. In the context of FQHE, $\mathcal{H}_{\text{topo}}$ consists of the ground state and all quasihole excitations that can be interpreted as edge excitations on a Hall manifold with a boundary. The existence of $\hat{H}_{\text{topo}}$ (not necessarily local) and a unique highest density state in $\mathcal{H}_{\text{topo}}$ (with electron density $\rho_{\max}$) implies incompressibility for the FQH system, which is the necessary condition for the plateau of the Hall conductance in the presence of disorder. Experimentally, the incompressible state can be realized by local Hamiltonians adiabatically connected to $\hat{H}_{\text{topo}}$.

When we smoothly go from $\hat{H}_{\text{topo}}$ to the realistic local Hamiltonian $\hat{H}_{\text{real}}$, we assume $\hat{H}_{\text{real}}$ is also incompressible at $\rho_{\max}$, albeit with a finite gap. This implies a finite energy gap for all eigenstates with $\rho < \rho_{\max}$, but there can be gapless excitations with $\rho > \rho_{\max}$. In particular the state with $\rho_{\max}$ does not necessarily have the lowest energy. For example, in FQH systems, quasiholes can have either lower or higher energy depending on the disorder or the edge confinement potential. We can model the smooth deformation as follows:

$$\hat{H} = (1 - \lambda)\hat{H}_{\text{topo}} + \lambda\hat{H}_{\text{real}}. \quad (1)$$

A number of things can happen when λ goes from 0 to 1, but in this work we will only focus on the following scenarios:

(1) The subspaces $\mathcal{H}_{\text{topo}}(\lambda)$ and $\bar{\mathcal{H}}_{\text{topo}}(\lambda)$ evolve adiabatically and are completely gapped for the entire range of λ; there is no level crossing between any two states from the different subspaces.

(a) All states in $\mathcal{H}_{\text{topo}}$ remain degenerate in the thermodynamic limit at $\lambda = 1$.

(b) Degeneracy of states in $\mathcal{H}_{\text{topo}}$ gets lifted, developing a finite bandwidth Δ_{topo} at $\lambda = 1$.

(2) Level crossing occurs between some states from the two subspaces, but the quantum sector of the highest-density state $|\psi_{\rho_{\max}}\rangle$ is adiabatically connected for λ between 0 and 1, which remains incompressible.

(3) A finite-energy gap opens up within $\mathcal{H}_{\text{topo}}$ at $\lambda = 1$ in the thermodynamic limit, with a new gapped ground state $|\psi_{\rho_0}\rangle$ at $\rho_0 < \rho_{\max}$. This implies no level crossing between $\mathcal{H}_{\text{topo}}$ and $\bar{\mathcal{H}}_{\text{topo}}$ in the quantum sector of $|\psi_{\rho_0}\rangle$ for λ between 0 and 1.

For simplicity, we only consider a single species of spinless fermions here, and the quantum sectors are labeled by the total number of electrons N_e, and the total angular momentum M on the disk geometry. The topological properties of $\hat{H}_{\text{real}}$ are thus completely determined by the behaviors listed above. If level crossing occurs in the quantum sector of the highest-density state, then $\hat{H}_{\text{topo}}$ and $\hat{H}_{\text{real}}$ are not topologically related in every sense. Statement 1 is definitely possible if $\hat{H}_{\text{real}}$ is a small perturbation from $\hat{H}_{\text{topo}}$. If statement 1(a) is the case, then $\hat{H}_{\text{topo}}$ and $\hat{H}_{\text{real}}$ are completely equivalent topologically. However, we should not take this for granted, as it is not fundamentally forbidden for statement 1(b) to be true. If that is the case, then one would expect certain physical properties considered universal at $\hat{H}_{\text{topo}}$ will be lost with the realistic Hamiltonian. These could include universal edge behaviors as predicted by the chiral Luttinger liquid theory [10–12], as well as the non-Abelian braiding of quasiholes [13] which fundamentally depends on the degeneracy of the quasihole manifold.

In numerical calculations, the most common behavior observed is actually statement 2, especially for FQH in higher LLs and for the non-Abelian FQH states. While this could well be the finite-size effect when realistic Hamiltonians are used, we should also take the possibility seriously that some level crossing could occur in the thermodynamic limit, while the ground state remains incompressible. The robustness of the Hall conductivity plateau only requires incompressibility and thus the adiabatic continuity of $|\psi_{\rho_{\max}}\rangle$, and this does not automatically imply the robustness of any or all of the other topological properties of the FQH state. If level crossing occurs despite the highest-density state being adiabatically connected, one could argue that $\hat{H}_{\text{topo}}$ and $\hat{H}_{\text{real}}$ no longer belong to the same universality class. Indeed, we would expect gapless boundary states to develop at the interface of the two Hamiltonians (or at certain values of λ when level crossing occurs). This could the be underlying difference between some of the known distinct FQH phases [14], although in this case all topological properties of the highest-density state (i.e., the ground state) are equivalent for the two phases.

Statements 3 and 2 are not mutually exclusive, though statement 3 implies a topological phase transition even for the ground state. Let $|\psi^{\alpha_1}_{\rho_{\max}}\rangle$ and $|\psi^{\alpha_2}_{\rho_0}\rangle$ be the two global ground states at $\lambda = 0$ and 1, respectively, with $\rho_0 < \rho_{\max}$, and α_1, α_2 are indices of quantum numbers from symmetries common to both $\hat{H}_{\text{topo}}$ and $\hat{H}_{\text{real}}$ (e.g., angular momentum M). If $|\rho_0 - \rho_{\max}|$ is subextensive (with respect to N_e), we still expect the two topological phases to occur at the same filling factor or the Hall plateau in the thermodynamic limit, and this could be the case for FQH phases that differ by the topological shift (related to Hall viscosity [15,16]). It is also interesting to consider the case that $|\alpha_1 - \alpha_2|$ is extensive, which implies both states can be (meta)stable for $0 < \lambda < 1$ since disorder will not be able to mix the two states in the thermodynamic limit. In analogy to the case of spontaneous symmetry breaking, we will have a spontaneous "topology" breaking, if the energies of $|\psi^{\alpha_1}_{\rho_{\max}}\rangle$ and $|\psi^{\alpha_2}_{\rho_0}\rangle$ are close as compared to disorder or temperature. A familiar example is the competition between the Pfaffnian and anti-Pfaffnian phase at half-filling, when the two-body interaction (which is particle-hole symmetric) dominates [17–20]. On the disk, the ground states of the two phases live in two angular momentum sectors that are infinitely apart in the thermodynamic limit because the two phases have different topological shifts ($s = -2$ for Pfaffnian and $s = +2$ for anti-Pfaffnian). Thus, local disorders alone will not be able to lift the degeneracy of the two states.

In this paper, we will discuss the physical properties of two rather special FQH candidates, the Gaffnian state and the Haffnian state, in the context of the three statements above. These two states are special because both from the effective theory description (e.g., the conformal field theory, or CFT) and the microscopic model perspective (e.g., three-body local model Hamiltonians), the Hilbert spaces $\mathcal{H}_{\text{topo}}$ and $\bar{\mathcal{H}}_{\text{topo}}$ can be unambiguously defined. However, the corresponding CFT models are nonunitary and/or irrational [12,21,22]. It is thus conjectured that there can be no *local* Hamiltonians to give $\bar{\mathcal{H}}_{\text{topo}}$ a finite-energy gap, while keeping all states in $\mathcal{H}_{\text{topo}}$ strictly degenerate (at zero energy). Thus, the Gaffnian and Haffnian states are considered to be related to some critical gapless phases in FQH systems [23], and it is generally dismissed as physically irrelevant to gapped FQH phases. We would like to examine these notions in more details in this work.

The organization of the paper will be as follows: In Sec. II, we give an overview of the well-known effective description of conformal invariance of the FQH systems, and give a rather different derivation of conformal invariance for the FQH quasihole subspace from the microscopic many-body states; in Sec. III we apply the microscopic derivation of conformal invariance to the Laughlin state and the model states from three-body pseudopotential interactions (including the Moore-Read, Gaffnian, and Haffnian states), leading to a number of interesting observations about the nature of their quasihole manifold; in Sec. IV we motivate the physical relevance of the Gaffnian and Haffnian states to gapped FQH phases, showing that they are model states for elementary excitations in the Laughlin phases. Interestingly, this also leads to a semirigorous microscopic argument on why the Haffnian model Hamiltonian is gapless in the thermodynamic limit, though the same arguments do not seem to apply to the Gaffnian model Hamiltonian; in Sec. V we move onto more realistic interactions, and argue that using the Gaffnian and Haffnian quasihole subspaces, a number of experimental results can be explained, and some more detailed predictions can be made. These include the physics related to the thermal Hall effect and the shot-noise and quasihole tunneling experiments; in Sec. VI, we give a summary with further discussions. In particular, we will summarize a number of detailed predictions from the analysis in this work, that can be directly related to experiments. Following is a list of various notations used in this paper:

Topological Hilbert space	$\mathcal{H}^I_{\text{topo}}$
Full Hilbert space	$\mathcal{H}^I_{\text{topo}} \otimes \bar{\mathcal{H}}^I_{\text{topo}}$
Bandwidth of $\mathcal{H}^I_t$	Δ^I_{topo}
Highest-density state in $\mathcal{H}^I_t$	$\lvert\psi^I_{\rho_{\max}}\rangle$
Electron number	N_e
Total angular momentum	M
A many-body quantum state in $\mathcal{H}^I_t$	$\psi_{k,I}$
A many-body primary state in $\mathcal{H}^I_t$	$\psi_{h_\alpha,I}$
The holomorphic part of $\psi_{k,I}$	$\phi_{k,I}$ or $\lvert\phi_{k,I}\rangle$
A descendant state at level N in $\mathcal{H}^I_t$	$\lvert\phi^{(N)}_{h_\alpha,I}\rangle$, $\lvert\phi^{(N)}_{h_I}\rangle$
A second primary state with $h=2,\ c=0$	$\lvert\tau_I\rangle$

II. CFT DESCRIPTION OF FQHE

We will now specialize to the FQH systems, so that $\mathcal{H}_{\text{topo}}$ is the Hilbert space of the FQH ground state and its quasihole excitations (degenerate states that are less dense than the ground state), while $\bar{\mathcal{H}}_{\text{topo}}$ is the Hilbert space of all gapped excitations, including the neutral and quasielectron excitations. On the disk or cylinder geometry where edges are present, each quasihole state can be reinterpreted as an edge excitation. This is because the insertion of magnetic fluxes (to create quasiholes) pushes electrons to the boundary, even if the insertion is deep in the bulk. For quasiholes created deep in the bulk, however, they correspond to edge excitations with very large momenta. Thus, those states are not necessarily important for the low-energy, long-wavelength limit of the edge theory.

Using the disk geometry, an important way of analyzing many FQH phases is to use the conformal field theory (CFT), which is particularly useful for two-dimensional critical systems with conformal symmetry [24,25]. The FQH systems are insulators that are gapped in the bulk. However, for a quantum Hall droplet with a boundary (which is also a more realistic scenario in the experiments), it is indeed a gapless system due to the gapless edge excitations. For all energy scales smaller than the bulk gap, we can thus treat the edge dynamics as a one-dimensional chiral system [12]. One can show that this one-dimensional system is conformally invariant in the $1+1$ space-time, if it is *maximally chiral*: all excitations travel at a common velocity v. Formally, for any local operator $\hat{O}$ at the edge, we need to have

$$\hat{O}(x,t) = \hat{O}(x - vt), \tag{2}$$

where x is the periodic spatial coordinate at the edge, and t is time. The CFT description is thus an effective theory for the Hilbert space of $\mathcal{H}_{\text{topo}}$ only. The states in $\mathcal{H}_{\text{topo}}$ are only degenerate in the limit of $v \to 0$. For any nonzero value of v, however, $\mathcal{H}_{\text{topo}}$ satisfies conformal symmetry with the assumption of Eq. (2).

Thus, if we project into the Hilbert space of $\mathcal{H}_{\text{topo}}$, assuming the linear spectrum and Eq. (2) can be realized with certain physical Hamiltonian, then all edge excitations can be described by an effective model satisfying conformal symmetry. Such models can also be analyzed and understood via the elegant machinery of CFT. There is thus a natural bulk-edge correspondence because each edge mode can also be understood as a quasihole excitation (i.e., conformally mapped to the spherical geometry as a bulk excitation in $\mathcal{H}_{\text{topo}}$). It has been discovered for some FQH phases that the microscopic models' wave functions on the disk geometry for all states in $\mathcal{H}_{\text{topo}}$ are given by the correlators or conformal blocks of certain CFT models [4,13,26]. Such models thus have to encode some information about the edge dynamics of the same FQH phases [27,28].

We now work under the assumption that every state in $\mathcal{H}_{\text{topo}}$ can be written as the correlator of a specific CFT model $\mathcal{M}_{\mathcal{H}_{\text{topo}}}$. This can be verified microscopically for a number of FQH models. It is thus pertinent to ask about the relationship between $\mathcal{H}_{\text{topo}}$ and the Hilbert space of $\mathcal{M}_{\mathcal{H}_{\text{topo}}}$. The CFT correlators that map to every microscopic state in $\mathcal{H}_{\text{topo}}$ only consist of primary fields in $\mathcal{M}_{\mathcal{H}_{\text{topo}}}$. Each quasihole corresponds to one primary field in the correlator with coordinates $\eta_i = x_i + iy_i$, where the subscript is the quasihole index, and x, y are the real-space coordinates of the two-dimensional disk. The degeneracy of the multiquasihole states after fixing their locations, which is important to the non-Abelian properties of the FQH phase, is also determined by the distinct correlators from the fusion rules of those primary fields. In fact, for all known FQH phases with exact model Hamiltonians (so that the ground state and quasiholes are well-defined zero-energy states), there is a one-to-one mapping of every state in $\mathcal{H}_{\text{topo}}$ to every possible correlator involving only the primary fields. These states are wave functions of the locations of the electrons and the quasiholes. The number of primary fields in the correlator corresponds to the number of electrons and quasiholes in the many-body wave function. Thus, from this perspective, the Hilbert space of $\mathcal{M}_{\mathcal{H}_{\text{topo}}}$ is much larger than $\mathcal{H}_{\text{topo}}$ since no descendant fields are involved in the construction of states in $\mathcal{H}_{\text{topo}}$. Note that this perspective is the CFT description of the FQH bulk properties, which is also valid on geometries with no boundaries, such as sphere or torus. For some hierarchical states with no known model Hamiltonians (but may have approximate ones [29]), there are attempts to write their many-body wave functions as CFT correlators involving descendant fields [30–32]. Here, we restrict ourselves to only ones with exact model Hamiltonians.

If we focus on the edge excitations as a dynamical system with linear dispersion (with the same Hilbert space $\mathcal{H}_{\text{topo}}$), then the CFT model $\mathcal{M}_{\mathcal{H}_{\text{topo}}}$ is an effective theory so far with no rigorous microscopic derivation. The coordinates of the effective theory are no longer real-space coordinates, but holomorphic and antiholomorphic coordinates $z = x + ivt$, $\bar{z} = x - ivt$. Here, x is the periodic coordinate along the edge of the disk, and t is the time, which we can take along the radial direction. The Hilbert space in $\mathcal{M}_{\mathcal{H}_{\text{topo}}}$ is thus generated by conformal generators $\hat{\mathcal{L}}_n$ satisfying the Virasoro algebra:

$$[\hat{\mathcal{L}}_n, \hat{\mathcal{L}}_m] = (n-m)\hat{\mathcal{L}}_{m+n} + \frac{c}{12}n(n^2-1)\delta_{m+n,0}, \tag{3}$$

where c is the central charge of the CFT model. The natural Hamiltonian is proportional to $\hat{\mathcal{L}}_0$, which is the dilation operator and thus the translation along the time (i.e., radial)

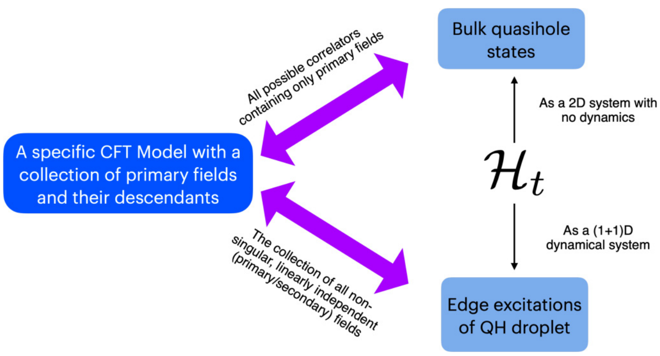

FIG. 1. Relationship between the effective CFT models and the microscopic many-body states, from the bulk and edge perspectives.

direction. We thus expect all states in $\mathcal{H}_{\text{topo}}$ to be mapped to the primary and descendant fields in $\mathcal{M}_{\mathcal{H}_{\text{topo}}}$, which are eigenstates of $\hat{\mathcal{L}}_0$ in the CFT description.

Thus, while it is microscopically equivalent for $\mathcal{H}_{\text{topo}}$ to be treated as bulk quasihole excitations, or as edge excitations at the boundary of the quantum Hall droplet, the corresponding CFT descriptions are rather distinctive (see Fig. 1). As quasiholes, $\mathcal{H}_{\text{topo}}$ is mapped to primary field correlators in the two-dimensional manifold (no dynamical information); as edge excitations, $\mathcal{H}_{\text{topo}}$ is mapped to all primary fields and descendant fields in one dimension, with a well-defined "conformal Hamiltonian." This bulk-edge correspondence, or the equivalence of the two descriptions, should in general not be true for any arbitrarily defined subspace. It is fundamentally due to the intrinsic topological or algebraic structures of $\mathcal{H}_{\text{topo}}$. The one-to-one mapping of the states in $\mathcal{H}_{\text{topo}}$ and the primary and descendant fields of the edge CFT can also be quite nontrivial, which is a particularly important issue when $\mathcal{M}_{\mathcal{H}_{\text{topo}}}$ is nonunitary or irrational.

A. Nonunitary and irrational CFT models

The CFT description and the bulk-edge correspondence seem to work quite well, if the CFT model $\mathcal{M}_{\mathcal{H}_{\text{topo}}}$ is rational and unitary. In these cases, $\mathcal{M}_{\mathcal{H}_{\text{topo}}}$ only contains a finite number of primary fields, and the norm of all primary and descendant fields in the model, as defined in CFT, are non-negative. The description becomes subtle when $\mathcal{M}_{\mathcal{H}_{\text{topo}}}$ is nonunitary and or irrational, which is relevant to the Gaffnian state [33] (nonunitary) and the Haffnian state [21,23] (nonunitary and irrational) we will focus on for the main part of this work.

It is generally argued that the nonunitary and irrational CFT models cannot be physical models in describing the dynamics of the conformally invariant one-dimensional edge systems [13,22]. This is because nonunitary CFT models contain fields with negative norm and thus diverging correlation functions at the edge. Irrational CFT models imply an infinite number of primary fields which seems unnatural. On the torus geometry, this implies the ground-state degeneracy (the number of highest-density states in $\mathcal{H}_{\text{topo}}$) is infinity in the thermodynamic limit, making it unlikely to describe a gapped bulk phase. The proposed resolution is that $\mathcal{H}_{\text{topo}}$ alone cannot describe the physical edge dynamics in these cases. The low-lying gapless excitations have to include states from $\bar{\mathcal{H}}_{\text{topo}}$, thus, the bulk gap has to close in the thermodynamic limit, so that the mapping to nonunitary and irrational CFT models will no longer hold.

To get a fuller picture of the scenario, we first note that $\mathcal{H}_{\text{topo}}$ is well defined with nonunitary and irrational CFT model on the disk geometry or local Hamiltonians, since the latter is gapless only in the thermodynamic limit, which is an asymptotic behavior. The one-to-one mapping of the null space to the primary field correlator, as well as to all of the primary and descendant fields, can be established. There is thus no ambiguity in defining $\mathcal{H}_{\text{topo}}$ as the null space of a particular $\hat{H}_{\text{topo}}$ with an infinite gap to $\bar{\mathcal{H}}_{\text{topo}}$, as long as we do not require $\hat{H}_{\text{topo}}$ to be local. Thus, the arguments about the inability to define $\mathcal{H}_{\text{topo}}$ as the gapped null space can only be applied to local Hamiltonians, which we will focus in the next section. On the other hand, it is not clear if the mapping between quantum states in $\mathcal{H}_{\text{topo}}$ and the primary and descendant fields in $\mathcal{M}_{\mathcal{H}_{\text{topo}}}$ goes beyond state counting. In particular, all states in $\mathcal{H}_{\text{topo}}$ have a well-defined, positive-definite quantum mechanical norm, even when their counterparts can have negative norms defined in CFT. The quantum mechanical quasihole correlation functions in $\mathcal{H}_{\text{topo}}$ also decay as a function of the distance between them, while in the CFT description the correlation diverges due to the negative conformal dimensions [34,35].

This apparent inconsistency implies it is important to understand what physical aspects of $\mathcal{H}_{\text{topo}}$ can be captured by the corresponding CFT model, in addition to the state counting and the linear spectrum. The linear spectrum from maximal chirality is also not intrinsic to $\mathcal{H}_{\text{topo}}$, but rather from a putative effective Hamiltonian on states in the restricted Hilbert space (the null space), that are expected to be only physically relevant in the long-wavelength limit of the edge system from the confining potential. Given that the only constraint or assumption here is the conformal invariance of $\mathcal{H}_{\text{topo}}$, the resolution could ultimately be about the derivation of effective CFT theory from the microscopic details of the Hilbert space.

B. Microscopic derivation of conformal invariance of $\mathcal{H}_{\text{topo}}$

We will show a rather crude attempt here that nevertheless leads to a number of interesting results relevant to the main focus of this work, but leave a detailed discussion for future works. To start from the familiar grounds, let us focus on the lowest Landau level (LLL) first, though the main results derived here apply to any Landau level as they should be. Every state in $\mathcal{H}_{\text{topo}}$ is a many-body wave function with holomorphic variables $z_i = x_i + iy$, where the subscript i is the electron index. We thus denote $\mathcal{H}_{\text{topo}} = \{\psi_k(z_1, z_2, z_3, \ldots)\}$, and the states can have any number of electrons. Without loss of generality, we can fix the total number of electrons to be N_e, assuming the electron number as a good quantum number. In the LLL, the many-body wave functions are given by

$$\psi_k(z_1, z_2 \ldots z_{N_e}) = \phi_k(z_1, z_2 \ldots z_{N_e}) e^{-\frac{1}{4}\sum_i z_i z_i^*}, \tag{4}$$

where $\phi_k(z_1, z_2 \ldots z_{N_e})$ is holomorphic in its variables, as a linear combination of the monomial basis. We can thus write $\phi_k = \sum_\lambda c_{k\lambda} m_\lambda$, where $m_\lambda = \text{Asy}(z_1^{n_{1\lambda}} z_2^{n_{2\lambda}} \ldots z_{N_e}^{n_{N_e\lambda}})$. The antisymmetrization Asy is over the electron indices. In addition, we assume rotational invariance on the disk geometry, thus, the total angular momentum M_λ is also a good quantum number, with $M_\lambda = \sum_i n_{i\lambda}$. Each state physically represents a quantum Hall droplet, the size of which is given by the highest power of z_i in the monomial basis of $\phi_k(z_1, z_2 \dots z_{N_e})$. The Gaussian factor in Eq. (4) is not important, so we will denote states in $\mathcal{H}_{\text{topo}}$ with ϕ_k.

It is also useful to have the second quantized representation of ϕ_k, so we can denote each monomial as follows:

$$m_\lambda = \text{Asy}\left(z_1^{n_{1\lambda}} z_2^{n_{2\lambda}} \dots z_{N_e}^{n_{N_e\lambda}}\right) \sim c^\dagger_{n_{1\lambda}} c^\dagger_{n_{2\lambda}} \dots c^\dagger_{n_{N_e\lambda}} |\text{vac}\rangle, \tag{5}$$

where $c_i^\dagger$ is the electron creation operator in the single-particle orbital indexed by i (i.e., z^i), satisfying the anticommutation relations $\{c_i, c_j^\dagger\} = \delta_{ij}$, $\{c_i, c_j\} = \{c_i^\dagger, c_j^\dagger\} = 0$. We can thus represent m_λ with a binary string. Each digit from left to right corresponds to a single-particle orbital from the center to the edge of the disk (i.e., indexed by the power of z). As an example with two electrons, we have $(z_1 - z_2)^3 = (z_1^3 - z_2^3) - 3(z_1^2 z_2 - z_1 z_2^2) = |1001000\dots\rangle - 3|0110000\dots\rangle$. One should note the second quantized representation is applicable to FQH in any LLs. All the results in this section can be derived from the second quantized representation, and they do not require the holomorphic wave functions that are specific to the LLL.

Given the conformal invariance of $\mathcal{H}_{\text{topo}}$, we should be able to identify one or more states $\phi_{h_\alpha} \in \mathcal{H}_{\text{topo}}$ as the primary states, indexed by α (as we will properly define later). These are states that are analogous to the "primary fields" in CFT. All other states in $\mathcal{H}_{\text{topo}}$ can be generated from the primary states by operators satisfying the familiar Virasoro algebra. To that end, we need to properly define the Virasoro generators acting on ϕ_k. Given that the classical version $\hat{l}_{-n} = \sum_i z_i^{n+1}\partial_{z_i}$ satisfies Eq. (3) with $c = 0$, we can define the following second quantized operators analog with $n \geqslant 0$:

$$\hat{L}_{-n} = \sum_{k=0}^{\infty} f_{k+n,k} \cdot k \hat{c}^\dagger_{k+n}\hat{c}_k, \tag{6}$$

$$\hat{L}_{n} = \sum_{k=0}^{\infty} f_{k,k+n} \cdot (k+n) \hat{c}^\dagger_{k}\hat{c}_{k+n}. \tag{7}$$

The function f_{k_1,k_2} comes from the single-particle state normalization that depends on the geometry. For example on the disk geometry, $f_{k_1,k_2} = \sqrt{k_1!/k_2!}$. It is easy to check that the Virasoro algebra is satisfied between $\hat{L}_m, \hat{L}_n$ if $nm \geqslant 0$. The commutation relation between the positive and negative modes is a bit more subtle:

$$[\hat{L}_m, \hat{L}_{-n}] = (n-m)\hat{L}_{m+n} + \hat{C}_{m,n}, \tag{8}$$

$$\hat{C}_{m,n} = \begin{cases} \sum_{k=0}^{m-1} f_{k,k+\Delta} \cdot (k-m)(k+\Delta)\hat{c}^\dagger_k \hat{c}_{k+\Delta}, & n \geqslant m \\ \sum_{k=0}^{n-1} f_{k+\Delta,k} \cdot k(k-n) c^\dagger_{k+\Delta}\hat{c}_k, & n \leqslant m \end{cases} \tag{9}$$

with $m, n \geqslant 0$, $\Delta = |m-n|$. Thus, the Virasoro algebra is not explicitly obeyed by the additional term $\hat{C}_{m,n}$.

On the other hand, we expect the conformal symmetry to be satisfied only in the thermodynamic limit, and the Virasoro algebra to be obeyed only within $\mathcal{H}_{\text{topo}}$. One or more states $\phi_{h_\alpha} \in \mathcal{H}_{\text{topo}}$ can be identified as the "primary states" in the following sense *in the thermodynamic limit*:

(a) $\hat{L}_n \phi_{h_\alpha} \in \bar{\mathcal{H}}_{\text{topo}}$ for $n > 0$,
(b) $\hat{L}_{-n} \phi_{h_\alpha} \in \mathcal{H}_{\text{topo}}$ for $n \geqslant 0$,
(c) $\hat{C}_{m,n} \phi_k \in \bar{\mathcal{H}}_{\text{topo}}$ for $m \neq n$.

In general, $\hat{L}_n \phi_{h_\alpha} \neq 0$ for $n > 0$, which violates the requirement for the primary fields in CFT. However, the assumption (a) implies ϕ_{h_α} are the highest weight states in $\mathcal{H}_{\text{topo}}$, thus qualifying them as the primary states in analogy to the primary fields. All descendant states that correspond to the descendant fields in CFT are within $\mathcal{H}_{\text{topo}}$ from (b). For the Virasoro algebra to hold, we need (c), so the Virasoro algebra is satisfied within $\mathcal{H}_{\text{topo}}$. Thus, the conformal invariance of $\mathcal{H}_{\text{topo}}$ is explicitly established.

We now look at the inner products and the norms of the states given by ϕ_k. Let us use $|\phi_k\rangle$ to denote ϕ_k, or the full wave function ψ_k with the Gaussian factor. We define $\hat{L}_n|\phi_k\rangle$ as $\hat{L}_n$ acting on the holomorphic part of ψ_k. For the primary states we can define the inner product using the usual quantum mechanical overlap as follows:

$$\langle \phi_{h_\alpha} | \phi_{h_\beta} \rangle = \int dz_1 dz_1^* \dots dz_{N_e} dz_{N_e}^* \psi^*_{h_\alpha} \psi_{h_\beta}. \tag{10}$$

A descendant state at level N is thus given by $|\phi^{(N)}_{h_\alpha}\rangle = \hat{L}_{-k_1}\hat{L}_{-k_2}\dots\hat{L}_{-k_n}|\phi_{h_\alpha}\rangle$, with $N = \sum_{i=1}^n k_i$. If the total angular momentum of $|\phi_{h_\alpha}\rangle$ is M_α, then the total angular momentum of $|\phi^{(N)}_{h_\alpha}\rangle$ is $M_\alpha + N$. We can now define the following norm:

$$\langle \phi^{(N)}_{h_\alpha} | \phi^{(N)}_{h_\alpha} \rangle = \langle \phi_{h_\alpha} | \hat{L}_{k_n}\hat{L}_{k_{n-1}} \dots \hat{L}_{n_1}\hat{L}_{-k_1}\hat{L}_{-k_2}\dots\hat{L}_{-k_n} | \phi_{h_\alpha} \rangle. \tag{11}$$

This is not equivalent to the quantum mechanical norm of $|\phi^{(N)}_{h_\alpha}\rangle$ since from Eqs. (6) and (7) we can see that $(\hat{L}_n)^\dagger \neq \hat{L}_{-n}$. However, Eq. (11) can be evaluated in a well-defined way by commuting all of the $\hat{L}_n$ with $n \geqslant 0$ to the right using Eq. (8), so that $\hat{L}_{k_n}\hat{L}_{k_{n-1}}\dots\hat{L}_{n_1}\hat{L}_{-k_1}\hat{L}_{-k_2}\dots\hat{L}_{-k_n}|\phi_{h_\alpha}\rangle$ is proportional to $|\phi_{h_\alpha}\rangle$. This is followed by the usual quantum mechanical overlap using Eq. (10).

We thus have two types of norm or overlap between two states in $\mathcal{H}_{\text{topo}}$. The first type is the usual quantum mechanical overlap from the integration over z_i, z_i^*, in the form of Eq. (10) for any two states. The other type is the so-called "conformal norm" and "conformal overlap," which is computed from the Virasoro algebra, the highest weight condition of the primary state, and the orthonormality of the primary states (equivalent to the quantum mechanical overlap, for the primary states only). This distinction is important since the linear dependence of a set of states depends entirely on the definition of the overlaps. At each level or total angular momentum sector, a set of states can be linearly dependent with the quantum mechanical overlap, but linearly independent with the conformal overlap, or vice versa. This is mainly because the Gram matrix of the conformal overlap is not positive definite. On the other hand, all physical quantities in principle should be derived from the quantum mechanical overlap.

The emergence of the central charge can be seen from the microscopic wave functions as follows:

$$\langle\phi_k|[\hat{L}_n,\hat{L}_{-n}]|\phi_k\rangle = \langle\phi_k|\hat{C}_{n,n}|\phi_k\rangle = \sum_{k'=0}^{n-1} k'(k'-n)\langle\phi_k|c^\dagger_{k'}\hat{c}_{k'}|\phi_k\rangle, \quad (12)$$

where $\langle\phi_k|c^\dagger_{k'}\hat{c}_{k'}|\phi_k\rangle$ gives the average occupation of electrons in the k'th orbital. Since $|\phi_k\rangle$ is a state with edge excitations, physically there is only density modulation near the edge. In the thermodynamic limit and for any finite value of k', we have $\langle\phi_k|c^\dagger_{k'}\hat{c}_{k'}|\phi_k\rangle = \nu$, the filling factor of the FQH phase. We thus have

$$\lim_{N_e\to\infty}\langle\phi_k|[\hat{L}_n,\hat{L}_{-n}]|\phi_k\rangle = \frac{2\nu}{12}(n^3-n) \quad (13)$$

and we can identify the central charge $c = 2\nu$. The importance of this result will be discussed in the later sections.

From the definitions of Eqs. (6) and (7), it is obvious all states $|\phi_k\rangle$ are eigenstates of $\hat{L}_0$, with eigenvalues given by the total angular momentum, which is also the conformal dimension. Thus, there is at least one primary state with conformal dimension that scales with N_e^2, and becomes infinity in the thermodynamic limit. The norm of the descendant states from this primary state thus is always positive, using the definition from Eq. (11). We will also discuss this potentially interesting point in the specific examples later.

III. CASE OF λ = 0

We will now first focus on the idealized case of $\hat{H}_{\rm topo}$, for which we assume the null space $\mathcal{H}_{\rm topo}$ is well defined. It turns out in many cases for the FQH systems, we can find model local Hamiltonians with well-defined null space, which allows us to scale the energies of states in $\bar{\mathcal{H}}_{\rm topo}$ to infinity for any *finite* systems. Thus, $\mathcal{H}_{\rm topo}$ can be realized by physically relevant Hamiltonians. For all the analysis in the previous section to apply in the thermodynamic limit, we also need the Hamiltonian to have a finite-energy gap in the thermodynamic limit. This is necessary because otherwise we cannot send the energies of the states in $\bar{\mathcal{H}}_{\rm topo}$ to infinity by rescaling the Hamiltonian.

A. Laughlin and Moore-Read phases

Let us look at two familiar examples of the Laughlin phase and the Moore-Read phase. The model Hamiltonian of the former is the $\hat{V}_1^{\rm 2bdy}$ Haldane pseudopotential, while for the latter is the three-body interaction Hamiltonian which we denote as $\hat{V}_3^{\rm 3bdy}$. In both cases, the CFT description of the null space (denoted by $\mathcal{H}^L_{\rm topo}$ for the Laughlin phase, and $\mathcal{H}^M_{\rm topo}$ for the Moore-Read phase) is well known, with unitary CFT models. For the Laughlin phase, the $\mathcal{H}^L_{\rm topo}$ can be mapped to the Hilbert space of the U(1) chiral bosons, while for the Moore-Read phase the $\mathcal{H}^M_{\rm topo}$ can be mapped to the Hilbert space of the U(1) chiral bosons with the additional Ising fermions [4,36].

We will now look into the explicit conformal invariance of $\mathcal{H}^L_{\rm topo}$. We can identify the Laughlin wave function $\phi_{h_L} = \prod_{i<j}(z_i-z_j)^q$ at filling factor $\nu = 1/q$ as the primary state, with $\hat{L}_1|\phi_{h_L}\rangle = 0$. While $\hat{L}_n|\phi_{h_L}\rangle$ does not vanish for $n > 1$, it clearly lives entirely in $\bar{\mathcal{H}}^L_{\rm topo}$ since $|\phi_{h_L}\rangle$ is the highest-density state in $\mathcal{H}^L_{\rm topo}$ with the minimal total angular momentum $M_L = q(N_e^2-N_e)/2$. The entire space of $\mathcal{H}^L_{\rm topo}$, which is spanned by Jack polynomials with $\alpha = -2/(q-1)$ and admissible root configurations (i.e., no more than one electron for every q consecutive orbital) [37], can be generated by repeated applications of $\hat{L}_{-n}$, $n > 0$ on $|\phi_{h_L}\rangle$, with the well-known Virasoro level counting of $1, 1, 2, 3, 5, 7, \ldots$ corresponding to the level $N = 0, 1, 2, 3, 4, 5, \ldots$. We have also numerically verified assumption (c) for $|\phi_k\rangle \in \mathcal{H}^L_{\rm topo}$ with finite-size scaling.

In this particular description, the conformal dimension of $|\phi_{h_L}\rangle$, or the eigenvalue of $\hat{L}_0$, is $h_L = \frac{q}{2}N_e(N_e-1)$. The central charge is given by $c = 2/q$. For $q \geqslant 3$ we have $c < 1$, and this corresponds to a nonunitary CFT for any finite N_e. However, states with negative conformal norm [as computed from Eq. (11)] can only occur at very large angular momenta, at which the finite-size effect comes in and the Virasoro counting is no longer obeyed. Thus, from the quantum mechanical point of view, those states with negative conformal norm are just linear combinations of other states in the same angular momentum sector. In the thermodynamic limit when $h_L \to \infty$, the Virasoro counting is obeyed at any arbitrarily large angular momentum sector, and all states will have positive conformal norm. It is thus in every sense a valid CFT description of $\mathcal{H}^L_{\rm topo}$ with $h_L = \infty, c = 2/q$, even though it is apparently quite different from the usual CFT description with chiral free bosons ($h = q,\ c = 1$).

To extend this description to the Moore-Read phase with $\mathcal{H}^M_{\rm topo}$, we can also identify the Pfaffian ground state $|\phi_{h_M}\rangle$ as the primary state with $h_M = N_e(N_e-\frac{3}{2}),\ \ c = 1$. However, its conformal family does not span the entire $\mathcal{H}^M_{\rm topo}$, which is the space of Jack polynomials with $\alpha = -3$ and admissible root configurations satisfying no more than two electrons in every four consecutive orbitals. In particular, at level $N = 2$, there are three linearly independent states in $\mathcal{H}^M_{\rm topo}$, while only two descendant states from $|\phi_{h_M}\rangle$, namely, $\hat{L}_{-2}|\phi_{h_M}\rangle$, $\hat{L}_{-1}\hat{L}_{-1}|\phi_{h_M}\rangle$. We can thus construct the state at $N = 2$ that is orthogonal to $\hat{L}_{-2}|\phi_{h_M}\rangle, \hat{L}_{-1}\hat{L}_{-1}|\phi_{h_M}\rangle$ (using the *quantum mechanical overlap*), and denote it as $|\phi_{h'_M}\rangle$. There is very strong numerical evidence that $|\phi_{h'_M}\rangle$ is annihilated by $\hat{L}_1, \hat{L}_2$ (and thus $\hat{L}_n$ with $n > 0$ following the Virasoro algebra), so this state can be identified as a second primary state in $\mathcal{H}^M_t$ (see Fig. 2).

We will thus have the following decomposition of $|\phi_{h'_M}\rangle$:

$$|\phi_{h'_M}\rangle = |\phi_{h_M}\rangle \otimes |\tau_M\rangle. \quad (14)$$

Since $|\phi_{h'_M}\rangle$ has conformal dimension $h_M + 2$, and that $\lim_{N_e\to\infty}\langle\phi_{h_M}|[\hat{L}_n,\hat{L}_{-n}]|\phi_{h_M}\rangle = \lim_{N_e\to\infty}\langle\phi_{h'_M}|[\hat{L}_n,\hat{L}_{-n}]|\phi_{h'_M}\rangle$ (the electron density at the center of the disk is not affected by excitations, or density modulations at the disk boundary), we then have a second primary state $|\tau_M\rangle$ with conformal dimension $h = 2$ and central charge $c = 0$. The fact that it has zero central charge is quite interesting. We will show it is a reasonable result that warrants further investigations.

The level counting of the primary state with $h = 2,\ c = 0$ can be computed explicitly, and listed in the third column of Table I. If we convolute the level counting from $|\tau_M\rangle$ with that of $|\phi_{h_M}\rangle$, the total number of states at $N > 5$ overcounts those in $\mathcal{H}^M_{\rm topo}$, indicating that not all states from the direct product of

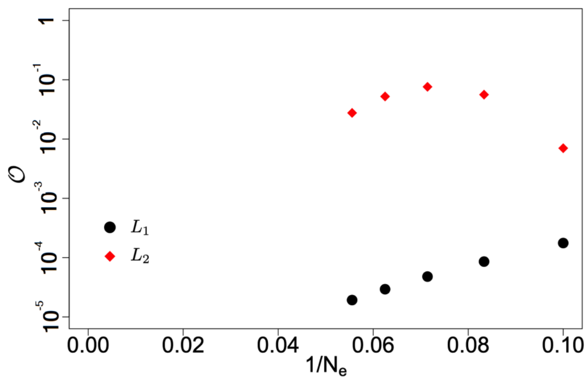

FIG. 2. The overlap with $\mathcal{H}_{\text{topo}}$ after applying L_1 (black plot) and L_2 (red plot) to the primary state $|\phi_{h'_M}\rangle$. The y axis is the log of the overlap, and the x axis is the inverse of the system size.

the two conformal families are linearly independent with the *quantum mechanical overlap* (see the fifth column of Table I). This again comes from the fact that the quantum mechanical overlap and the conformal overlap are not equivalent. Thus, in principle, this mismatch does not contradict the conformal invariance of $\mathcal{H}^M_{\text{topo}}$.

On the other hand, the important discovery here is that if we only count the number of unitary states at each level using the conformal overlap (states with positive conformal norms), its convolution with the Virasoro counting of $|\phi_{h_M}\rangle$ agrees *exactly* with the counting of the Pfaffian quasiholes, or the Hilbert space of $\mathcal{H}^M_{\text{topo}}$, for all the system sizes we can check numerically (see the sixth and seventh columns of Table I). This is also reasonable because we should be able to map the CFT description here to the well-known CFT models with U(1) bosons and Ising fermions [the $\mathcal{M}(4,3)$ minimal model], which is unitary. We conjecture this observation can be generalized to the entire Read-Rezayi series, which we will discuss in more details elsewhere.

B. Gaffnian and Haffnian phases

The Gaffnian and Haffnian states are closely related to the Pfaffian states, as they are all the highest-density zero-energy states of the leading three-body pseudopotential interactions. Let us denote the null space that contains the Gaffnian and Haffnian quasiholes as $\mathcal{H}^G_{\text{topo}}$ and $\mathcal{H}^H_{\text{topo}}$, respectively. While the model Hamiltonian for the Pfaffian is $\hat{H}_{\text{mr}} = \hat{V}_3^{\text{3bdy}}$, that of the Gaffnian is $\hat{H}_{\text{gf}} = \hat{V}_3^{\text{3bdy}} + \hat{V}_5^{\text{3bdy}}$, and that of the Haffnian is $\hat{H}_{\text{hf}} = \hat{V}_3^{\text{3bdy}} + \hat{V}_5^{\text{3bdy}} + \hat{V}_6^{\text{3bdy}}$. There have been no rigorous claims on if $\hat{H}_{\text{gf}}$ and $\hat{H}_{\text{hf}}$ are gapped in the thermodynamic limit (and no rigorous statement can be made for $\hat{H}_{\text{mr}}$ for that matter), but even if they are gapless (as supported by a number of arguments), their null spaces are still well defined for any system sizes. $\mathcal{H}^G_{\text{topo}}$ is spanned by Jack polynomials with $\alpha = -\frac{3}{2}$ and admissible root configurations satisfying no more than two electrons for any five consecutive orbitals. The Haffnian states (and their quasiholes) are no longer Jack polynomials, but they can be uniquely determined using the local exclusion condition (LEC) formalism. There is thus still a one-to-one correspondence from $\mathcal{H}^H_{\text{topo}}$ and root configurations satisfying no more than two electrons for any six consecutive orbitals.

In both cases, the highest-density Gaffnian state and Haffnian state can be identified as the primary state, which we denote as $|\phi_{h_G}\rangle$ and $|\phi_{h_H}\rangle$, respectively. There is also an additional primary state with $h = 2,\ c = 0$, just like the Moore-Read case, so we will also denote with $|\tau_G\rangle$ and $|\tau_H\rangle$. There is, however, an important difference here. While $\mathcal{H}^M_{\text{topo}}$ only consists of descendant states of $|\tau_M\rangle$ with positive conformal norms, this is no longer the case for $\mathcal{H}^G_{\text{topo}}$ and $\mathcal{H}^H_{\text{topo}}$. The latter contain descendant states from $|\tau_G\rangle$ or $|\tau_H\rangle$ with both positive and negative conformal norms. In fact, based on extensive numerical evidence, we conjecture this is the case for the null space of all three-body interaction Hamiltonians of the following form (note there is no three-body pseudopotential $\hat{V}_4^{\text{3bdy}}$):

$$\hat{H}_{\text{3bdy}} = \sum_{k=3}^{k_0} \hat{V}_k^{\text{3bdy}}, \tag{15}$$

which has been extensively analyzed in Simon *et al.* [22]. In all these cases, the highest-density zero-energy state is the primary state with infinite conformal dimension in the thermodynamic limit and central charge 2ν, which we can collectively denote as $|\phi_{h_I}\rangle$. Its conformal family thus gives

TABLE I. Level counting of the CFT model $|\psi_{h_I}\rangle$ with $h = \infty,\ c = 2\nu$, and $|\tau\rangle$ with $h = 2,\ c = 0$. $\{\bar{\tau}\rangle\}$ denotes the collection of the primary and descendant states in the conformal family with positive conformal norms. The fifth column gives the upper bound of the counting. The sixth and seventh columns have the identical counting.

N	$\{\|\phi_{h_I}\rangle\}$	$\{\|\tau_I\rangle\}$	$\{\|\bar{\tau}_I\rangle\}$	$\{\|\phi_{h_I}\rangle \otimes \|\text{vac}\rangle, \|\phi_{h_I}\rangle \otimes \|\tau_I\rangle\}$	$\{\|\phi_{h_I}\rangle \otimes \|\text{vac}\rangle, \|\phi_{h_I}\rangle \otimes \|\bar{\tau}_I\rangle\}$	$\mathcal{H}^M_{\text{topo}}$	$\mathcal{H}^G_{\text{topo}}$	$\mathcal{H}^H_{\text{topo}}$
0	1	0	0	1	1	1	1	1
1	1	0	0	1	1	1	1	1
2	2	1	1	3	3	3	3	3
3	3	1	1	5	5	5	5	5
4	5	2	2	10	10	10	10	10
5	7	2	2	16	16	16	16	16
6	11	4	3	29	28	28	29	29
7	15	4	3	45	43	43	45	45
8	22	7	5	75	70	70	74	75
9	30	8	5	115	105	105	113	115
10	42	12	7	181	161	161	176	180

the full Virasoro counting that is linearly independent quantum mechanically with positive conformal norm. There is also one additional primary state with $h = 2, c = 0$, which we collectively denote as $|\tau_I\rangle$. All states at level N are given as follows:

$$\left|\phi_{h_\alpha,I}^{(N)}\right\rangle = \hat{L}_{-k_1}\hat{L}_{-k_2}\dots\hat{L}_{-k_n}\left|\phi_{h_I}\right\rangle \otimes |\text{vac}\rangle, \tag{16}$$

$$\left|\phi_{h_\beta,I}^{(N)}\right\rangle = \hat{L}_{-k'_1}\hat{L}_{-k'_2}\dots\hat{L}_{-k'_{n'}}\left|\phi_{h_I}\right\rangle \otimes \hat{L}_{-k''_1}\hat{L}_{-k''_2}\dots\hat{L}_{-k''_{n''}}|\tau_I\rangle, \tag{17}$$

where $N = \sum_{i=1}^{n} k_i = \sum_{i=1}^{n'} k'_i + \sum_{i=1}^{n''} k''_i + 2$, α, β are indices of states at level N, and both $|\phi_{h_\alpha,I}^{(N)}\rangle, |\phi_{h_\beta,I}^{(N)}\rangle$ are microscopic wave functions that can be explicitly obtained (in the LLL they are of holomorphic variables $z_1, z_2, \dots, z_{N_e}$).

The conformal family of $|\tau_I\rangle$ has singular descendant states (descendant states with zero conformal norm). By removing those singular states, $|\phi_{h_\alpha,I}^{(N)}\rangle, |\phi_{h_\beta,I}^{(N)}\rangle$ are linearly independent with respect to the conformal overlaps. However, $|\phi_{h_\alpha,I}^{(N)}\rangle, |\phi_{h_\beta,I}^{(N)}\rangle$ are overcomplete and linearly dependent with respect to the quantum mechanical overlap for finite k_0, so the counting of the linearly independent microscopic wave functions in the null spaces of Eq. (15) is bounded from above by the counting given by Eqs. (27) and (29) (after the singular fields are removed). We have checked explicitly for $k_0 \leqslant 8$.

The conformal family of $|\tau_I\rangle$ also contains fields with negative conformal norms because the associated CFT model ($h = 2, c = 0$) is nonunitary and irrational. However, if we denote $\{|\bar{\tau}_I\rangle\}$ as a subset of the conformal family that contains only states with positive conformal norm, then the remaining counting from Eqs. (27) and (29) (after removing the singular and negative norm states) agrees with the counting of $\mathcal{H}_{\text{topo}}^{M}$, which is the null space of Eq. (15) with $k_0 = 3$. As we increase k_0 to 5,6, etc. (corresponding to $\mathcal{H}_{\text{topo}}^{G}$, $\mathcal{H}_{\text{topo}}^{H}$, and so on), more and more states from Eq. (29) with negative conformal norm are needed to match the counting of the null space. We thus conjecture the full conformal counting (all negative conformal norm states) is needed to account for the counting of the null space in the limit of $k_0 \to \infty$.

One of the most important aspects of the FQH edge dynamics is the thermal Hall conductance [34,38,39], normally related to the central charge of the conformal edge theory and should be completely determined by the null space $\mathcal{H}_{\text{topo}}$. It seems in the description described here, the central charge of 2ν and 0 comes from the bulk properties at the center of the quantum Hall droplet, thus unrelated to what happens at the edge. However, the thermal Hall conductance is completely determined by the "density of states" or the counting of quasihole states in each angular momentum sector [34]. For the Laughlin and Moore-Read states, the counting can be completely determined by the $h = \infty, c = 2\nu$ and $h = 2, c = 0$ CFT models in the new description here. For "nonunitary" states like the Gaffnian and Haffnian states, we are not sure at this stage how the nonunitary counting of the $h = 2, c = 0$ model is gradually incorporated, as k_0 in Eq. (15) increases. Yet, this is also the case for the conventional CFT description with the nonunitary and/or irrational CFT models, as the central charge is different from the "effective central charge" [12] obtained from the counting of $\mathcal{H}_{\text{topo}}^{G}$ and $\mathcal{H}_{\text{topo}}^{H}$. The latter can only be obtained microscopically (i.e., from the properties related to the Jack polynomials).

In this sense, there is no obvious disadvantage with the new CFT description presented here. While we do not have a rigorous proof available, we believe the description here can be mapped to the usual CFT descriptions with minimal or irrational models. The results also show the strong relevance of the $h = 2, c = 0$ CFT model for the three-body projection Hamiltonians, including the Moore-Read, Gaffnian, and Haffnian states. It illustrates that the conformal invariance of the topological null space can be described from different perspectives with effective CFT models. Many physical features of the null space, which are related to the edge dynamics, however, depend on the quantum mechanical properties of the many-body wave functions that could be fundamentally different from the CFT descriptions with their own definitions of Hermitian conjugates, norms, and overlap, as well as quasihole correlations. After all, the Gaussian factors in the many-body wave functions play an important role for the quantum mechanical norm, overlaps, and thus linear dependence of those wave functions. Such Gaussian factors are not accounted for in the CFT descriptions, and they explicitly break conformal invariance with the presence of the magnetic length. We thus need to be more careful in characterizing the physical properties of the Gaffnian and Haffnian phases solely from the effective CFT descriptions.

IV. GAFFNIAN AND HAFFNIAN STATES AS ELEMENTARY EXCITATIONS

To further motivate the physical relevance of the Gaffnian and Haffnian phases to the gapped FQH systems, we look at the familiar Laughlin phase at $\nu = \frac{1}{3}$, and focus on its gapped elementary excitations (namely, the neutral and quasielectron excitations). Such excitations are well studied in the composite fermion picture and the Jack polynomial formalism [40–45]. The elementary low-lying neutral excitation is the magnetoroton mode. In the long-wavelength limit, it is a quadrupole excitation from the geometric deformation of the ground state. At large momenta, it is a dipole excitation consisting of a pair of separated Laughlin quasielectron and quasihole. The energy gap of the magnetoroton mode defines the incompressibility, and thus the robustness of the Hall plateau, of the Laughlin phase.

From the microscopic point of view, the best way to understand the physical properties of such gapped excitations is to construct good model wave functions. Unlike the null space of model Hamiltonians (the zero-energy ground states and quasiholes), in general, microscopic wave functions for the gapped excitations are nonuniversal, and different approaches lead to (slightly) different model wave functions. The magnetoroton mode is special in the sense that the composite fermion approach and the Jack polynomial formalism give exactly identical model wave functions. This is true for excitations consisting of only one Laughlin quasielectron (thus including single quasielectron states that are also gapped). For states containing more than one quasielectron, the two approaches yield different model wave functions, though their overlaps are generally quite high.

Given the uniqueness of the magnetoroton model wave functions, we show here that the quadrupole excitations are exact zero-energy states of the Haffnian model Hamiltonians, thus are states within $\mathcal{H}_{\text{topo}}^{H}$. In contrast, the dipole excitations, as well as the single Laughlin quasielectron states, are exact zero-energy states of the Gaffnian model Hamiltonian, thus living within $\mathcal{H}_{\text{topo}}^{G}$ [46,47]. To see that, let us write the root configuration of the magnetoroton modes as follows [40]:

$$110000100100100100\cdots \quad \Delta M=-2 \quad \in \mathcal{H}_{\text{topo}}^{H}, \tag{18}$$

$$110001000100100100\cdots \quad \Delta M=-3 \quad \in \mathcal{H}_{\text{topo}}^{G}, \tag{19}$$

$$110001001000100100\cdots \quad \Delta M=-4 \quad \in \mathcal{H}_{\text{topo}}^{G}, \tag{20}$$

$$110001001001000100\cdots \quad \Delta M=-5 \quad \in \mathcal{H}_{\text{topo}}^{G}, \qquad \vdots \tag{21}$$

Here, the solid and open circles beneath the digits indicate the locations of quasiparticles (of charge $e/3$, when three consecutive orbitals contain more than one electron) and quasiholes (of charge $-e/3$, when three consecutive orbitals contain fewer than one electron). Each root configuration represents a many-body wave function, where only bases "squeezed" from the root configuration have no-zero coefficients. These nonzero coefficients can also be uniquely determined using the method in [40], from which the model wave functions are obtained.

The easiest way to see that the $\Delta M=-2$ state is a zero-energy state of the Haffnian model Hamiltonian is that it satisfies the local exclusion condition (LEC) [48,49] of $\{4,2,4\}$ at the center of the disk (corresponding to the north pole of the sphere). Similarly, we know the $\Delta M<-2$ states are the zero-energy states of the Gaffnian model Hamiltonian because they satisfy the LEC of $\{2,1,2\}\vee\{5,2,5\}$. Note that near the filling factor of $\nu=\frac{1}{3}$, there are an extensive number of Haffnian or Gaffnian quasiholes.

Let us denote the subspace of Laughlin ground state and quasiholes (the null space of $\hat{V}_1^{\text{2bdy}}$ pseudopotential) to be $\mathcal{H}_{\text{topo}}^{L}$, then we clearly have the relationship that $\mathcal{H}_{\text{topo}}^{L}\in\mathcal{H}_{\text{topo}}^{H}\in\mathcal{H}_{\text{topo}}^{G}$. We can thus reinterpret the elementary neutral excitations of the Laughlin phase as the quantum fluids of interacting Haffnian or Gaffnian quasiholes [47]. Similarly, if we look at the model wave functions of a single Laughlin quasielectron, it has the following root configuration:

$$11000100100100100\cdots \quad \Delta M=-N_e/2 \quad \in \mathcal{H}_{\text{topo}}^{G}. \tag{22}$$

It is also a zero-energy state of the Gaffnian model Hamiltonian. For multiple quasielectrons that are far away from each other, they can all be considered as some locally bound states of Gaffnian quasiholes.

A. Gaplessness of the model Haffnian Hamiltonian

The model Hamiltonian of the Haffnian state is the special case of Eq. (15) with $k_0=6$. It in fact is given by a family of the following Hamiltonian:

$$\hat{H}_{\text{hf}}=\hat{V}_3^{\text{3bdy}}+\lambda_1\hat{V}_5^{\text{3bdy}}+\lambda_2\hat{V}_6^{\text{3bdy}} \tag{23}$$

with $\lambda_1,\lambda_2>0$. The null space of Eq. (23) is $\mathcal{H}_t^{H}$. One should note that if we define the the Laughlin ground states and quasiholes space (i.e., the null space of $\hat{V}_1^{\text{2bdy}}$) as $\mathcal{H}_{\text{topo}}^{L}$, we then have $\mathcal{H}_{\text{topo}}^{L}\in\mathcal{H}_{\text{topo}}^{H}$.

What we are able to show here is that the incompressibility of $\hat{V}_1^{\text{2bdy}}$ at $\nu=\frac{1}{3}$ (more specifically at $N_o=3N_e-2$, where N_o is the number of orbitals on the sphere or disk geometry) implies that Eq. (23) is gapless at $N_o=3N_e-4$ (where the Haffnian state is the highest-density zero-energy state), for any positive values of λ_1,λ_2. This is because Laughlin quasielectrons, which are orthogonal to the Haffnian ground state in the thermodynamic limit, do not have a finite-energy gap with Eq. (23).

The incompressibility of $\hat{V}_1^{\text{2bdy}}$ implies both the quadrupole and dipole excitations cost a finite amount of energy in the thermodynamic limit. We now look at a two-dimensional subspace spanned by two states of the following root configuration:

$$110000110000110000\cdots11000011, \tag{24}$$

$$11000100100100\cdots100100100011. \tag{25}$$

The first state of Eq. (24) is the Haffnian model state, which is the unique zero-energy state in $L=0$ with $N_o=3N_e-4$ from Eq. (23). The second state of Eq. (25) is the Laughlin state with two quasielectrons in the same quantum sector. It can be constructed using the method in [47]. The model state has very high overlap with the exact ground state of $\hat{V}_1^{\text{2bdy}}$ in $L=0$ with $N_o=3N_e-4$ [the arguments here also apply if we use this exact ground state in place of Eq. (25)]. Since the Haffnian state contains an extensive number of quadrupole excitations, its variational energy with respect to $\hat{V}_1^{\text{2bdy}}$ is also extensive. On the other hand, the variational energy of the two-quasielectron state of Eq. (25) is finite in the thermodynamic limit [see Fig. 3(b)], which is double the Laughlin charge gap at $\nu=\frac{1}{3}$.

We now look at the spectrum of Eq. (23) within this two-dimensional subspace and argue that the two energies have to be *degenerate* in the thermodynamic limit. The overlap of the two states (24) and (25) quickly decays with the system size [see Fig. 3(c)], so for all purposes we can treat them as the eigenstates in this subspace, with the Haffnian state as the zero-energy ground state. If Eq. (25) has a finite-energy gap in the thermodynamic limit, then we can consider perturbing Eq. (23) with an infinitesimal amount of $\hat{V}_1^{\text{2bdy}}$ as follows:

$$\hat{H}=\hat{H}_{\text{hf}}+\lambda\hat{V}_1^{\text{2bdy}}. \tag{26}$$

There will be a level crossing no matter how small λ is since the Haffnian state will have infinite energy in the thermodynamic limit, while Eq. (25) will have finite energy. This is not possible unless Eqs. (24) and (25) are degenerate at $\lambda=0$, implying that Eq. (23) is gapless in the thermodynamic limit. In another word, if $\hat{H}_{\text{hf}}$ is gapped in the thermodynamic limit, then an infinitesimally small perturbation can close the gap and lead to level crossing.

The argument can be generalized as follows. Let $\hat{H}_1,\hat{H}_2$ be two local Hamiltonians with the null spaces $\mathcal{H}_1,\mathcal{H}_2$, respectively. Let $|\psi_1\rangle\in\mathcal{H}_1,|\psi_2\rangle\in\mathcal{H}_2$ be the highest-density states (with densities $\rho_1\geqslant\rho_2$) in their respective null spaces.

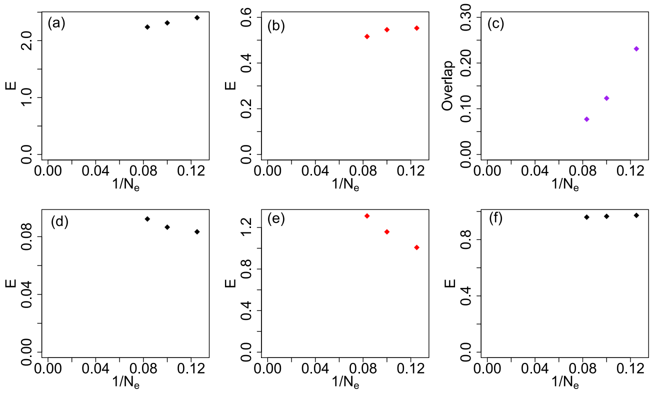

FIG. 3. The x axis is the inverse of system size (number of electrons), and the y axis represents (a) variational energy of the two-quasielectron Laughlin state with respect to $\hat{H}_{\mathrm{hf}}$; (b) variational energy of the two-quasielectron Laughlin state with respect to $\hat{V}_1^{2\mathrm{bdy}}$; (c) overlap of the two-quasielectron Laughlin state and the Haffnian model state; (d) variational energy of the $\hat{V}_{\mathrm{LLL}}$ ground state at $\nu = \frac{2}{5}$ with respect to $\hat{H}_{\mathrm{gf}}$; (e) variational energy of the $\hat{V}_{\mathrm{LLL}}$ ground state at $\nu = \frac{2}{5}$ with respect to $\hat{V}_1^{2\mathrm{bdy}}$; (f) overlap between the $\hat{V}_{\mathrm{LLL}}$ ground state at $\nu = \frac{2}{5}$ and the Gaffnian state.

If there exists a state $|\psi_1'\rangle$ with density ρ_1' and finite values $\Delta_1 \geqslant 0$, $\Delta_2 \geqslant 0$ such that following conditions are satisfied,

$$\rho_1' \geqslant \rho_1, \tag{27}$$

$$\lim_{N_e\to\infty} \langle\psi_1'|\hat{H}_1|\psi_1'\rangle = \Delta_1, \tag{28}$$

$$\lim_{N_e\to\infty} \langle\psi_1|\hat{H}_2|\psi_1\rangle \to \infty, \tag{29}$$

$$\lim_{N_e\to\infty} \langle\psi_1'|\hat{H}_2|\psi_1'\rangle = \Delta_2, \tag{30}$$

then we have

$$\lim_{N_e\to\infty} \langle\psi_1|\hat{H}_1|\psi_1\rangle = \lim_{N_e\to\infty} \langle\psi_1'|\hat{H}_1|\psi_1'\rangle \tag{31}$$

implying $\hat{H}_1$ is gapless with neutral excitations (if $\rho_1 = \rho_1'$) or charged excitations (if $\rho_1 < \rho_1'$). Note that $|\psi_1\rangle$ and $|\psi_1'\rangle$ are two eigenstates of $\hat{H}_1$. If their energies are gapped in the thermodynamic limit, an infinitesimally small amount of perturbation of $\hat{H}_2$ to $\hat{H}_1$ will close the gap if Eqs. (27)–(30) are satisfied. A simple schematic illustration of the argument can be found in Fig. 4.

We do not require $\hat{H}_2$ to be gapped in the above arguments. However, in the case of $\hat{H}_1 = \hat{H}_{\mathrm{hf}}, \hat{H}_2 = \hat{V}_1^{2\mathrm{bdy}}$, we have $\rho_1 > \rho_2$ but $\lim_{N_e\to\infty}(\rho_2 - \rho_1) = 0$. Thus, we can find the states $|\psi_1'\rangle$ with $\rho_1' \geqslant \rho_1$ such that they are the Laughlin ground state plus a finite number of Laughlin quasielectrons, which are exponentially localized excitations. These states satisfy Eqs. (27), (28), and (30) (note the Laughlin ground state is also in the null space of $\hat{H}_{\mathrm{hf}}$). From the fact that $\hat{V}_1^{2\mathrm{bdy}}$ is gapped, we know its quadrupole excitation is gapped in the thermodynamic limit. Given $|\psi_1\rangle$, or the Haffnian state, contains an extensive number of quadrupole excitations, Eq. (29) is also satisfied. Thus, $\hat{H}_{\mathrm{hf}}$ has both gapless neutral and charged excitations and is compressible. All states that physically represent the Laughlin ground state with a finite number of quasielectrons are degenerate with the Haffnian

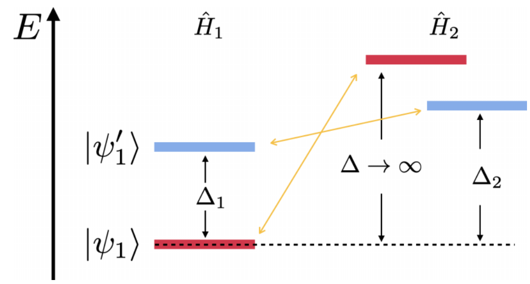

FIG. 4. We can set the energy of $|\psi_1\rangle$ with respect to $\hat{H}_1$ to be zero, so that of $|\psi_1'\rangle$ is Δ_1. The variational energy of $|\psi_1'\rangle$ with respect to $\hat{H}_2$ is Δ_2, while that of $|\psi_1\rangle$ is Δ, which goes to infinity in the thermodynamic limit. An infinitesimally small perturbation of $\hat{H}_2$ to $\hat{H}_1$ will lead to a level crossing between the two states indicated by the yellow arrow.

ground state in the thermodynamic limit if the interaction Hamiltonian is $\hat{H}_{\mathrm{hf}}$.

B. Gaffnian state at $\nu = \frac{2}{5}$

The argument above does not apply to the Gaffnian state because there are no known local Hamiltonians with well-defined null spaces nearby $\hat{H}_{\mathrm{gf}}$, playing the role of $\hat{V}_1^{2\mathrm{bdy}}$ to $\hat{H}_{\mathrm{hf}}$. From the three-body interactions, we know that $\bar{\mathcal{H}}_{\mathrm{topo}}^M$ contains states with clusters of three particles having total relative angular momentum 3. The Hilbert space of $\mathcal{H}_{\mathrm{topo}}^M\backslash\mathcal{H}_{\mathrm{topo}}^G$ contains states with clusters of three particles having total relative angular momentum 5. The Hilbert space of $\mathcal{H}_{\mathrm{topo}}^G\backslash\mathcal{H}_{\mathrm{topo}}^H$ contains states with clusters of three particles having total relative angular momentum 6. These subspaces will thus be affected by individual three-body pseudopotentials differently as shown in Fig. 5. In Fig. 5, it is generally believed that $\lim_{N_e\to\infty}\Delta_M$ is finite, and we have argued that $\lim_{N_e\to\infty}\Delta_H = 0$, with the gap closing from excitations in the subspace of $\mathcal{H}_{\mathrm{topo}}^G\backslash\mathcal{H}_{\mathrm{topo}}^H$ (although there could be other gap-

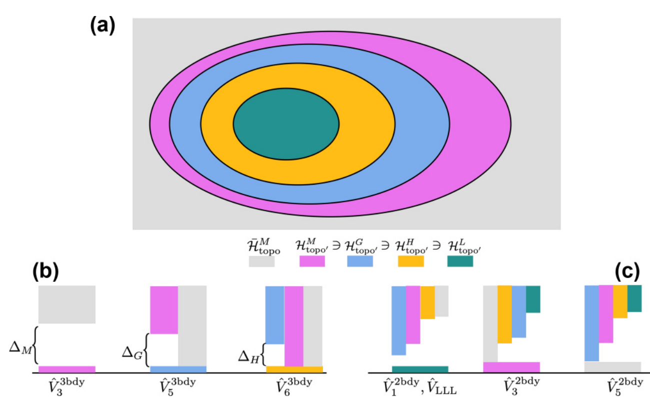

FIG. 5. (a) The relationship between different Hilbert spaces defined in the main text. (b) The variational energies of different Hilbert spaces with three-body pseudopotentials. (c) The variational energies of different Hilbert spaces with two-body interactions. The variational energies are computed numerically from finite-size systems. While the relative strength of different Hilbert spaces is consistent for different system sizes, they should be just indicative of what could happen in the thermodynamic limit.

less modes). Thus, for $\hat{H}_{\rm gf}$ to be gapless, with positive $\hat{V}_3^{\rm 3bdy}$ the gapless mode can only be in the subspace of $\mathcal{H}_{\rm topo}^M\backslash\mathcal{H}_{\rm topo}^G$, which in particular are the zero-energy states of $\hat{V}_3^{\rm 3bdy}$. One competing state for the Gaffnian state is the Abelian Jain state at the same filling factor and topological shift, with the following root configuration:

$$11001001010010100100\dots10010011 \tag{32}$$

which is unsqueezed from the root configuration of the Gaffnian state. The Jain state *cannot* be uniquely determined by any known local operators, and there are several highest-weight states [50] supported by the basis squeezed by Eq. (32) (in contrast, there is a unique highest-weight state supported by the basis squeezed from the Gaffnian root configuration). For finite systems, the Jain state has very high overlap with the Gaffnian state, suggesting the basis *unsqueezed* from the Gaffnian root configuration but *squeezed* from Eq. (32) plays a minor role. The Jain root configuration also shows that the Jain state contains two Gaffnian neutral excitations (one at the north pole, the other at the south pole).

Let us denote the Jain state as $|\psi_{J,2/5}\rangle$. From the root configuration, the basis of the Jain state clearly satisfies the LEC condition of $\{3, 2, 3\}$, it is thus a zero-energy state of $\hat{V}_3^{\rm 3bdy}$, i.e., $|\psi_{J,2/5}\rangle \in \mathcal{H}_{\rm topo}^M$. It is generally believed at the critical point [33,51,52] of $\hat{H}_{\rm gf}$, $|\psi_{J,2/5}\rangle$ and $|\psi^G_{\rho_{\max}}\rangle$ are degenerate in the thermodynamic limit (the latter is the Gaffnian state), based on the CFT conjecture. It is thus natural to consider the possibility of letting $|\psi_{J,2/5}\rangle$ play the role of $|\psi_1'\rangle$, with $|\psi^G_{\rho_{\max}}\rangle = |\psi_1\rangle$ in Eqs. (27) to (30). Even though there is no known model Hamiltonian for the Jain state, we note that the arguments above do not require $\hat{H}_2$ to be a local Hamiltonian. We can thus assume the existence of $\hat{H}_2$ such that Eq. (30) is satisfied.

However, it is not clear if such $\hat{H}_2$ can satisfy Eq. (29), given that $|\psi_{J,2/5}\rangle$ and $|\psi^G_{\rho_{\max}}\rangle$ have very high overlap for finite systems (comparable to the overlap between the Laughlin model state and the LLL Coulomb interaction ground state). A more serious issue is the strong numerical evidence against Eq. (28). From Fig. 3(d) we see that the variational energy of the ground state with the *lowest Landau level Coulomb interaction* with the Gaffnian model Hamiltonian seems to be extensive. This is qualitatively the same if the exact Jain $\frac{2}{5}$ state is used [53], i.e., $\langle\psi_{J,2/5}|\hat{H}_{\rm gf}|\psi_{J,2/5}\rangle \sim O(N_e)$. While we can only access relatively small system sizes here, the numerics does suggest that as far as the ground-state properties are concerned, the Gaffnian state and the Jain state seem to be indistinguishable topologically. We will discuss about the universal properties of their respective quasihole states in the next section.

V. CASE OF $\lambda > 0$

We now move on to more realistic interactions with $\lambda > 0$ in Eq. (1). Several possible scenarios can happen as we outlined in Sec. I, and we look at these possibilities with the particular focuses at filling factor $\nu = \frac{2}{5}$ (where the Gaffnian state is located) and $\nu = \frac{1}{3}$ (where the Haffnian state is located). When we move away from the model Hamiltonian, the conformal invariance of the null spaces is broken, so in principle the connection to the CFT models is no longer valid. The two well-known FQH phases at these two filling factors are the Abelian Jain state and the Laughlin state. We would like to show, however, the physical relevance of the null spaces (i.e., quasihole subspaces) at these two filling factors. In particular, the goal is to see if the observed experimental data can be explained using their respective quasihole subspaces, and what new experimental results we can predict from the perspective of quasihole subspaces.

A. Gaffnian and Jain phases

One important question to ask is if the Gaffnian state and the Jain ground state at $\nu = \frac{2}{5}$ are topologically equivalent: that any topological indices computed from these two microscopic wave functions are identical [14]. If $\hat{H}_G$ is gapped in the thermodynamic limit in the $L = 0$ sector, then the statement has to be true even if the gap closes in some other L sector. This is because the two states are adiabatically connected, and without gap closing in $L = 0$ sector any physical properties computed from the two states have to go smoothly from one value to another, implying the invariance of topological indices.

We now assume $\hat{H}_G$ is gapless in the thermodynamic limit in the $L = 0$ sector. Since we know $\hat{V}_3^{\rm 3bdy}$ gaps out $\bar{\mathcal{H}}_{\rm topo}^M$, the gap can only close within $\mathcal{H}_{\rm topo}^M$, which is where the Jain trial wave function from the composite fermion construction is located. Let us define $\mathcal{H}_{\rm topo'}^M = \mathcal{H}_{\rm topo}^M\backslash\mathcal{H}_{\rm topo}^G$, then the Gaffnian ground state $|\psi_{0,G}\rangle$ is orthogonal to $\mathcal{H}_{\rm topo'}^M$. The main question here is which subspace describes the low-energy physics as λ increases from zero. Let $\{|\psi_M\rangle\} \in \mathcal{H}_{\rm topo}^M$ be the set of states degenerate with $|\psi_{0,G}\rangle$ in the thermodynamic limit. If we perturb $\hat{H}_G$ with an infinitesimal amount of $\hat{V}_1^{\rm 2bdy}$, we can apply the degenerate perturbation theory to the first order, and diagonalize $\hat{V}_1^{\rm 2bdy}$ in the subspace of $\{|\psi_M\rangle\} \cup |\psi_{0,G}\rangle$. Since $\{|\psi_M\rangle\}$ is orthogonal to $|\psi_{0,G}\rangle$, we expect the diagonal matrix elements to be extensive (thus going to infinity), while matrix elements between $\{|\psi_M\rangle\}$ and $|\psi_{0,G}\rangle$ to vanish, in the limit of large system sizes. From Fig. 5 we expect $\langle\psi_{0,G}|\hat{V}_1^{\rm 2bdy}|\psi_{0,G}\rangle < E_M$, where E_M is the ground state of $\hat{V}_1^{\rm 2bdy}$ within $\mathcal{H}_{\rm topo'}^M$. We expect this to be true in the thermodynamic limit. Thus, with the following Hamiltonian

$$\hat{H} = (1-\lambda)\hat{H}_G + \lambda\hat{V}_1^{\rm 2bdy} \tag{33}$$

there is no level crossing between $|\psi_{0,G}\rangle$ and $\mathcal{H}_{\rm topo'}^M$ when we increase λ from 0. Note that at $\lambda = 0$, even if the Hamiltonian is gapless, the variational energies of $\hat{H}_G$ in $\mathcal{H}_{\rm topo'}^M$ can only approach zero asymptotically. Since it is generally believed that the ground state of Eq. (33) with $\lambda > 0$ is adiabatically connected to the Abelian Jain state [51], we argue that the Gaffnian *ground state* is indeed adiabatically connected, and thus topologically equivalent, to the Jain *ground state* from the composite fermion construction.

1. Thermal Hall effect and the quasihole bandwidth

The arguments above suggest that as far as the ground states are concerned, the Gaffnian phase and the Jain phase are topologically equivalent. The two phases, however, are *not* topologically equivalent with regard to the universal properties of the low-lying excitations, which in particular dictates the (non-)Abelian-ness of the FQH phase. For a dilute gas of Gaffnian quasiholes, we also expect no level crossing between $\mathcal{H}^G_{\text{topo}}$ and $\mathcal{H}^M_{\text{topo}'}$ as λ increases. We thus conjecture statement 1(b) (from the Introduction) should capture the adiabatic tuning from the Gaffnian model Hamiltonian to the realistic short-range interaction that attributes to the Hall plateau observed at $\nu = \frac{2}{5}$. We do not expect to see non-Abelian braiding of the quasiholes because with realistic interaction a large bandwidth of the quasihole manifold will develop, lifting the required degeneracy [54].

This effect should also be reflected in thermal quantum Hall measurement [38,39], which is given by the heat capacity of the chiral edge at the boundary of the quantum Hall fluid [34]. Let the partition function of the 1D edge system be given as follows:

$$\mathcal{Z} = \sum_{N=0}^{\infty} g(N,\beta)e^{-\beta\epsilon_N}, \tag{34}$$

where $\beta = 1/k_BT$, and $g(N, 0)$ is the number of the quasihole states at level N, i.e., the density of state as a function of the angular momentum. The "kinetic energy" ϵ_N accounts for the energy of the states at the same level, which can be contributed from the confining potential of the quantum Hall droplet near the edge. Let N have the unit of angular momentum, for general cases we can expand it as follows:

$$\epsilon_N = c_1N + c_2N^2 + \cdots, \tag{35}$$

where $c_1 = \frac{v_F}{2\pi L}$, with v_F the Fermi velocity and L the circumference of the droplet. At finite temperature we have

$$g(N,\beta) = \sum_{\alpha} e^{-\beta\epsilon_{N,\alpha}}, \tag{36}$$

where α is the index of the states in a single level, and $\epsilon_{N,\alpha}$ are the additional energy costs from the creation energies of quasiholes as well as interaction between quasiholes. We can also absorb the nonlinearity of the kinetic energy into the density of the states part, and rewrite the partition function as follows:

$$\mathcal{Z} = \sum_{N=0}^{\infty} \tilde{g}(N,\beta,c_2)e^{-\beta\epsilon_N}, \tag{37}$$

$$\tilde{g}(N,\beta,c_2) = \sum_{\alpha} e^{-\beta(\epsilon_{N,\alpha}+c_2N^2+\cdots)}. \tag{38}$$

Conformal invariance implies $\tilde{\epsilon}_{N,\alpha} = \epsilon_{N,\alpha} + c_2N^2 + \cdots = 0$ identically. With this assumption, the thermal Hall conductance is given by $\kappa = \frac{c\pi^2k_B^2T}{3h}$, where c is the central charge of chiral Luttinger liquid. In the composite fermion picture, the Jain state consists of two occupied CF levels analogous to the IQHE with $\nu = 2$, so the central charge is $c = 2$. One should note that even with the assumption of conformal invariance, the value of $c = 2$ is not yet supported by the microscopic CF theory. This is because the quasihole excitations from each CF level with the CF construction are not orthonormal with each other, and there are missing states after the projection into the LLL [14,42]. These missing states will effectively reduce the density of states at each momentum sector. The effective field theories from the CF construction, on the other hand, are generally formulated from the CF theory before LLL projection, ignoring the missing states. Thus, strictly speaking the effective field theories from the CF construction predict an upper bound of $c = 2$ for the thermal Hall conductance. If we do not consider the nonuniversal factors in actual experiments that break the chiral Luttinger liquid description of the quantum Hall edge, the actual central charge from the thermal Hall conductance measurement should also be bounded above by $c = 2$.

For the Gaffnian model Hamiltonian, the null space has an effective central charge $c = 1 + 3/5$ from its Virasoro counting, which can be computed from Eq. (37) by taking $\tilde{g}(N,\beta,c_2) = \tilde{g}(N,0,0)$. It is important to note this effective central charge is different from the negative central charge predicted from the $\mathcal{M}(5,3)$ minimal model conventionally associated with the Gaffnian phase [12,34]. The negative central charge (thus the thermal Hall conductance) comes from the unphysical negative conformal norms. Such contributions have to be corrected since all physical quasihole states have positive quantum mechanical norm and will contribute positively to the thermal Hall conductance.

With the realistic interaction and confining potential, conformal invariance is explicitly broken. This is reflected by $\tilde{\epsilon}_{N,\alpha} \neq 0$, which effectively modifies the density of state $\tilde{g}(N,\beta,c_2)$. From this perspective, the thermal Hall conductance is only universal in the presence of conformal symmetry, when it is independent of the Fermi velocity v_F. We thus believe while the composite fermion description is a good effective theory at $\nu = \frac{2}{5}$, the thermal Hall conductance will not in general be quantized with $c = 2$. It can be computed by assuming all edge excitations are Gaffnian quasiholes. The actual value, however, will depend strongly on the realistic interaction, which is known to split the quasihole bands due to the nonzero creation energies of quasiholes [54].

Experimentally, the $\nu = \frac{2}{5}$ plateau is observed in the lowest Landau level, where the Coulomb interaction is more long ranged as compared to $\hat{V}_1^{\text{2bdy}}$. Thus, apart from the confining potential, insertion of the fluxes and the creation of quasiholes will cost negative amount of energy, i.e., $\epsilon_{N,\alpha} < 0$. In the long-wavelength limit if we ignore the nonlinearity of the confining potential (i.e., $c_2 = 0$), then we have $\tilde{\epsilon}_{N,\alpha} < 0$ and $\tilde{g}(N,\beta,c_2) > \tilde{g}(N,0,0)$. A higher density of state leads to an increase of the edge heat capacity, and we thus expect $\kappa > \frac{8}{5}\frac{\pi^2k_B^2T}{3h}$. In contrast, stronger confining potential [e.g., sharper edge of the two-dimensional electron gas (2DEG) in the experiments] generally leads to larger c_2 and reduced effective density of states, leading to the suppression of the thermal Hall conductivity. These are some qualitative behaviors we can predict about the experiments based on the simple analysis here, and more detailed calculations will be presented elsewhere.

2. *Quasihole tunneling and shot noise*

In addition to the thermal Hall conductance, we can also explore the topological nature of the quantum Hall fluid at $\nu = \frac{2}{5}$ by looking at charge tunneling between counterpropagating edges at the quantum point contact (QPC). In the composite fermion picture, quantum fluid consists of two CF levels in analogy to the two LLs of the IQHE, contributing to the two tunneling channels at the QPC. Like the IQHE, the outer channel has full transmission since it is further apart, and backscattering mainly comes from the inner channel, with the charge carriers each carrying the charge of $q = e/5$. This can be extracted from the relationship of $S = 2qI_B$, where S is the spectral function of the shot noise, and I_B is the backscattering current [55,56].

While the tunneling of $e/5$ charge carriers has been confirmed in a number of experiments, it has also been discovered at very low temperature: the tunneling charge is $2e/5$ instead of $e/5$ [56]. This rather interesting phenomenon illustrates the richness of edge dynamics at $\nu = \frac{2}{5}$, that cannot be readily explained using the composite fermion theory. Here, we show this phenomenon can be naturally explained by looking into the dynamics of the Gaffnian quasiholes, which are noninteracting with $\hat{H}_{\mathrm{gf}}$, but are no longer the case with realistic interactions.

If we insert one magnetic flux to the Gaffnian ground state and create two Gaffnian quasiholes, the states from the following two root configurations are degenerate with $\hat{H}_{\mathrm{gf}}$:

$$ {}_{\circ\circ}01100011000\ldots1100011, \tag{39}$$

$$ {}_{\circ}1010010100\ldots10100101_{\circ}, \tag{40}$$

where in Eq. (39) we have two Gaffnian quasiholes forming a bound state with charge $2e/5$, piled at the north pole, in the $L_z = N_e/2$ sector. In contrast, Eq. (40) is the state with two unbounded Gaffnian quasiholes, one at the north pole and the other at the south pole, in the $L_z = 0$ (for N_e even) or $L_z = 1$ (for N_e odd) sector. Here, we have a Gaffnian quasihole when for five consecutive orbitals, we have one (instead of two) electron, as determined by the admission rule for the Gaffnian ground state. For quasiholes at the north or south pole, the number and the location of the quasiholes can be determined by the position of the inserted flux and the symmetry of the root configuration. We can also apply the admission rule by embedding the root configuration in the ground-state fluid [e.g., for Eq. (40), the quasiholes can be located with the admission rule for the configuration of $\ldots 110001100010100010100\ldots 101001010001100011\cdots$]. With a realistic interaction, we can evaluate their corresponding variational energy to determine if it is more energetically favorable for the two quasiholes to be bounded or unbounded.

From Fig. 6 we can see from finite-size analysis that $\hat{V}_1^{\mathrm{2bdy}}$ prefers bound quasihole states, while $\hat{V}_3^{\mathrm{2bdy}}$ prefers unbound quasihole states. Realistic interactions such as LLL Coulomb interaction (i.e., $\hat{V}_{\mathrm{LLL}}$) is known to be quite close to $\hat{V}_1^{\mathrm{2bdy}}$, we thus expect it to prefer bound quasihole states as well, as supported by numerical evidence. This also implies the two Gaffnian quasiholes can pull away from each other at finite temperature. Given that the energy difference between the bound and unbound quasihole states seems quite small with $\hat{V}_{\mathrm{LLL}}$, in realistic samples their separation can be quite large, leading to tunneling of single quasihole from one edge to another if the QPC is narrow.

Thus, the tunneling at the QPC can be illustrated in Fig. 7. At the same temperature, there will always be a higher density of bound quasiholes with charge $2e/5$, as compared to (loosely) unbounded ones carrying the charge of $e/5$. Let the density of the $2e/5$ quasiholes available for tunneling be $n_{2e/5}$, and the density of the $e/5$ quasiholes available for tunneling be $n_{e/5}$. We thus have

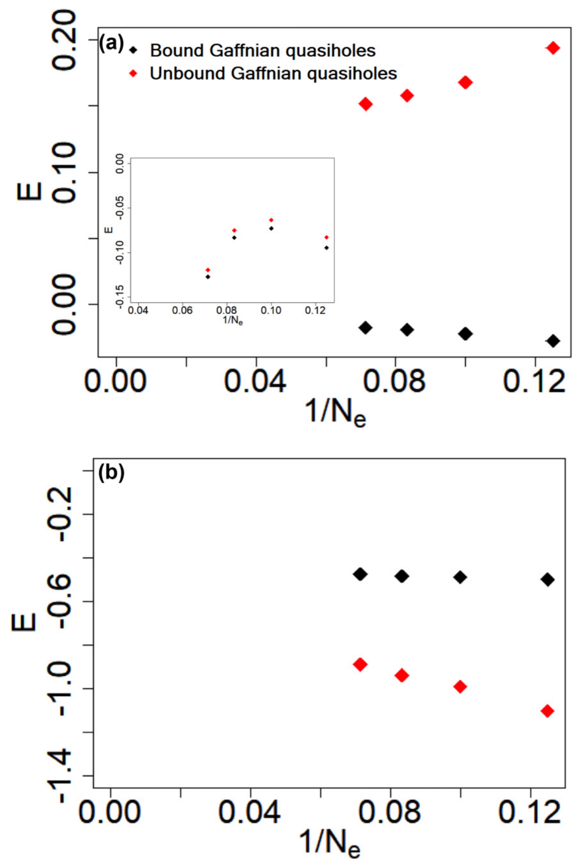

FIG. 6. The energy cost of creating a pair of bounded (black plot) and unbounded (red plot) Gaffnian quasiholes, after inserting one magnetic flux to the ground state. The x axis is the inverse of the system size. (a) $\hat{V}_1^{\mathrm{2bdy}}$ interaction; (b) $\hat{V}_3^{\mathrm{2bdy}}$ interaction. The inset shows the same plots with $\hat{V}_{\mathrm{LLL}}$ Coulomb interaction, with greater finite-size effects. For both $\hat{V}_1^{\mathrm{2bdy}}$ and $\hat{V}_{\mathrm{LLL}}$ interactions, the unbounded pair of quasiholes has higher variational energy.

$$\frac{n_{e/5}}{n_{2e/5}} \sim e^{-\beta\delta E}, \tag{41}$$

where δE is the characteristic energy difference between bounded and unbounded Gaffnian quasiholes. On the other hand, as shown in Fig. 7, the $e/5$ excitations, being at higher energy, have shorter tunneling distance. In general, the tunneling amplitude, given by the overlap of the (localized) edge excitations, is suppressed exponentially by the state separation. Thus, at low temperature, there are predominantly $2e/5$ excitations at the edge, leading to the shot-noise experiment detecting the quantized charge of $2e/5$. As temperature increases, a substantial amount of $e/5$ excitations are present, which dominates the tunneling process since their tunneling amplitude is much larger as compared to the $2e/5$ excitations, when the quantum point contact is narrow. Thus, there will be a crossover as temperature increases, and the

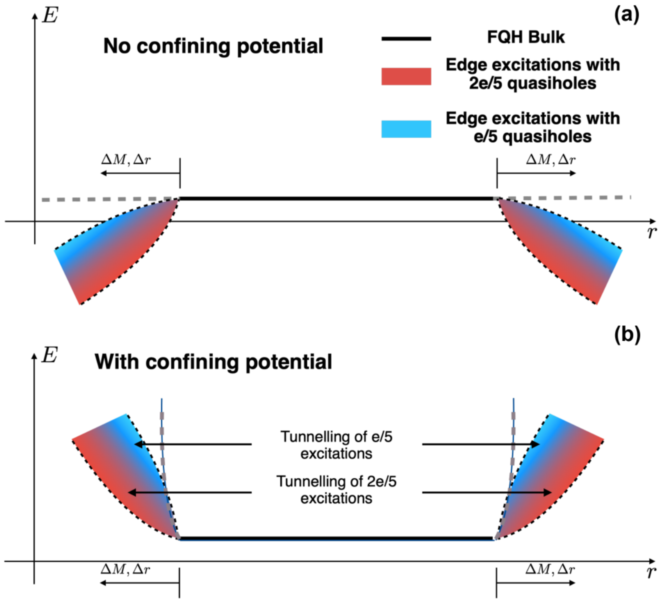

FIG. 7. Schematic distribution of the $e/5$ and $2e/5$ density at the edge of $\nu = \frac{2}{5}$ FQH fluid, based on the numerical analysis with short-range interactions. (a) Without confining potential; (b) with confining potential.

shot-noise measurement will detect mostly quantized charge of $e/5$. This qualitative argument agrees with the experimental observation quite well, and a detailed quantitative analysis with experimental parameters will be performed elsewhere.

B. Haffnian physics at $\nu = \frac{1}{3}$

We have already shown that $\hat{H}_{\mathrm{hf}}$ is gapless in the thermodynamic limit with microscopic arguments that are not readily applicable to the Gaffnian model Hamiltonian. One can then wonder if it is possible to perturb $\hat{H}_{\mathrm{hf}}$ so that we can have a gapped ground state inheriting some of the topological properties of the Haffnian model state, like what we have shown for the Gaffnian phase. From Fig. 5, however, we can see that any small perturbation by short-range interactions likely leads to level crossing between $\mathcal{H}^H_{\mathrm{topo}'}$ and $\bar{\mathcal{H}}^H_{\mathrm{topo}'}$ in the thermodynamic limit. For interactions dominated by $\hat{V}_1^{\mathrm{2bdy}}$, the ground-state and low-lying excitations at $N_o = 3N_e - 4$ are Laughlin quasielectrons. Increasing $\hat{V}_3^{\mathrm{2bdy}}$ pushes up the Laughlin quasielectrons, but the low-lying excitations are dominated by $\bar{\mathcal{H}}^M_{\mathrm{topo}}$, which is contained in $\bar{\mathcal{H}}^H_{\mathrm{topo}'}$. Thus, in simple realistic systems, we do not expect gapped ground states that are topologically equivalent to the Haffnian state, in contrast to the Gaffnian case.

This, however, does not rule out the possibility of such a gapped phase. If we assume that $\hat{H}_{\mathrm{hf}}$ gaps out $\bar{\mathcal{H}}^M_{\mathrm{topo}}$, which is reasonable, we then need a local Hamiltonian to gap out both $\mathcal{H}^M_{\mathrm{topo}'}$ and $\mathcal{H}^G_{\mathrm{topo}'}$ for $N_o \leqslant 3N_e - 4$, for us to have a gapped phase topologically equivalent to the Haffnian model state, and thus distinct from the Laughlin phase. The topological phase transition from this Haffnian-type phase to the Laughlin phase is also accompanied by the fractionalization of the Laughlin quasiholes into "Haffnian" quasiholes [47] carrying charge of $e/6$. Given that $\mathcal{H}^L_{\mathrm{topo}} \in \mathcal{H}^G_{\mathrm{topo}}$, each Laughlin quasihole can be understood as a bound state of two Gaffnian quasiholes. In the Laughlin phase, the bound state is energetically favorable. Pulling the two Gaffnian quasiholes apart costs an energy that is proportional to the distance between them. In the Haffnian-type phase, the unbounded Gaffnian quasiholes are energetically favorable and they emerge as "Haffnian" quasiholes (note that each Gaffnian quasihole carries the charge of $e/6$ at $\nu = \frac{1}{3}$, and $\mathcal{H}^H_{\mathrm{topo}} \in \mathcal{H}^G_{\mathrm{topo}}$).

If no Haffnian-type gapped phase is possible for any local Hamiltonian, then the only known topological phase at $\nu = \frac{1}{3}$ for a single-fermion species is the Laughlin phase. Even in this case, $\mathcal{H}^H_{\mathrm{topo}}$ can still play a relevant physical role. It is the Hilbert space of the quadrupole excitations of the Laughlin phase, and there are both numerical and experimental evidence that the quadrupole excitations can go soft in the Laughlin phase [46,47,57–60]. As long as the charge gap is maintained, the quantum phase still has a robust Hall conductance plateau, though finite-temperature transport can be modified by the quadrupole excitations, leading to the so-called nematic FQHE phase. If the quadrupole excitations become gapless, this could imply that the Laughlin state is compressible and degenerate with the Haffnian state, which is a uniform gas of quadrupole excitations. Thus, the likely scenario in the experiment is a small energy gap of the quadrupole excitation as compared to the finite temperature. The Laughlin state is still incompressible due to the presence of the charge gap, and that the Haffnian state has an extensive energy gap.

Another possible scenario is for the quadrupole excitation gap to scale as $\sim 1/N_e$, so it becomes gapless in the thermodynamic limit. There is now a finite variational energy gap of the Haffnian model state, which contains $\sim N_e$ number of quadrupole excitations. More interesting, there will be a mode with linear dispersion, with the energy scaling linearly with the number of quadrupole excitations. Given that the gaplessness of the quadrupole gap leads to the nematic FQH phase, this linear mode has been described in the effective field theory as the gapless nematic Goldstone mode [47,59]. One should note that in the thermodynamic limit, a single quadrupole excitation does not have any density modulation, which is unlike the dipole excitations at large momenta. Thus indeed, this linear dispersion in the long-wavelength limit does not come from the density fluctuation, but from the spatial modulation of the nematic director. It would be interesting to see if this Goldstone mode can be measured experimentally, as it is not clear if it can be realized with realistic interactions.

VI. SUMMARY AND DISCUSSIONS

Using the null space of model Hamiltonians as the preferred degrees of freedom, in this work we argue that the Gaffnian and Haffnian states (as well as their quasihole states) have rich physical properties that can play interesting roles in familiar and exotic gapped FQH systems. The model Hamiltonians of the Gaffnian and the Haffnian state may be gapless in the thermodynamic limit, and we have provided in this work a microscopic argument that the Haffnian model Hamiltonian is indeed gapless against Laughlin quasielectron excitations.

On the other hand, the quasihole subspaces are still spanned by well-defined microscopic many-body wave functions. In particular, we show at $\nu = \frac{2}{5}$ the Gaffnian quasihole subspace can explain many, if not all, of the topological and nonuniversal behaviors observed in the experiments with the introduction of the realistic Hamiltonian. This reinforces the previous notion that the Gaffnian formalism and the composite fermion picture are describing the two sides of the same coin [14,54] at $\nu = \frac{2}{5}$. With realistic interaction there is no bulk-edge correspondence. The Gaffnian ground state and the Jain ground state are argued to be topologically equivalent, while the Gaffnian quasihole manifold is split into bands with Coulomb interactions. The dynamics of Gaffnian quasiholes can explain the shot-noise and tunneling experiments, and its prediction of the dependence of the thermal Hall conductance with different tuning parameters can also be checked in experiments.

Both the Gaffnian and Haffnian quasiholes play important roles in the low-lying excitations of the Laughlin phase. The quadrupole excitations, which are neutral, are made of Haffnian quasiholes. In contrast, the dipole and quasielectron excitations are made of Gaffnian quasiholes. The energetic competitions between the Haffnian and Gaffnian quasiholes thus give a unifying description of the dynamics of the Laughlin phase at the finite temperature. These include the nematic FQH phase [46,47,57–60], which is a topological phase with nontrivial geometric properties, as well as potential fractionalization of the Laughlin quasiholes at finite temperature [47]. We also show that with realistic two-body interactions, there is generally no Haffnian-type ground state similar to the case of the Gaffnian state. This is because the Haffnian quasiholes have very high variational energy as compared to other sub-Hilbert spaces with known realistic interactions. It is still interesting to explore if there exist local Hamiltonians that gap out all other sub-Hilbert spaces from the Haffnian quasiholes, so as to realize an incompressible Haffnian-type phase with a distinct topological shift as compared to the Laughlin phase at $\nu = \frac{1}{3}$.

It is also important to understand how the results in this work reconcile with the arguments from the effective CFT description, regarding the relevance of the Gaffnian and Haffnian states to gapped FQH phases. We would first like to note that, strictly speaking, the results proposed in this work do not contradict the CFT arguments. In those arguments, the gaplessness of the model Hamiltonians for the Gaffnian and Haffnian states requires the fundamental assumption of the conformal invariance of their respective null spaces. Realistic interactions that break the conformal symmetry can still retain topological properties of some (may not be all) of those from the model Hamiltonians. Moreover, the CFT arguments do not prevent the Gaffnian and Haffnian quasiholes to be the useful degrees of freedom for low-lying excitations of other incompressible FQH phases.

Moreover, we have shown an alternative derivation of the conformal invariance of model Hamiltonian null spaces from the microscopic wave functions. In contrast to the effective theories, this derivation shows explicitly how conformal invariance is obeyed by the Hilbert space in the thermodynamic limit, and how the conformal dimension and central charge emerge from the many-body wave functions themselves. The delicate structure of the conformal invariance, which we reveal by focusing on the null spaces of three-body pseudopotential interactions, shows that there could be a "simple" way of manifest conformal symmetry from the microscopic perspective. The negative conformal norm that can be computed microscopically does not lead to unphysical quantum mechanical behaviors of the many-body wave functions. It would be interesting to see how different CFT descriptions of the FQH edges are related to each other, which can potentially give us a deeper understanding of how CFT reveals the dynamical properties of both the bulk and edge of the FQH systems.

We end this section with a number of detailed predictions based on the analysis in this work, related to the thermal Hall conductance and the shot-noise and quasihole tunneling experiments. At filling factor $\nu = \frac{2}{5}$ in the LLL, we predict the coefficient of the thermal Hall conductance κ to be nonuniversal and bounded between $\frac{8}{5}$ (with exact Gaffnian quasihole degeneracy) and 2 (as predicted by the composite fermion theory). With more short-ranged interaction (e.g., greater sample thickness or with screening), smaller quasihole creation energy will generally lead to smaller κ. Nonlinear confinement potentials at the edge will also reduce the effective density of state and thus reduce κ, and we expect that to be the case with a sharper edge.

The tunneling experiments at $\nu = \frac{2}{5}$ will involve quasparticles of both $e/5$ and $2e/5$ charge (the latter can be considered as a bound state of two Gaffnian quasiholes). At very low temperature, only $2e/5$ quasiholes will be present for the tunneling. At higher temperature, the edge excitations will consist of a mixture of $e/5$ and $2e/5$ quasiholes, and the former has shorter tunneling distance. Thus, for narrow quantum point contact (QPC), at higher temperature the tunneling can predominantly involve $e/5$ quasiholes. For wider QPC, however, the tunneling amplitude for $e/5$ quasiholes will be less dominant, and there will be no clean experimental signals of a particular quasihole charge, and we expect some averaged quantities between $e/5$ and $2e/5$ from experiments involving tunneling between counterpropagating edge currents.

At $\nu = \frac{1}{3}$, the role of Haffnian quasiholes and the experimental raminifcations are mostly predicted in Ref. [47]. We do not expect a gapped FQH phase with a topological shift $S = -4$ with the LLL or SLL interaction, though this phase is not ruled out in principle. In contrast, there can be a finite-temperature phase transition of the quasihole manifold, especially near the nematic FQH phase and at the edge of the Hall plateau (when there is a relatively large quasihole density). This is the remnant of the "Haffnian phase" with non-Abelian $e/6$ quasiholes that cannot be fully realized due to the gaplessness of the Haffnian model Hamiltonian. Such a phase transition, and the fractionalization of the $e/3$ Laughlin quasiholes into $e/6$ quasiholes, can in principle be detected by single-electron tunneling experiments, or the shot-noise and inteferometry experiments. Above the critical temperature for the phase transition, we also expect the thermal Hall conductance coefficient no longer quantized at $\kappa = 1$.

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