Intrinsic Hamiltonian of composites in many-fermion systems

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I determine the intrinsic Hamiltonian of the relative motion of the constituent fermions of bosonic composites in superfluid fermion systems, assuming the effective fermion-fermion potential to be a sum of separable terms. The derivation is based on an expansion that treats composites, quasiparticles, and their interactions on the same footing. The intrinsic Hamiltonian of the composites is expressed in terms of the solution of the gap equation and of the form factors of the potential. It has the Brillouin-Wigner form, namely it is Hermitian and energy-dependent. In such a context, a solid justification is given for discarding negative energies of the composites. As a check, I rederive the dispersion law of the Bogoliubov-Anderson mode and the spectrum of the Bardasis and Schrieffer Hamiltonian. The possible occurrence of gapped modes depending on the form of the fermion-fermion interaction is discussed. The intrinsic wave functions are derived explicitly in the long wave limit, while their full determination requires numerical calculations.

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I. INTRODUCTION

Several fermion systems, both relativistic and nonrelativistic, have a spectrum showing bosonic excitations. There have been several ways to introduce such degrees of freedom in fermion theories, which essentially amount to projecting the fermion Hilbert space on the space of bosonic composites. Irrespective of the way one would expect that the resulting intrinsic Hamiltonian, namely the Hamiltonian describing the relative motion of the constituent fermions in the composites, should be of the Brillouin-Wigner form,

$$H_{\text{projected}}(\mathcal{E})|\mathcal{P}\psi\rangle = \mathcal{E}|\mathcal{P}\psi\rangle, \tag{1}$$

where

$$H_{\text{projected}}(\mathcal{E}) = \mathcal{P}H\mathcal{P} + \mathcal{P}H\mathcal{Q}(H-\mathcal{E})^{-1}\mathcal{Q}H\mathcal{P}. \tag{2}$$

In the above equations, H, ψ are the Hamiltonian and wave function of the constituent fermions, and $\mathcal{P}$, $\mathcal{Q}$ are the projection operators on the subspace of composites and its complement. The projected Hamiltonian is Hermitian, but it depends on the energy $\mathcal{E}$ of the composites.

An explicit expression of such a Hamiltonian is not needed to find the energy of the bosonic excitations. Such energies, however, usually come out in pairs of opposite sign, and the negative ones are discarded without a solid justification. In the formalism of the intrinsic Hamiltonian developed here instead, it is clear that they violate a necessary condition for unitarity of Bogoliubov transformations. Moreover, such a formalism is necessary to determine the structure of the composites in terms of the constituent fermions. Indeed, while the structure of Cooper pairs in the condensates has been studied extensively, the wave functions of the constituent fermions in dynamical composites have received less attention. They are mostly assumed or come out to be pointlike, while with attractive interactions they must have a structure extended in space. I think that pointlike composites can arise only with repulsive interactions, of which I will report an example at the end of this work.

A brief survey of the historical key steps in the study of bosonic excitations in fermion systems is in order to put the present investigation into perspective. Bogoliubov [1] and Anderson [2] found in superconducting systems by random phase approximation (RPA) calculations a gapless mode, and Bardasis and Schrieffer [3] found in a BCS-type model [4], whose interaction is expanded in multipoles, a number of excitons whose biggest gap is smaller than twice the quasiparticle gap. Such excitons are dynamical Cooper pairs with intrinsic structure extended in space. The spontaneous breaking of symmetry occurring in the BCS model was further studied by Nambu [5], while Goldstone [6] constructed a model field theory that shows such a mechanism, in which one massless and one massive boson appeared.

Finally, always inspired by the BCS theory, Nambu and Jona-Lasinio [7] constructed a relativistic model of fermions with spontaneous breaking of chiral invariance, in which one massless and several massive bosons appeared.

There exists now an extensive literature about bosonic excitations [8,9] in Fermi systems that use phenomenological or effective Hamiltonians. Based on the analogy with the Goldstone model, the gapless and gapped modes were often called Nambu-Goldstone and Higgs mode, respectively, and they were accordingly related to the phase and amplitude oscillations of the order parameter. Excitons with energy gap smaller than twice the fermionic quasiparticle gap were also discussed [10].

Several experiments reported signals attributed to the gapped mode [11]. When its energy gap is equal to twice the quasiparticle gap, however, it should be strongly damped [12] and its existence should require appropriate conditions.

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Very recently, pair-breaking excitations were studied, both of quasiparticle [13] and collective nature [14].

The size of the composites in the condensate is very important in several systems. One example is provided by nuclear physics. After Bohr, Mottelson, and Pines [15] suggested that deformed atomic nuclei might be superconducting, the problem arose as to whether Cooper pairs of nucleons would be small enough to be contained in an atomic nucleus. At the beginning, this appeared unlikely [16,17]. Afterward, it was observed [18] that in the pairing of protons and neutrons, one should take into account the tensor force, which is very strong. When the tensor force is taken into account, the energy gap becomes anisotropic, depending on the angle between the relative momentum of the nucleons and the axis of spin quantization. Taking the average value of the energy gap over this angle, the rmsr of the quasideuteron in symmetric nuclear matter turned out to be smaller than the radius of large nuclei.

A more recent example of the importance of the size of condensed Cooper pairs is provided by the BEC-BCS crossover, which is governed by the varying size of these pairs. The features of superfluidity common to ultracold atoms and nuclear matter are reviewed by Strinati *et al.* [19].

The size of the pairs in the condensate was also studied recently [20], and the existence of giant pairs in a two-band superfluid gas was predicted [21]. A systematic study of the structure of excited composites as well as of the interactions between composites and fermionic quasiparticles instead is as far as I know missing.

In the present work, I will derive an intrinsic Hamiltonian that has the form (2) using the nilpotency expansion. Such an expansion is based on the idea that the number of fermion states in a composite, called the nilpotency index, must be large because it is a measure of the collectivity of the composite, so that its inverse is a natural expansion parameter. The higher the collectivity, the better the approximation.

The nilpotency expansion treats composites, quasiparticles, and their interactions on the same footing, opening the possibility of further investigations of excitations of energy higher than twice the gap of both single particle and collective nature.

The idea of the nilpotency expansion was first proposed for atomic nuclei [22] as a total projection of the nucleon space on that of bosonic composites. The projection was performed respecting fermion number conservation. The expansion was performed after a subtraction, but the resulting expansion parameter was related to the inverse of the index of nilpotency only on average.

The method (with the inverse of the index of nilpotency as an exact expansion parameter) was then tested [23] on a relativistic four-fermion interaction model [24] whose spectrum in the limit of a large number of flavors is exactly reproduced, and afterward it was extended to include fermionic quasiparticles [25] and applied to quantum gauge theories [26]. There are now several applications [27], and very recently it was tested successfully [28] on QCD in two dimensions. Moreover, while in previous papers composites were described as fluctuations of a background field, in the latter one they were constructed as bound states of fermionic quasiparticles in the background field, with the same results to leading order in the nilpotency index. I also used the nilpotency expansion in a preliminary study of collective excitations in superconducting systems [29] in which, however, I made a too drastic approximation, which I will discuss later.

The paper is organized in the following way. In Sec. II, I report the partition function of relativistic field theories that I will use, and I compare it in Appendix A with that of a nonrelativistic fermion system after Hubbard-Stratonovich linearization. In Sec. III, I introduce dynamical composites and quasiparticles by a time-dependent Bogoliubov transformation on the fermions. In Sec. IV, I perform the nilpotency expansion in the partition function deriving the Hamiltonian of composites and quasiparticles to zero order. Such a Hamiltonian has quite general features, it is similar to other Hamiltonians of composites, so that the reader interested only in the intrinsic Hamiltonian can ignore its derivation and start from this point. In Sec. V, I perform a Bogoliubov transformation on the composites, in Sec. VI, I write the eigenvalue equation for the composites, and finally in Sec. VII, I derive the intrinsic Hamiltonian. In Sec. VIII, I use the above results to derive energies and eigenfunctions of the Bogoliubov-Anderson modes, of the Bardasis-Schrieffer modes, and of other modes whose existence depends on the form of the fermion-fermion interaction, assuming that it can be given as a sum of separable terms. In Sec. IX, I briefly report the results of Ref. [23] in order to give an analytic example of pointlike composites. Section X contains a summary.

II. PARTITION FUNCTIONS

I will use for nonrelativistic systems the nilpotency expansion devised for relativistic field theories on a lattice [27]. I choose such a formulation for future comparison with relativistic theories. The composites studied in such theories are fermion-antifermion pairs. To use their results in the simplest way, I will regard spin-up and spin-down fermions as different species,

$$\hat{u}_k = \hat{c}_{k\uparrow}, \quad \hat{v}_k = \hat{c}_{k\downarrow}, \tag{3}$$

where $\hat{c}_{ks}$ are the original fermion operators. In this way, I restrict myself to composites of zero third component of the spin.

I will define the fermion Hamiltonian I am studying, and I will report the expression of the partition function for relativistic field theories that I will apply. I will show that it takes the nonrelativistic expression reported in Appendix A in the nonrelativistic limit so that I can use its results taking this limit at the end.

I will consider fermion Hamiltonians of the type

$$H_F = \sum_q \xi(q)\,(\hat{u}_q^\dagger\,\hat{u}_q + \hat{v}_q^\dagger\,\hat{v}_q) + \sum_{p_1p_2p_3p_4} \hat{u}_{p_1}^\dagger\,\hat{v}_{p_2}^\dagger V_{p_1p_2p_3p_4}\,\hat{v}_{p_4}\hat{u}_{p_3}. \tag{4}$$

In the above equation (setting $\hbar = 1$), the first term is the kinetic energy,

$$\xi_q = \frac{q^2}{2m} - \mu, \tag{5}$$

where m, μ are the mass and the chemical potential of the fermions. The second term is the interaction, which I assume to be a sum of separable terms,

$$V = \sum_{P\nu} \mathrm{v}_{P\nu} \sum_{qq'} f_\nu(Pq) f_\nu(Pq') \times \hat{u}^\dagger_{\frac{1}{2}P+q} \hat{v}^\dagger_{\frac{1}{2}P-q} \hat{v}_{\frac{1}{2}P-q'} \hat{u}_{\frac{1}{2}P+q'}, \tag{6}$$

whose definitions can be found in Appendix A.

For nonrelativistic theories, the manipulations relating the partition function to the Hamiltonian usually involve an expansion in the temporal spacing τ. In relativistic theories, such an expansion is not permissible in general, because powers of τ are accompanied by divergent integrals. Renormalizable field theories in four dimensions, however, are quadratic in the fermion fields, and for such theories a formulation is possible, which does not require the expansion in τ. In such cases, instead of the Hamiltonian one uses the transfer matrix [30]

$$\mathcal{T}_t = \exp(\tau \hat{u}^\dagger N_t^\dagger \hat{v}^\dagger) \exp(-2\hat{u}^\dagger M \hat{u} - 2\hat{v}^\dagger M \hat{v}) \exp(\tau \hat{v} N_t \hat{u}), \tag{7}$$

where

$$M = -\tfrac{1}{2} \ln(1 - \tau\xi) \tag{8}$$

and $N^\dagger, N$ are functions of the elementary bosonic fields coupled to the fermions, e.g., gauge fields. The nonrelativistic form (A8) of the partition function can be obtained, as shown in the next section, by inserting the identity $\hat{\mathcal{I}}$ written in terms of fermionic coherent states according to

$$Z = \int d\mu(N^\dagger, N) \prod_t \mathrm{Tr}^{\mathrm{Fock}} (\hat{\mathcal{I}} \hat{\mathcal{T}}_t), \tag{9}$$

where the measure on the fields $N^\dagger, N$ depends on the particular system. So the relativistic formulation yields the partition function, as reported in Appendix A, of a nonrelativistic system after the Hubbard-Stratonovich transformation with the measure

$$d\mu(N^\dagger, N) = [dN^* dN]. \tag{10}$$

III. COMPOSITES IN THE PARTITION FUNCTION

Of course one can get the projected Hamiltonian on the composite subspace by a suitable definition of the projection operator $\mathcal{P}$ and an evaluation of the necessary matrix elements [22]. Another possibility [25], which I follow here, consists in introducing composites in the Euclidean partition function. To this end, I performed an independent Bogoliubov transformation at each time defined by the matrices $\mathcal{F}_t^\dagger, \mathcal{F}_t$,

$$\begin{aligned}\hat{\alpha}_t &= (1 + \mathcal{F}_t^\dagger \mathcal{F}_t)^{-\frac{1}{2}} (\hat{u} - \mathcal{F}_t^\dagger \hat{v}^\dagger),\\ \hat{\beta}_t &= (\hat{v} + \hat{u}^\dagger \mathcal{F}_t^\dagger)(1 + \mathcal{F}_t \mathcal{F}_t^\dagger)^{-\frac{1}{2}}.\end{aligned} \tag{11}$$

Because the spins have been eliminated the fermionic variables are labeled only by the momenta that are also the entries of the matrices $\mathcal{F}_t$. These matrices will be related to dynamical composite fields.

I construct coherent fermionic states at each time,

$$|\alpha_t \beta_t \mathcal{F}_t\rangle = \exp(\alpha_t \hat{\alpha}^\dagger + \beta_t \hat{\beta}^\dagger) |\mathcal{F}_t\rangle, \tag{12}$$

where α_t, β_t are Grassmann variables, and $|\mathcal{F}_t\rangle$ is the vacuum of the quasiparticle operators,

$$|\mathcal{F}_t\rangle = \exp(u^\dagger \mathcal{F}_t^\dagger v^\dagger)|0\rangle. \tag{13}$$

Next I define a realization of the identity at each time,

$$\hat{\mathcal{I}}(\mathcal{F}_t) = \int d\alpha_t^* d\alpha_t d\beta_t^* d\beta_t |\alpha_t \beta_t \mathcal{F}_t\rangle\langle \alpha_t \beta_t \mathcal{F}_t|. \tag{14}$$

Notice that such an operator is defined for given matrices $\mathcal{F}_t^\dagger, \mathcal{F}_t$. Now I rewrite the fermion partition function inserting the identity operator at each time,

$$Z_F' = \mathrm{Tr}^{\mathrm{Fock}} \prod_t (\hat{\mathcal{I}}(\mathcal{F}_t) \hat{\mathcal{T}}_t) = \int [dN^\dagger dN] \times [d\alpha^* d\alpha][d\beta^* d\beta] \exp \sum_t (\mathcal{L}_B' + \mathcal{L}_{\mathrm{q.p.}}), \tag{15}$$

where

$$\mathcal{L}_B'(\mathcal{F}^\dagger, \mathcal{F}, N^\dagger, N) = -\mathrm{Tr} \ln(R_t E_{t,t+1}) \tag{16}$$

is the purely bosonic Lagrangian, and $\mathcal{L}_{\mathrm{q.p.}}$ is the Lagrangian of fermionic quasiparticles interacting with the composite bosons whose expression is given in Eq. (26) of Ref. [27], but it will be neglected here. In the above equations,

$$\begin{aligned}R_t &= (1 + \mathcal{F}_t^\dagger \mathcal{F}_t)^{-1},\\ E_{t,t+1} &= (\mathcal{F}_{Nt+1})^\dagger e^{2M} \mathcal{F}_{Nt} + (\mathcal{F}_{t+1})^\dagger e^{-2M} \mathcal{F}_t,\\ \mathcal{F}_{Nt} &= 1 + \tau N_t^\dagger \mathcal{F}_t.\end{aligned} \tag{17}$$

The expression (15) of Z_F is exact, no matter which are the matrices $\mathcal{F}_t^\dagger, \mathcal{F}_t$. Therefore, I can integrate on them with an arbitrary measure getting

$$Z_F = \int d\mu(\mathcal{F}^\dagger, \mathcal{F}) Z_F'. \tag{18}$$

In the present paper, I chose the simple measure

$$d\mu(\mathcal{F}^*, \mathcal{F}) = [d\mathcal{F}^* d\mathcal{F}]. \tag{19}$$

Integrating over the fields $N^\dagger, N$, I get a new composite Lagrangian that depends only on the fields $\mathcal{F}^\dagger, \mathcal{F}$,

$$\mathcal{L}_B(\mathcal{F}^\dagger, \mathcal{F}) = \mathrm{Tr}\{R[\mathcal{F}^\dagger \nabla_t \mathcal{F} + \mathcal{F}^\dagger \xi\, \mathcal{F} + \mathcal{F}^\dagger \mathcal{F} \xi]\} + \sum_\nu \mathrm{v}_\nu \mathrm{Tr}\{R\, \mathcal{F}^\dagger f_\nu\} \mathrm{Tr}\{f_\nu^\dagger\, \mathcal{F} R\}. \tag{20}$$

Such a Lagrangian respects all the symmetries of the fermion Hamiltonian.

IV. NILPOTENCY EXPANSION

The nilpotency expansion is performed here as an expansion with respect to the fluctuations of the fields around a background (see, however, [28]) classified according to the powers of the index of nilpotency.

The background is determined by a saddle point calculation. Let me consider the composites Lagrangian (20) at fields constant in time, which I will overline,

$$\overline{\mathcal{L}}_B = \mathrm{Tr}\{\overline{R}[\overline{\mathcal{F}}^\dagger \xi\, \overline{\mathcal{F}} + \overline{\mathcal{F}}^\dagger \overline{\mathcal{F}} \xi]\} + \sum_\nu \mathrm{v}_\nu \mathrm{Tr}\{\overline{R}\, \overline{\mathcal{F}}^\dagger f_\nu\} \mathrm{Tr}\{f_\nu^\dagger\, \overline{\mathcal{F}}\, \overline{R}\}. \tag{21}$$

Variation of $\overline{\mathcal{L}}_B$ with respect to $\overline{\mathcal{F}}$ gives the standard gap equation. In its solution and in the following, I adopt the standard approximation,

$$\sum_k f(k^2, \hat{k}) \approx \rho_F \frac{1}{4\pi} \int d\hat{k} \int_{-\omega}^{+\omega} d\xi \, f(2m(\mu+\xi), \hat{k}), \quad (22)$$

where $\hat{k}$ is the unit vector in the k-direction, and ρ_F is the density of fermion states of each species at the Fermi surface.

Assuming for simplicity

$$\overline{\mathcal{F}}^\dagger = \overline{\mathcal{F}}, \quad [\overline{\mathcal{F}}, \xi] = 0, \quad (23)$$

the solution of the gap equation is

$$\overline{\mathcal{F}}_{k_1 k_2} = \delta_{k_1, -k_2} \overline{\mathcal{F}}_{k_1}, \quad (24)$$

where

$$\overline{\mathcal{F}}_k = \frac{1}{\Delta}[E_k - \xi_k], \quad (25)$$

$$E_k = \sqrt{\Delta^2 + \xi_k^2}, \quad \Delta = 2\omega \exp(1/g), \quad g = |\mathrm{v}|\rho_F. \quad (26)$$

Such a solution exists only if for some quantum number $\overline{\nu}$, $\mathrm{v}_{\overline{\nu}} < 0$.

Let me then introduce the time-dependent fluctuations,

$$\mathcal{F}_t = \overline{\mathcal{F}} + \delta\mathcal{F}_t, \quad (27)$$

and consider the creation operator of the fermion composite,

$$\hat{\mathcal{F}}^\dagger = \hat{u}^\dagger \, \delta\mathcal{F}^\dagger \hat{v}^\dagger. \quad (28)$$

A characteristic feature of such a composite is its index of nilpotency, which is the maximum number Ω of such a composite compatible with the Pauli principle,

$$(\hat{\mathcal{F}}^\dagger)^\Omega \neq 0, \quad (\hat{\mathcal{F}}^\dagger)^{\Omega+1} = 0, \quad (29)$$

$$\Omega = \tfrac{1}{2}\,\mathrm{rank}(\delta\mathcal{F}). \quad (30)$$

The index of nilpotency of the composites provides a measure of their collectivity, and a value of $\Omega \gg 1$ is obviously necessary for the composite to be approximated by an elementary boson. Therefore, its inverse is a natural parameter to control the bosonization of the system.

Now I assume, and I will verify at the end, that it will turn out that $\delta\mathcal{F}_t = O(\Omega^{-\frac{1}{2}})$. In the estimate of the order of the terms in the expansion, one must take into account that $\overline{\mathcal{F}} = O(1)$, and $\sum_q$ gives a factor of order Ω.

It is convenient to perform the change of variables

$$B_t = \sqrt{R}\,\delta\mathcal{F}_t\sqrt{R}. \quad (31)$$

The time-independent part of the bosonic Lagrangian, $\overline{\mathcal{L}}_B$, is of order Ω. The part of $\mathcal{L}_B$ of order $\Omega^{\frac{1}{2}}$ vanishes because of the saddle point equations. To write the part of zero order, I perform the expansion

$$R_t = \overline{R} + R_t^{(1)} + R_t^{(2)} \quad (32)$$

in which the superscript refers to the order in $\Omega^{-\frac{1}{2}}$. $\overline{R}$ is time-independent, diagonal, and of order zero,

$$\overline{R}_{k_1,k_2} = \delta_{k_1,k_2}\overline{R}_{k_1}, \quad \overline{R}_k = \frac{E_k + \xi_k}{2E_k}, \quad (33)$$

while

$$R_t^{(1)} = -\sqrt{\overline{R}}\,[\,\overline{\mathcal{F}}^\dagger B_t + B_t^\dagger \overline{\mathcal{F}}]\sqrt{\overline{R}}, \quad (34)$$

$$R_t^{(2)} = -\sqrt{\overline{R}}\,[\,B_t^\dagger B_t + \overline{\mathcal{F}}^\dagger B_t B_t^\dagger \overline{\mathcal{F}} + B_t^\dagger \overline{\mathcal{F}} B_t^\dagger \overline{\mathcal{F}} + \overline{\mathcal{F}}^\dagger B_t \overline{\mathcal{F}}^\dagger B_t]\,\sqrt{\overline{R}}. \quad (35)$$

With these variables,

$$\sum_t \mathrm{Tr}\,(\mathcal{F}_t^\dagger \nabla_t \mathcal{F}_t) = \sum_t \mathrm{Tr}\,(B_t^\dagger \nabla_t B_t) + O(\Omega^{-\frac{1}{2}}), \quad (36)$$

showing that to zero order, B^*, B are conjugate to each other. The time derivative terms of order $\Omega^{-\frac{1}{2}}$ can be relevant when one considers the coupling with fermionic quasiparticles.

I must now determine the order of the potential terms. First, I observe that defining the potential form factors f_ν to be of order 1, the strength of the potential is proportional to the inverse of the square root of the volume,

$$\mathrm{v} = O(\Omega^{-\frac{1}{2}}). \quad (37)$$

Next, I find it convenient to perform a change of the entries of the matrices: from the momenta of the fermions k_1, k_2 to the total momentum of the pair $P = k_1 + k_2$ and its internal momentum $q = \frac{1}{2}(k_1 - k_2)$. So I can write the bosonic Hamiltonian to zero order, omitting the quantum number ν,

$$\begin{aligned} H_B(P) = \sum_{q_1 q_2} \{ & [2\overline{E}(Pq_1)\delta_{q_1 q_2} + \mathrm{v}[C_1(Pq_1)C_1(Pq_2) \\ & + C_2(P, q_1)C_2(Pq_2)]]B^*(Pq_1)B(Pq_2) \\ & + \mathrm{v}\,\delta_{q_1 q_2} C_2(P, q_1)C_2(Pq_2) \\ & + B^*(Pq_1)B^*(-Pq_2)\,\mathrm{v}\,C_2(P, q_1)C_1(Pq_2) \\ & + B(-Pq_2)B(Pq_1)\,\mathrm{v}\,C_1(Pq_2)C_2(Pq_1).\}, \end{aligned} \quad (38)$$

where

$$\overline{E}(Pq) = \frac{1}{2}(E_{q+\frac{1}{2}P} + E_{q-\frac{1}{2}P}),$$

$$C_1(Pq) = f(Pq)\sqrt{\frac{[E_{q+\frac{1}{2}P} - \xi_{q+\frac{1}{2}P}][E_{q-\frac{1}{2}P} - \xi_{q-\frac{1}{2}P}]}{2\,E_{q+\frac{1}{2}P}E_{q-\frac{1}{2}P}}},$$

$$C_2(Pq) = f(Pq)\sqrt{\frac{[E_{q+\frac{1}{2}P} - \xi_{q+\frac{1}{2}P}][E_{q-\frac{1}{2}P} - \xi_{q-\frac{1}{2}P}]}{2\,E_{q+\frac{1}{2}P}E_{q-\frac{1}{2}P}}}. \quad (39)$$

H_B is the Hamiltonian of a bosonic superfluid whose interaction is given by the sum of two separable potentials. The c-number term comes from normal ordering, and the last two terms are the so-called "dangerous" terms.

A comment is in order here. In quantum gauge theories, the matrices $N^\dagger, N$ are functions of the gauge fields, and at the saddle point the background fields $\overline{\mathcal{F}}^\dagger = \overline{\mathcal{F}}^\dagger(N^\dagger, N)$, $\overline{\mathcal{F}} = \overline{\mathcal{F}}(N^\dagger, N)$ become functions of the gauge fields and transform in such a way that local gauge invariance is conserved, but they do not have in general a particle interpretation [27]. In the present application to many-body systems, instead, the background fields are constant, thus breaking fermion number conservation.

In the work in which the nilpotency expansion was proposed [22], the background field was defined as

$$\overline{\mathcal{F}} = b\,\phi, \tag{40}$$

where b is a dynamical variable related to the occupation number of composites in the ground state, and ϕ is their structure function, under the condition $\langle b^* b\rangle = \frac{1}{2}N_F$, N_F being the number of fermions. The expansion was performed with respect to the gauge invariant quantity $\mathcal{F}^\dagger\mathcal{F} - r^2$, which can be made of order $1/\sqrt{\Omega}$ on the average by chosing the parameter r in a given way. So at the cost of such an approximation, the fermion number was exactly conserved.

V. BOGOLIUBOV TRANSFORMATION ON COMPOSITES

I introduce creation annihilation operators for the composites satisfying canonical commutation relations

$$[\hat{B}(Pq), \hat{B}^*(P'q')] = \delta_{P,P'}\,\delta_{q,q'}. \tag{41}$$

The operators $\hat{B}^*$, $\hat{B}$ can be written

$$\hat{B}(Pq) = \sum_n \hat{a}_n(P)\psi^n(Pq), \tag{42}$$

where the $\psi^n(Pq)$'s are a complete set of functions

$$\sum_n [\psi^n(Pq)]^*\psi^n(Pq') = \delta_{qq'}, \tag{43}$$

$$\sum_q [\psi^m(Pq)]^*\psi^n(Pq) = \delta_{mn}, \tag{44}$$

and $\hat{a}(P)^*$, $\hat{a}(P)$ are canonical creation annihilation operators

$$[\hat{a}_m(P)^*, \hat{a}_n(P')] = \delta_{mn}\delta_{PP'}. \tag{45}$$

To diagonalize H_B, I perform a (bosonic) Bogoliubov transformation

$$\hat{a}_m(P) = \sum_n u_P^{mn}\,\hat{b}_n(P) - v_P^{mn}\,\hat{b}_n^\dagger(-P), \tag{46}$$

where the matrices u_P, v_P satisfy the following relationships:

$$\begin{aligned}
&(\tilde{u}_P u_P - \tilde{v}_P v_P)^{mn} = \delta_{mn},\\
&(u_P\tilde{u}_P - v_P\tilde{v}_P)^{mn} = \delta_{mn},\\
&(\tilde{u}_P v_P - \tilde{v}_P u_P)^{mn} = 0,\\
&(u_P\tilde{v}_P - v_P\tilde{u}_P)^{mn} = 0.
\end{aligned} \tag{47}$$

I can then write

$$\begin{aligned}
\hat{B}(Pq) &= \sum_n \hat{b}_n(P)\phi_+^n(Pq) - \hat{b}_n^\dagger(-P)\phi_-^n(Pq),\\
\hat{B}^\dagger(Pq) &= \sum_n \hat{b}_n^\dagger(P)\phi_+^n(Pq) - \hat{b}_n(-P)\phi_-^n(Pq),
\end{aligned} \tag{48}$$

where

$$\begin{aligned}
\phi_+^m(Pq) &= \sum_n u_P^{nm}\psi_n(Pq),\\
\phi_-^m(Pq) &= \sum_n v_P^{nm}\psi_n(Pq).
\end{aligned} \tag{49}$$

It should be clear that $\phi_\pm$ are not independent from one another, since both can be expressed in terms of the matrices u, v and the internal wave functions ψ^n.

For simplicity, *I restrict myself to ψ's and ϕ's that are real and of positive parity*,

$$\phi_\pm(Pq) = \phi_\pm(-P, -q). \tag{50}$$

After the Bogoliubov transformation, the composites Hamiltonian (38) becomes

$$\begin{aligned}
H_B = \sum_P &\{\hat{b}_m^\dagger(P)\hat{b}_n(P)\,\mathcal{E}^{mn}(P) + [\hat{b}_m^\dagger(P)\hat{b}_n^\dagger(-P)\\
&+ \hat{b}_n(-P)\hat{b}_m(P)]T^{mn}(P)\}.
\end{aligned} \tag{51}$$

All the above quantities are functions of the composites momentum,

$$\begin{aligned}
\mathcal{E}^{mn} &= E_{++}^{mn} + E_{--}^{mn} + \mathrm{v}\big(\chi_{+-}^m\chi_{+-}^n + \chi_{-+}^m\chi_{-+}^n\big),\\
T^{mn} &= \tfrac{1}{2}\big[E_{+-}^{mn} + E_{+-}^{nm} + \mathrm{v}(\chi_{+-}^m\chi_{-+}^n + \chi_{+-}^n\chi_{-+}^m)\big],
\end{aligned} \tag{52}$$

where

$$\begin{aligned}
E_{\sigma\tau}^{mn}(P) &= \sum_q 2\overline{E}(Pq)\,\phi_\sigma^m(Pq)\phi_\tau^n(Pq),\\
\chi_{\sigma\tau}^m(P) &= \sum_q [C_2(Pq)\phi_\sigma^m(Pq) - C_1(Pq)\phi_\tau^m(Pq)],\\
\sigma, \tau &= \pm.
\end{aligned} \tag{53}$$

In the classic Bogoliubov model, one can impose the vanishing of the dangerous terms, and such a condition determines the parameters of the Bogoliubov transformation. In the present case, the solution of the conditions $T^{mn} = 0$ is in general difficult, and moreover the constraint matrix T^{mn} does not commute with the boson conserving matrix $\mathcal{E}^{mn}$.

VI. EIGENVALUE EQUATION FOR THE COMPOSITES ENERGY

Diagonalization of H_B can be facilitated by [31] incorporating the dangerous terms in the auxiliary energy functionals

$$\mathcal{E}_\pm^{mn} = \mathcal{E}^{mn} \pm T^{mn}, \tag{54}$$

which are equal to each other and to the energy matrix $\mathcal{E}^{mn}$ when the constraints are satisfied, in which case I can write

$$H_B = \sum_P \mathcal{E}_\pm^{mn}(P)\,\hat{b}_m^\dagger(P)\hat{b}_n(P). \tag{55}$$

The expressions of $\mathcal{E}^{mn}$, T^{mn} simplify by introducing the auxiliary functions [32]

$$\lambda_\pm^n(Pq) = \phi_+^n(Pq) \pm \phi_-^n(Pq). \tag{56}$$

Commutation relations of all the quantum operators and completeness of the functions $\phi_\pm^n(Pq)$ yield the following relationships:

$$\sum_q \lambda_\pm^m(Pq)\lambda_\mp^n(Pq) = \delta^{mn}, \tag{57}$$

$$\sum_m \lambda_\pm^m(Pq)\lambda_\mp^m(Pq') = \delta_{qq'}. \tag{58}$$

In terms of the $\lambda_{\pm}^{n}$'s,

$$\mathcal{E}_{\pm}^{mn}(P) = \sum_{qq'} \lambda_{\pm}^{m}(Pq)\mathcal{H}_{\mp}(Pqq')\lambda_{\pm}^{n}(Pq'),$$
$$T^{mn}(P) = \frac{1}{4}\sum_{q} 2\overline{E}(Pq)[\lambda_{+}^{m}(Pq)\lambda_{+}^{n}(Pq) - \lambda_{-}^{m}(Pq)\lambda_{-}^{n}(Pq)] + \frac{1}{4}v[X_{+}^{m}(P) \times X_{+}^{n}(P) - X_{-}^{m}(P)X_{-}^{n}(P)], \tag{59}$$

with

$$\mathcal{H}_{\mp}(Pq\,q') = 2\overline{E}(Pq)\ \delta_{q\,q'} + \mathrm{v}_P\, C_{\mp}(Pq)C_{\mp}(Pq') \tag{60}$$

and

$$C_{\mp}(Pq) = C_2(Pq) \mp C_1(Pq), \tag{61}$$

$$X_{\pm}^{n}(P) = \sum_{q} C_{\mp}(Pq)\lambda_{\pm}^{n}(Pq). \tag{62}$$

The composites energies are given by the expectation values

$$\sum_{qq'} \lambda_{\pm}^{m}(Pq)\mathcal{H}_{\mp}(Pqq')\lambda_{\pm}^{n}(Pq') = \mathcal{E}_P^{n}\,\delta^{mn}, \tag{63}$$

which, when the dangerous terms vanish, must take one and the same value. Notice that a simplification has been achieved because the kernels $\mathcal{H}_{\pm}$ contain *only one separable potential* for each P. The auxiliary functions $\lambda_{\pm}^{n}(Pq')$, however, are not eigenfunctions of such operators. Indeed, by multiplying Eq. (63) by $\lambda_{\mp}^{m}(P)$, summing over m, and using Eqs. (58), one gets

$$\sum_{q'} \mathcal{H}_{\mp}(Pqq')\lambda_{\pm}^{n}(Pq') = \mathcal{E}_P^{n}\lambda_{\mp}^{n}(Pq), \tag{64}$$

which shows that the operators $\mathcal{H}_{\mp}(P)$ change $\lambda_{\pm}$ into $\lambda_{\mp}$. It is easy to check that the above equations guarantee the vanishing of the "dangerous" tensor T^{mn} and the orthogonality of the $\lambda_{\pm}^{m}$, namely the validity of Eqs. (57) for $m \neq n$. The first statement can be proved by subtracting Eqs. (63) with the $\pm$ sign from one another; the second one can be proved by subtracting from one another Eqs. (63) written for different energies m, n. The condition (57) for $m = n$ remains to be imposed.

To evaluate the energy of the composites, I first solve Eqs. (64),

$$\lambda_{\pm}(Pq) = 2\overline{E}(Pq)\mathcal{A}_{\mp}(Pq)X_{\pm}(P) + \mathcal{E}_P\mathcal{A}_{\pm}(Pq)X_{\mp}(P), \tag{65}$$

where

$$\mathcal{A}_{\pm}(Pq) = -\mathrm{v}_P\, D(Pq)^{-1}\, C_{\pm}(Pq) \tag{66}$$

with

$$D(Pq) = 2\overline{E}(Pq)^2 - \mathcal{E}_P^2. \tag{67}$$

Remember, however, that the $X_{\pm}$ are functionals of the $\lambda_{\pm}$. Multiplying Eq. (65) with the $\pm$ sign by $C_{\mp}$ and summing over q, I get the homogeneous coupled equations

$$(A_{\mp} - 1)X_{\pm} + BX_{\mp} = 0, \tag{68}$$

where

$$A_{\pm}(P) = \sum_{q} 2\overline{E}(Pq)\mathcal{A}_{\pm}(Pq)C_{\pm}(Pq),$$
$$A(P) = \mathcal{E}_P \sum_{q} \mathcal{A}_{\pm}(Pq)C_{\mp}(Pq). \tag{69}$$

Their compatibility condition is

$$(A_{+} - 1)(A_{-} - 1) - A^2 = 0. \tag{70}$$

This condition determines the energy of the composites, which is the only unknown quantity appearing in it. *It is essentially the equation that gives the poles in the propagator in functional formulations*. The energy eigenvalues occur in pairs of opposite values, as a consequence of regarding the $\phi_{\pm}$ as independent. Condition (57), however, is not satisfied by the states of negative energy, as I will show below. Notice that Eqs. (64) can be written in matrix form, with a non-Hermitian matrix of skew diagonal elements $\mathcal{H}_{\pm}$, a feature discussed by several authors who, regarding the $\phi_{\pm}$ as independent, find it to be a shortcoming.

An important result follows from the condition that $D(Pq) > 0$, namely $\mathcal{E}_P < 2\Delta$, so that if $\mathcal{E}_0 = 2\Delta$, the kinetic contribution must be negative.

VII. INTRINSIC HAMILTONIAN OF THE COMPOSITES

I determined the eigenvalue equation for the energy of the composites. Such an equation is essentially the same as the equations derived by Bardasis and Schrieffer and others. But its energy solutions are not eigenvalues of $\mathcal{H}_{\pm}$ because, as already noticed, the $\lambda_{\pm}$ are not eigenfunctions of such operators. So I do not know which are the composite wave functions, namely the wave functions describing the relative motion of the constituent fermions. Therefore, I want to determine an operator reproducing the spectrum of (70), which I will call the intrinsic Hamiltonian of the composites, and see how its eigenfunctions are related to $\lambda_{\pm}$. After some manipulations reported in Appendix B, I found

$$\mathcal{H}_{\mathrm{intr}}(\mathcal{E}) = \tfrac{1}{2}(\mathcal{H}_{+} + \mathcal{H}_{-}) - \tfrac{1}{2}(\mathcal{H}_{+} - \mathcal{H}_{-}) \times \left[\tfrac{1}{2}(\mathcal{H}_{+} - \mathcal{H}_{-}) + \mathcal{E}\right]^{-1}\tfrac{1}{2}(\mathcal{H}_{+} - \mathcal{H}_{-}), \tag{71}$$

which has eigenvalues $\pm\mathcal{E}$, but one of them, as shown in Appendix B, must be discarded as a consequence of the normalization Eqs. (57). Notice that violation of such a condition implies that the Bogoliubov transformation is not unitary.

Notice also that the intrinsic Hamiltonian is expressed in terms of the form factors of the potential and the solution of the gap equation. It is Hermitian but energy-dependent and therefore its eigenfunctions belonging to different eigenvalues need not be orthogonal to one another.

I come now to the comparison with the Brillouin-Wigner formulation, Eq. (2). The expression of $\mathcal{H}_{\mathrm{intr}}$ derived here (71) is not based on a total projection on the space of composites. Indeed, the total Lagrangian appearing in Eq. (15) contains, in addition to the Lagrangian of the composites, also the Lagrangian of quasiparticles interacting with the composites. To get the projected Hamiltonian of Eq. (2), one should integrate over quasiparticles, thus missing the relative part of the spectrum, which instead I want to investigate in a future work. As a consequence, the term

$$\tfrac{1}{2}(\mathcal{H}_+ - \mathcal{H}_-) \tag{72}$$

has nonvanishing diagonal matrix elements and cannot be identified with the terms

$$\mathcal{PHQ} \text{ and } \mathcal{QHP} \tag{73}$$

of the projected Hamiltonian. The two Hamiltonians are similar to one another, but they differ also by the sign in front of the energy $\mathcal{E}$ in the denominators.

VIII. ENERGY AND WAVE FUNCTIONS OF COMPOSITES

First I rewrite the expressions of the various quantities in terms of integrals,

$$\begin{aligned} A_\pm(P) &= \frac{1}{8\pi}\, g \int d\hat{q} \int d\xi\, \frac{2\overline{E}}{\Delta}\, \mathcal{A}_\pm C_\pm, \\ A(P) &= \frac{1}{8\pi}\, g\, \frac{\mathcal{E}}{\Delta} \int d\hat{q} \int d\xi\, \mathcal{A}_\pm C_\mp(Pq). \end{aligned} \tag{74}$$

In their evaluation, I will set

$$\begin{aligned} \eta_P &= \frac{\mathcal{E}_P}{2\Delta}, \\ g_P &= \rho_F |\mathrm{v}_P| \end{aligned} \tag{75}$$

and

$$\begin{aligned} x_P &= \overline{x}_P \gamma, \\ \overline{x}_P &= \frac{q_F P}{2m\Delta} = \frac{1}{2} r_{\mathrm{Pipp}} P, \quad \gamma = \frac{q \cdot P}{qP}, \end{aligned} \tag{76}$$

where r_{Pipp} is the Pippard coherence length.

A. The case $P = 0$

At $P = 0$, A vanishes because its integrand is odd, so that Eqs. (68) decouple,

$$\begin{aligned} (A_+ - 1)X_- &= 0, \\ (A_- - 1)X_+ &= 0. \end{aligned} \tag{77}$$

Evaluation of $A_\pm(0)$ yields

$$\begin{aligned} A_+ - 1 &\approx g_0 \frac{\eta_0}{\sqrt{1-\eta_0^2}} \arcsin \eta_0, \\ A_- - 1 &\approx -g_0 \frac{\sqrt{1-\eta_0^2}}{\eta_0} \arcsin \eta_0, \end{aligned} \tag{78}$$

which have the solutions

$$\begin{aligned} &\eta = 0, \;\; A_+ = 1, \;\; A_- \neq 1, \;\;\; X_+ = 0, \\ &\eta = 1, \;\; A_- = 1, \;\;\; A_+ \neq 1, \;\; X_- = 0. \end{aligned} \tag{79}$$

In the first case, the eigenvalue equations (65) become

$$\begin{aligned} \lambda_+(0q) &= 0, \\ 2E_q \lambda_-(0q) &= -\mathrm{v}_0\, C_-(0q) X_-(0) \end{aligned} \tag{80}$$

so that the normalization condition Eq. (57) is not fulfilled and such a state does not exist. This can be understood on physical grounds because according to the first of the above equations, $\phi_+ = -\phi_-$, and then according to Eqs. (48), $\hat{B}(0q) = \hat{B}^\dagger(0q)$, implying that these are not quantum variables.

In the second case, the eigenvalue equations (65) become

$$\begin{aligned} 2E_q\lambda_+(0q) - 2\Delta\lambda_-(0q) &= -\mathrm{v}_0\, C_-(0q) X_+(0), \\ -2\Delta\lambda_+(0q) + 2E_q\lambda_-(0q) &= 0. \end{aligned} \tag{81}$$

The integral appearing in the normalization condition (57) is divergent, so that for $P = 0$ also such a state does not exist. In conclusion, the energies $\mathcal{E}_0 = 0, 2\Delta$ can at most be limit values of energies of true states. In the following subsections, I will therefore study the spectrum at $\mathcal{E} \gtrsim 0, \lesssim 2\Delta$.

B. The excitons of Bardasis and Schrieffer

Bardasis and Schrieffer [3] expand the potential in multipoles. Because of the present restriction to positive parity and reality of the form factors, in the comparison with their investigation I must restrict myself to even multipoles with zero third component of angular momentum. Restricting further to states with $P = 0$, in the present notation the form factors are

$$f_l(k) = Y_{l0}(\hat{k}), \tag{82}$$

where the quantum number $\nu = l$ is the orbital angular momentum. Since at $P = 0$, $A = 0$, I then get from Eqs. (68)

$$\begin{aligned} A_+ - 1 &= g_l \left(\frac{1}{g_l} - \frac{1}{g_0} + \frac{\eta_l}{\sqrt{1-\eta_l^2}} \arcsin \eta_l \right) = 0, \\ A_- - 1 &= g_l \left(\frac{1}{g_l} - \frac{1}{g_0} - \frac{\sqrt{1-\eta_l^2}}{\eta_l} \arcsin \eta_l \right) = 0, \end{aligned} \tag{83}$$

which are exactly Eqs. 3.15 of Bardasis and Schrieffer [3].

C. The Bogoliubov-Anderson mode

Now I look for solutions to the compatibility equations (70) in which $\mathcal{E}_P \to 0$ for $P \to 0$, namely $\eta_P \approx \overline{x}_P << 1$. Evaluation of the quantities $A_\pm, A$ then yields

$$\begin{aligned} A_+(P) - 1 &\approx g_P \left[\frac{1}{g_0} - \frac{1}{g_P} + \left(\eta_P^2 - \frac{1}{3}\overline{x}_P^2 \right) \right] + O(P^3), \\ A_-(P) - 1 &\approx g_P \left[\frac{1}{g_0} - \frac{1}{g_P} + \frac{1}{3}\left(\eta_P^2 + \frac{15}{48}\overline{x}_P^2 \right) \right] + O(P^3), \\ A(P) &= O(P^3). \end{aligned} \tag{84}$$

Neglecting terms of order P^3, the compatibility equation (70) then requires $A_+(P) = 1$, namely

$$\mathcal{E}_P \approx \pm 2\Delta \sqrt{\frac{1}{g_P} - \frac{1}{g_0} + \frac{1}{3}\frac{v_F^2 P^2}{4\Delta^2}}, \tag{85}$$

where $v_F = q_F/m$ is the Fermi velocity. Notice that for $P \neq 0$, X_+ does not vanish, but it is given by Eq. (B5). It remains to impose the normalization, Eq. (57), which yields

$$\sum_{\mathbf{q}} \lambda_+(Pq)\lambda_-(Pq) \approx -g_P \mathrm{v}_P \mathcal{E}_P X_-^2(P) = 1. \tag{86}$$

Because $v_P < 0$, the composites energy must be positive and one must choose the plus sign in Eq. (85). For g_P independent of P one recovers the standard energy of the Bogoliubov-Anderson mode. If $g_0 > g_P$, there is a correction to such energy. It remains the case that $g_0 < g_P$. I notice that such a condition does not necessarily imply instability: indeed, a state with Cooper pairs condensed with nonvanishing total momentum should have density fluctuations, which would have a kinetic energy cost. Stability of the BCS state then depends on the balance between kinetic and potential contributions, at variance with condensation of Cooper pairs with nonvanishing intrinsic angular momentum and vanishing total momentum [3].

So the BCS ground state might be stable provided $g_P - g_0$ is small enough. If such a condition is verified and the energy is real, again there is a correction to the standard Bogoliubov-Anderson dispersion law. If for some values of P the energy becomes imaginary, the normalization condition (57) cannot be satisfied, so that the corresponding state does not exist. I remind the reader that violation of the orthogonality conditions implies that the Bogoliubov transformation is not unitary, so that the composite Hamiltonian is not equivalent to any projection of the original fermion Hamiltonian.

One can see that $X_-(P)$ diverges as $\sqrt{|P|}^{-1}$ for $P \to 0$. Then Eq. (B5) gives $X_+ = O(|P|^{\frac{5}{2}})$. Finally from Eqs. (65) one can see that the intrinsic wave function of the Bogoliubov-Anderson particles is polarized along the total momentum P, and for small values of such momentum it approaches the wave function of the Cooper pairs in the condensate

$$\phi_+(Pq) = \tfrac{1}{2}[\lambda_+(Pq) + \lambda_-(Pq)] \to [2\,\overline{E}(Pq) - \mathcal{E}_{\mathbf{P}}]^{-1} \quad (87)$$

so that one can conclude that the Bogoliubov-Anderson particles are Cooper pairs going out of the condensate.

D. Other modes?

Now I investigate whether there are collective energies $\mathcal{E} \lesssim 2\Delta$. In a preliminary study [29] (in which I determined the saddle point in the presence of the Hubbard-Stratonovich fields, and I integrated over their fluctuations), I derived exactly the Hamiltonian (51), but in the determination of its energy eigenvalues I made the too drastic simplification of retaining only one term in the sum (42), even though such an approximation violates the completeness condition Eq. (43). Such an approximation gives indeed the incorrect result for the energy gap,

$$\mathcal{E}_0 = 2\frac{\Delta}{\sqrt{1 + (\pi g_0)^2}}. \quad (88)$$

It is perhaps not surprising that by relaxing a condition I got a lower energy.

I evaluate now the energy with the full expansion in the sum (42) looking for solutions to the compatibility equations with $\mathcal{E} \leqslant 2\Delta$. Setting

$$\eta_P^2 = 1 - \sigma_P^2 \quad (89)$$

and assuming $\sigma_P \sim \overline{\xi}_P$ for $P \to 0$, evaluation of $A_\pm(P), A(P)$ then gives

$$\begin{aligned} A_+(P) - 1 &\approx g_P \left\{ \frac{1}{g_0} - \frac{1}{g_P} + \frac{\pi}{2}\frac{1}{\sigma_P} f_+\left(\frac{\overline{x}_P}{\sigma_P}\right) \right\}, \\ A_-(P) - 1 &\approx g_P \left\{ \frac{1}{g_0} - \frac{1}{g_P} - \frac{\pi}{4}\sigma_P f_-\left(\frac{\overline{x}_P}{\sigma_P}\right) \right\}, \\ A(P) &\approx \frac{\pi}{4} g_P \frac{P^2}{4m\Delta}\frac{\sqrt{1-\sigma_P^2}}{\sigma_P} f_+\left(\frac{\overline{x}_P}{\sigma_P}\right), \end{aligned} \quad (90)$$

where

$$\begin{aligned} f_+(\rho) &= \frac{1}{\rho}\ln\left(\rho + \sqrt{1+\rho^2}\right), \\ f_-(\rho) &= \sqrt{1+\rho^2} + f_+(\rho). \end{aligned} \quad (91)$$

Notice that $f_\pm(1) = O(1)$. Because $f_\pm > 0$, there are three possibilities,

$$\begin{aligned} &g_P < g_0, \quad A_+ - 1 = 0, \quad A_- - 1 \neq 0, \\ &g_P = g_0, \quad A_+ - 1 \neq 0, \quad A_- - 1 \neq 0, \\ &g_P > g_0, \quad A_+ - 1 \neq 0, \quad A_- - 1 = 0. \end{aligned} \quad (92)$$

In the first case, the solution should be

$$\sigma_P = \frac{\pi}{2}\frac{g_0 g_P}{g_0 - g_P} f_+\left(\frac{\overline{x}_P}{\sigma_P}\right), \quad (93)$$

which is unacceptable because σ_P diverges for $P \to 0$ (assuming v_P is a continuous function).

Also in the second case, a solution to the compatibility equation does not exist, because $(A_+ - 1)(A_- - 1) \sim 1$ while $A(P)^2 \sim \overline{x}_P^2$, contrary to the result of [29] due, as already explained, to an unacceptable approximation.

In the third case, there is the solution

$$\sigma_P = \frac{4}{\pi}\frac{g_P - g_0}{g_0 g_P}\frac{1}{f_-\left(\frac{\overline{x}_P}{\sigma_P}\right)} \quad (94)$$

provided $g_P - g_0 \sim \overline{x}_P$ as a consequence of the assumption that $\sigma_P \sim \overline{x}_P$ in the derivation of the above result (but again one should discuss the stability of the BCS ground state). If such a state exists, it should be strongly mixed with states of two quasiparticles that are almost degenerate.

IX. POINTLIKE COMPOSITES IN A RELATIVISTIC FOUR-FERMION INTERACTION MODEL

In all the previous examples, the intrinsic wave functions are extended in space. I think that pointlike composites can be generated only by repulsive fermion-fermion interactions. To give an example of pointlike composites with repulsive fermion-fermion interaction, I report briefly the results of Ref. [23] in which the properties of the relativistic four-fermion interaction model [24], namely the value of the mass of the composite boson and the logarithmic divergences of the wave functions, are exactly reproduced by the nilpotency expansion for a large number of flavors. The action of the model in continuum Euclidean space reads

$$S = \int d^4x \left[\overline{\psi}(x) \left(m + \sum_{\mu=1}^{4} \gamma_\mu \partial_{x_\mu} \right) \psi(x) + \frac{1}{2} \frac{g^2}{4N_f} [\bar{\psi}(x)\psi(x)]^2 \right], \tag{95}$$

where m is the mass parameter, γ_μ are the Dirac matrices, g^2 is the coupling constant, and ψ is the fermion field, which has N_f degenerate flavors. This model has a discrete chiral symmetry at $m = 0$:

$$\psi \rightarrow -\gamma_5 \psi, \quad \bar{\psi} \rightarrow \bar{\psi}\gamma_5. \tag{96}$$

For large N_f, this symmetry is spontaneously broken [24], with generation of a boson whose mass is twice that of the fermion quasiparticles. There is no Goldstone-Nambu boson because the chiral symmetry is not continuous. The model can be linearized introducing an auxiliary scalar field $\sigma(x)$ whose integration generates the four-fermion coupling:

$$S' = \int d^4x \left[\overline{\psi}(x) \left(m + \sigma(x) + \sum_{\mu=1}^{4} \gamma_\mu \partial_{x_\mu} \right) \psi(x) + \frac{4N_f}{2g^2} \sigma^2(x) \right] \tag{97}$$

so that the partition function can be rewritten

$$\mathcal{Z} = \int [d\sigma][d\overline{\psi} d\psi] \exp[-S']. \tag{98}$$

It can be regularized on a lattice that provides the necessary cutoff replacing the continuous derivatives by discrete derivatives. I do not discuss here the problems related with such a replacement for fermions. One can then put the partition function in the form (A8) if in the transfer matrix (7) one sets

$$N_t = -2\left[\gamma_0(m + \sigma_t) + \sum_{i=1}^{3} (\gamma_0 \gamma_i \partial_{x_i})_{\text{lattice}} \right], \quad M = 0. \tag{99}$$

Now one can perform time-dependent Bogoliubov transformations and determine the values of the time-independent fields $\overline{B}, \overline{\sigma}$ that minimize the action. At this point, one introduces the field fluctuations

$$B_t^\dagger = \overline{B}^\dagger + \delta B_t^\dagger, \quad B_t = \overline{B} + \delta B_t, \tag{100}$$

$$\sigma(t, \mathbf{x}) = \overline{\sigma} + \eta(t, \mathbf{x}). \tag{101}$$

The η-field is homologous of the fluctuations $\delta N^\dagger, \delta N$ in the BCS model in the formulation [23]. But at variance with the BCS model in which the physical composite is described by the fluctuations $\delta B_t^\dagger, \delta B_t$, the physical composite of the relativistic model is described by the local field η and it is therefore pointlike. Its action reads

$$S_\eta = \int \frac{d\omega}{2\pi} \int \frac{d^3p}{(2\pi)^3} \eta(\omega, \mathbf{p}) \eta(-\omega, -\mathbf{p}) \times Z(\overline{\sigma}^2)(\omega^2 + \mathbf{p}^2 + (2\overline{\sigma})^2), \tag{102}$$

where

$$Z(\overline{\sigma}^2) \simeq -\frac{1}{(2\pi)^2} \ln \overline{\sigma}^2. \tag{103}$$

The above equations show that the composite boson has mass $2\overline{\sigma}$, while the fermionic quasiparticles have mass $\overline{\sigma}$. The cutoff is contained in $\overline{\sigma}$, which is defined in lattice units, but the factor $\ln \overline{\sigma}^2$ that diverges when the lattice cutoff is removed going to the continuum temporal limit is absorbed in the wave-function renormalization.

The composite boson of the relativistic model is generated by a potential of the form $[\overline{\psi}(x)\psi(x)]^2 = (n_x - 1)^2$, where n_x is the fermion number at point x. Such a potential induces an in-site fermion occupation and then a pointlike composite boson, and it is similar to the ϕ^4 potential in the theory of a scalar with the "wrong" sign of the mass. These features are related to the fact that while in the BCS model the energy spectrum depends explicitly on the cutoff, in the relativistic model the cutoff is eliminated absorbing the logarithmic divergencies in the renormalization of the wave function.

A further comment concerns the comparison between superconducting systems and the Higgs model. The latter is constructed in terms of an *elementary* scalar field, and the Higgs field is given by the fluctuation of its amplitude. The elementary fields in a superconducting system are instead fermions, and therefore such a system is more similar to a chiral system of quarks in particle physics, and the gapped mode to the σ-meson. In this case, part of the problem is to determine the structure of the composite bosons, which are the pions and the σ meson, in terms of the fermion fields, which are the quarks fields. The corresponding problem of determining the structure of the composite bosons as well as the occurrence of several such modes in superconducting systems is altogether neglected in several approaches to collective modes based on effective Lagrangians.

X. SUMMARY AND CONCLUSIONS

I have determined the intrinsic Hamiltonian of composites in superfluid fermion systems whose interaction is a sum of separable terms, using the nilpotency expansion, which is an expansion in the inverse of the nilpotency index of the composites. Such an index counts the number of fermion states in the composite wave functions, and therefore it is a measure of their collectivity. Obviously the more collective a state is, the better its approximation as a particle can be. The formalism provides also the Hamiltonian of fermion quasiparticles and their interaction with composites, which are therefore treated on the same footing. I hope in a future work to complete the present investigation with the study of excitations of energy greater than 2Δ, about which very interesting results already exist [13,14], in a unified framework.

The intrinsic Hamiltonian of the composites is of the Brillouin-Wigner-type, namely Hermitian and energy-dependent. It is expressed in terms of the background field, namely the solution of the gap equation, and of the form factors of the fermion interaction. It is not equal to the projected Hamiltonian (2), because it does not act in a totally projected space, but in a space in which there exist also fermionic quasiparticles coupled to the composites.

The formalism shows that one of the two energy eigenvalues of opposite sign usually arising in such a context must be discarded because it does not satisfy a condition necessary for the unitarity of the Bogoliubov transformation. Moreover, it provides a systematic way to determine the wave functions of the constituent fermions in the composites. Such wave functions are necessarily extended in space.

As a check, I rederived the dispersion law of the Bogoliubov-Anderson composite and determined explicitly its intrinsic wave function in the long wave limit, while its full determination requires a numerical calculation. In the same way, I was able to reproduce the results of Bardasis and Schrieffer [3] and I investigated the possibility of other modes depending on the specific form of the fermion-fermion interaction.

By comparison, I reported previous results about the pointlike composite arising in a relativistic four-fermion model [28]: In this model, indeed, the fermion-fermion interaction is repulsive, and the formation of the condensate is due to a mechanism analogous to the breaking of chiral invariance in the Goldstone model [6], namely a mass with the "wrong" sign. I think that pointlike composites are possible only with repulsive fermion-fermion interactions.

Finally, I noticed that in the original formulation [22] the nilpotency expansion was performed respecting fermion number conservation, while in the procedure adopted in the present paper, the fermion number is explicitly broken. Let me emphasize, however, that such a procedure has been applied to relativistic theories without breaking the fermion number. The reason is that while in nonrelativistic models the fluctuations δB are associated with dynamical Cooper pairs of fermions, and therefore they carry fermion number 2, in relativistic theories they are associated with fermion-antifermion pairs, so that they carry fermion number zero. An exception is the case of diquarks in QCD [28]. But also in this case, due to the peculiar solution of the saddle point equations for the Bogoliubov transformations, the fermion number is not violated.

APPENDIX A: PARTITION FUNCTION FOR NONRELATIVISTIC SYSTEMS

The effective fermion-fermion potential in terms of the original creation-annihilation operators is

$$\hat{V} = \sum_{p_1 s_1 p_2 s_2 p_3 s_3 p_4 s_4} \frac{1}{2}\, \hat{c}^\dagger_{p_1 s_1} \hat{c}^\dagger_{p_2 s_2} V_{p_1 s_1 p_2 s_2 p_3 s_3 p_4 s_4}\, \hat{c}_{p_4 s_4} \hat{c}_{p_3 s_3}, \quad \text{(A1)}$$

where

$$V_{p_1 s_1 p_2 s_2 p_3 s_3 p_4 s_4} = \sum_S \left\langle \frac{1}{2} s_1 \frac{1}{2} s_2 | S0 \right\rangle \left\langle \frac{1}{2} s_3 \frac{1}{2} s_4 | S0 \right\rangle \times V^{(S)}_{p_1 p_2 p_3 p_4}, \quad \text{(A2)}$$

$$V_{p_1 p_2 p_3 p_4} = \sum_S V^{(S)}_{p_1 p_2 p_3 p_4}, \quad \text{(A3)}$$

$\langle \frac{1}{2} s_1 \frac{1}{2} s_2 | S0 \rangle$ being Clebsh-Gordan coefficients. Obviously,

$$V^{(S)}_{p_1 p_2 p_3 p_4} = (-)^S V^{(S)}_{p_2 p_1 p_3 p_4} = (-)^S V^{(S)}_{p_1 p_2 p_4 p_3}. \quad \text{(A4)}$$

Restricting myself to spin singlet composites and to interactions that are sums of separable terms, I can write

$$V = \sum_{P\nu} \mathrm{v}_{P\nu} \sum_{qq'} f_\nu(Pq) f_\nu(Pq') \times \hat{u}^\dagger_{\frac{1}{2}P+q} \hat{v}^\dagger_{\frac{1}{2}P-q} \hat{v}_{\frac{1}{2}P-q'} \hat{u}_{\frac{1}{2}P+q'}. \quad \text{(A5)}$$

In the above equation, P, q are the total and relative momenta of the fermion pairs, ν represents all the additional quantum numbers, and $\mathrm{v}_{P\nu}$ and $f_\nu(Pq)$ are the corresponding strengths and form factors, respectively. The quantum number ν will be mostly omitted.

Regarded as matrices in momentum space, the form factors must be written

$$[f_\nu(P)]_{k_1 k_2} = \delta_{k_1+k_2,P} f_\nu(P,k), \quad k = \tfrac{1}{2}(k_1 - k_2). \quad \text{(A6)}$$

In the present work, I will assume, with the exception of Sec. VIII B,

$$f_\nu(P,k) = 1 \ \text{ for } \xi(k) < \omega \quad \text{(A7)}$$

and zero otherwise, where ω is an energy cutoff.

After a Hubbard-Stratonovich transformation, the nonrelativistic partition function with discrete (imaginary) time can be written

$$Z = \int [dN^* dN][du^* du][dv^* dv] \exp\Big(-\tau \sum_t u_t^* \nabla_t u_t + v_t^* \nabla_t v_t + u_t^* \xi\, u_{t-1} + v_t^* \xi\, v_{t-1} - u_{t-1} N_t^* v_{t-1} - v_t^* N_t u_t^* - N_t^\dagger V^{-1} N_t \Big) \quad \text{(A8)}$$

with the notation

$$[dadb] = \prod_t da_t db_t. \quad \text{(A9)}$$

τ is the time spacing, V^{-1} is the inverse of the potential matrix appearing in Eq. (A5), u^*, u, v^*, v are Grassmann variables, and

$$\nabla_t f_t = \frac{1}{\tau}(f_t - f_{t-1}). \quad \text{(A10)}$$

In the above equations, the fermion fields are defined on lattice sites, and I go to momentum space according to

$$N_{x_1 x_2} = \mathcal{N}_s^{-3} \sum_{p_1 p_2} N_{p_1 p_2} \exp\left(i(p_1 x_1 + p_2 x_2)\right),$$
$$D_{x_1 x_2} = \mathcal{N}_s^{-3} \sum_p D_p \exp\left(i[p\,(x_1 - x_2)]\right), \quad \text{(A11)}$$

where $\mathcal{N}_s^{\,3}$ is the number of spatial sites of the lattice. The second definition applies to matrices diagonal in momentum space.

APPENDIX B: DERIVATION OF THE INTRINSIC HAMILTONIAN

To find the expression of the intrinsic Hamiltonian, I first rewrite Eqs. (64) in terms of the $\phi_\pm$'s,

$$\sum_{q'} \mathcal{H}_\mp(Pqq')[\phi_+(Pq') \pm \phi_-(Pq')] = \mathcal{E}_P[\phi_+(Pq) \mp \phi_-(Pq)]. \quad \text{(B1)}$$

Summing and subtracting, I get

$$[\mathcal{H}_+ \pm \mathcal{H}_-]\phi_+ - [\mathcal{H}_+ \mp \mathcal{H}_-]\phi_- = 2\mathcal{E}\phi_\pm \quad \text{(B2)}$$

from which I find

$$\phi_\pm(Pq) = \sum_{q'q''}[(\mathcal{H}_+ + \mathcal{H}_- \mp 2\mathcal{E}_P)^{-1}](qq') \times [\mathcal{H}_+(Pq'q'') - \mathcal{H}_-(Pq'q'')]\phi_\mp(Pq''). \quad \text{(B3)}$$

Putting Eqs. (B3) into Eqs. (B2), I get

$$\mathcal{H}_{\text{intr}}(\pm\mathcal{E})\phi_\pm = \pm\mathcal{E}\phi_\pm, \quad \text{(B4)}$$

where $\mathcal{H}_{\text{intr}}$ is given by Eq. (71).

Next, using Eqs. (68),

$$X_\pm = -\frac{A}{A_\mp - 1}X_\mp, \quad \text{(B5)}$$

I can write both $\lambda_\pm$ in terms of X_+,

$$\lambda_+ = \left(\mathcal{A}_- - \frac{\mathcal{E}}{2\overline{E}}\mathcal{A}_+\frac{A}{A_+ - 1}\right)X_+,$$

$$\lambda_- = \left(\frac{\mathcal{E}}{2\overline{E}}\mathcal{A}_- - \mathcal{A}_+\frac{A}{A_+ - 1}\right)X_+, \quad \text{(B6)}$$

or in terms of X_-,

$$\lambda_+ = \left(\frac{\mathcal{E}}{2\overline{E}}\mathcal{A}_+ - \mathcal{A}_-\frac{A}{A_- - 1}\right)X_-,$$

$$\lambda_- = \left(\mathcal{A}_+ - \frac{\mathcal{E}}{2\overline{E}}\mathcal{A}_-\frac{A}{A_- - 1}\right)X_-. \quad \text{(B7)}$$

Because $\mathcal{A}_\pm$ and $X_\pm^2$ are even functions of $\mathcal{E}$, Eq. (57) selects one sign of the energy, which, in the cases I considered, turned out to be positive.

When the λ's satisfy Eqs. (64), I can write

$$\mathcal{E}^{mn} = \mathcal{E}_+^{mn} = \mathcal{E}_-^{mn} = \phi_+^m\mathcal{H}_{\text{intr}}(\mathcal{E})\phi_+^n = \mathcal{E}^n\delta^{mn}, \quad \text{(B8)}$$

which shows that $\mathcal{H}_{\text{intr}}(\mathcal{E})$ has the energy spectrum of the composites.

From the above equations, one gets the expressions of the $\phi_\pm$'s in terms of X_+ or X_-,

$$\phi_\pm = \frac{1}{2}\left[\left(1 \pm \frac{\mathcal{E}}{2\overline{E}}\right)\mathcal{A}_+ - \left(\frac{\mathcal{E}}{2\overline{E}} \pm 1\right)\mathcal{A}_+\frac{A}{A_\pm - 1}\right]X_+,$$

$$\phi_\pm = \frac{1}{2}\left[\left(\frac{\mathcal{E}}{2\overline{E}} \pm 1\right)\mathcal{A}_+ - \left(1 \pm \frac{\mathcal{E}}{2\overline{E}}\right)\mathcal{A}_-\frac{A}{A_\pm - 1}\right]X_+. \quad \text{(B9)}$$

Their normalization is determined by the values of the $X_\pm$, which in turn are fixed by the normalization condition (57).

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