Prethermal quasiconserved observables in Floquet quantum systems

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Prethermalization, by introducing emergent quasiconserved observables, plays a crucial role in protecting periodically driven (Floquet) many-body phases over an exponentially long time, while the ultimate fate of such quasiconserved operators can signal thermalization to infinite temperature. To elucidate the properties of prethermal quasiconservation in many-body Floquet systems, here we systematically analyze infinite-temperature correlations between observables. We numerically show that the late-time behavior of the autocorrelations unambiguously distinguishes quasiconserved observables from nonconserved ones, allowing one to single out a set of linearly independent quasiconserved observables. By investigating two Floquet spin models, we identify two different mechanisms underlying the quasiconservation law. First, we numerically verify energy quasiconservation when the driving frequency is large, so that the system dynamics is approximately described by a static prethermal Hamiltonian. More interestingly, under moderate driving frequency, another quasiconserved observable can still persist if the Floquet driving contains a large global rotation. We show theoretically how to calculate this conserved observable and provide numerical verification. Having systematically identified all quasiconserved observables, we can finally investigate their behavior in the infinite-time limit and thermodynamic limit, using autocorrelations obtained from both numerical simulation and experiments in solid-state nuclear magnetic resonance systems.

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I. INTRODUCTION

Controlling quantum systems using a periodic (Floquet) drive has emerged as a powerful tool in the field of condensed-matter physics and quantum information science. It has been used to realize Hamiltonians that are not accessible in a static system, such as modifying the tunneling and coupling rates [1–6], inducing nontrivial topological structures [7–17], creating synthetic gauge fields [18–22], and spin-orbit couplings [23]. On a quantum computer, Floquet engineering also enables universal quantum simulation via the Trotter-Suzuki scheme [24–30]. Floquet systems also possess interesting dynamical phenomena, ranging from the discrete-time crystalline phase [31–35] to dynamical localization [36,37], dynamical phase transitions [38,39], and coherent destruction of tunneling [40–42].

While the connection to an effective time-independent Hamiltonian is appealing, the active drive leads to energy absorption by the Floquet many-body system, which is then expected to heat up to infinite temperature. The heating is detrimental to any quantum application, as no local quantum information is retained and all interesting phenomena mentioned above disappear [43–45]. It has been shown theoretically [46–50] and experimentally [51,52] that even when the system heats up, the thermalization time can be exponentially long in the drive parameters (typically the frequency of a rapid drive). Then, a long-lived *prethermal* quasiequilibrium is established, which allows exploitation of the engineered Floquet Hamiltonian for quantum simulation [53–55]. The emergent symmetries and conserved observables in the prethermal state distinguish it from the fully thermalized state and underpin the existence of novel Floquet phases [34,35,50]. Even more surprisingly, some numerical studies have shown that the emergent conserved observables might not display thermalizing behavior even in the infinite-time limit [53–56]. Many-body localization [32,57–63], dynamic localization [53,55,64], and some fine-tuned driving protocols [54,56,65] provide a way to escape the thermalization fate, which could also be absent in finite-size systems. Indeed, distinguishing the long-lived prethermal state from an eventual thermal state is challenging. Numerical studies are bound to finite-size (and often small) systems, while experiments can only probe finite times, before the external environment induces thermal relaxation.

Here we tackle this problem by a numerical and experimental study of two Floquet models in spin chains, namely, the kicked dipolar model (KDM) and the alternating dipolar model (ADM). While most studies on spin chain dynamics have focused on the evolution of pure states, here we propose to study Floquet prethermalization using infinite-temperature correlations. This metric provides information about quasi-

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conserved observables across the whole spectrum and serves as a direct measurable quantity in nuclear magnetic resonance (NMR) experiments. In Sec. II, we show that the existence of long-lived quasiconserved observables can be unambiguously identified using late-time behavior of the correlations, based on which we provide a method to systematically search for all linearly independent local quasiconserved quantities. Then we provide both numerical and analytical tools to investigate such prethermal conserved observables and their origins. We first show that the prethermal Hamiltonian $H_{\rm pre}$ obtained from the Magnus expansion under rapid drive yields a quasiconserved observable in each model, in Sec. III A. We further show, in Sec. III B, that when the driving Hamiltonian contains a large global rotation, the Floquet propagator can induce an additional conserved observable, as shown by going beyond the usual Magnus expansion. With all the quasiconserved observables at hand, we investigate, in Sec. IV, whether they exist in the thermodynamic limit and infinite-time limit by looking at the dependence of autocorrelations on system size (numerically) and on time (experimentally). Both methods indicate that quasiconserved observables vanish and the system thermalizes to infinite temperature.

II. QUASICONSERVED OBSERVABLES

A. Hamiltonians and correlations

In this paper, we use the Trotter-Suzuki scheme for the driving protocol, where the time-dependent Hamiltonian is piecewise constant in one driving period. However, our results are general for any form of periodic driving. The evolution of the system that we study is given by the unitary propagator in one period, $U_F = e^{-iH_2\tau}e^{-iH_1\tau}$, where in each period we consider the system to be under the Hamiltonian H_1 for a time τ, and then under H_2 for another duration τ. Motivated by NMR experiments, we consider two models of an L-site spin-1/2 chain: the kicked dipolar model (KDM), where $H_1^{(K)} = JD_y$ and $H_2^{(K)} = hZ$, and the alternating dipolar model (ADM), where $H_1^{(A)} = JD_y$ and $H_2^{(A)} = JD_x$. Here, $D_\alpha = \sum_{j<k}\frac{1}{2}(3S_\alpha^j S_\alpha^k - \vec{S}_j\cdot\vec{S}_k)/|j-k|^3$ is the dipolar interaction operator in an arbitrary direction set by α ($\alpha = x, y, z$), where S_α^j are spin-1/2 operators of the jth spin ($j = 1, \dots, L$) and $\vec{S}_j = (S_x^j, S_y^j, S_z^j)^T$. As shown in Ref. [35], the $1/r^3$ interaction is sufficiently short range in one dimension (1D) to yield no qualitative difference with respect to the nearest-neighbor interaction, and thus for simplicity in numerical and analytical studies we only keep the nearest-neighbor interaction unless explicitly mentioned. $Z = \sum_j S_z^j$ is the collective magnetization operator along the z axis, and below we will also use $X = \sum_j S_x^j$, $Y = \sum_j S_y^j$. J and h are the strength of the dipolar interaction and the collective z field, respectively, and we fix $h = J$ throughout the paper. In numerics, we assume periodic boundary conditions.

To investigate the quasiconservation properties, we use infinite-temperature correlations as our metric, $\langle\mathcal{O}(t)\mathcal{O}'\rangle_{\beta=0} \equiv \mathrm{Tr}[U_t\mathcal{O}U_t^\dagger\mathcal{O}']/(\|\mathcal{O}\|\|\mathcal{O}'\|)$, where U_t is the unitary evolution during time t, $\mathcal{O}$ and $\mathcal{O}'$ are observables, and the norm is defined as $\|\mathcal{O}\| \equiv \sqrt{\mathrm{Tr}\mathcal{O}^2}$ [66]. Note that early works [67] used this metric to determine whether a system is ergodic or integrable. Here we show that we can also use these correlations to identify quasiconservation in prethermal systems, even if they are expected to be ergodic.

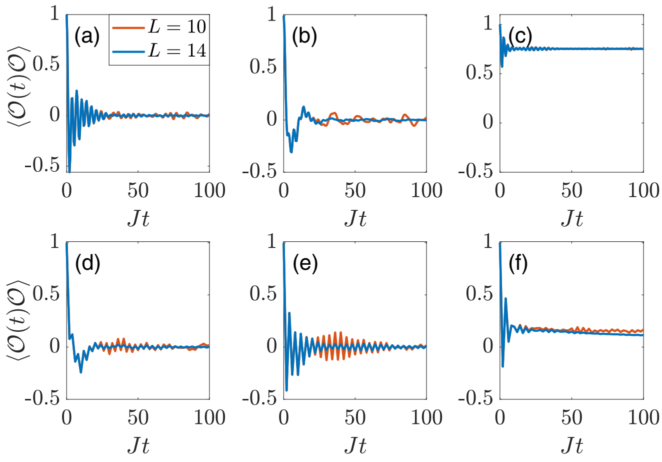

FIG. 1. Typical dynamics of $\langle\mathcal{O}(t)\mathcal{O}'\rangle$ in a Floquet spin chain. Here we choose KDM and $\mathcal{O} = \mathcal{O}'$. (a)–(c) $J\tau = 0.5$; (d)–(f) $J\tau = 2$. (a),(d) $\mathcal{O} = X$; (b),(e) $\mathcal{O} = Y$; (c),(f) $\mathcal{O} = Z$. Different colors correspond to different system size L, as shown in the legend.

Figure 1 shows numerical simulations of some exemplary correlations, i.e., the magnetization along three axes $\mathcal{O} = \mathcal{O}' = Z, X, Y$ in KDM (the qualitative behavior is general for other observables and models.) The autocorrelations of X and Y display oscillations around 0 and damping, which originate from the z field and the dipolar interaction, respectively. Instead, $\langle Z(t)Z\rangle$ exhibits a more interesting behavior. For small $J\tau$, it quickly equilibrates at a nonzero value independent of L, and it remains constant afterwards. For relatively large $J\tau$, there is a slow decay of $\langle Z(t)Z\rangle$ toward a final value that decreases with increasing L. We thus expect the final value to be zero in the thermodynamic limit, corresponding to an infinite-temperature final state. Indeed, the observable Z displays the defining characteristics of what we deem a quasiconserved observable in the prethermal regime: the autocorrelation of a quasiconserved observable is nonzero in the prethermal regime, but goes to zero in the fully thermalized state. In simulations, autocorrelations of quasiconserved observables still have nonzero value at infinite time due to the small system size [e.g., $\langle Z(t)Z\rangle$ in Fig. 1], while for nonconserved observables, autocorrelations are zero [e.g., $\langle X(t)X\rangle$ in Fig. 1]. These distinct behaviors serve as a direct metric to identify quasiconserved observables. As any observable that overlaps with a quasiconserved observable would have nonzero infinite-time autocorrelation, we want to find a linearly independent, orthogonal set of *eigen*quasiconserved observables.

B. Eigenquasiconserved observables

We design a systematic procedure to search for the set of eigenquasiconserved observables, $\{\mathcal{E}_\mu\}$, starting from the infinite-time correlations $\langle\mathcal{O}(\infty)\mathcal{O}'\rangle \equiv \lim_{T\to\infty}(1/T)\int_0^T\langle\mathcal{O}(t)\mathcal{O}'\rangle dt$. We note that eigenvectors $\{E_\mu\}$ of the Floquet (super)propagator $\hat{U}_F$ form an orthogonal vector basis for the space of operators (here, $\hat{U}[\mathcal{O}] = U\mathcal{O}U^\dagger$) $|\langle E_j(\infty)E_k\rangle| \propto \delta_{jk}$, which we can call "eigenobservables." However, this operator basis is, in general, highly nonlocal

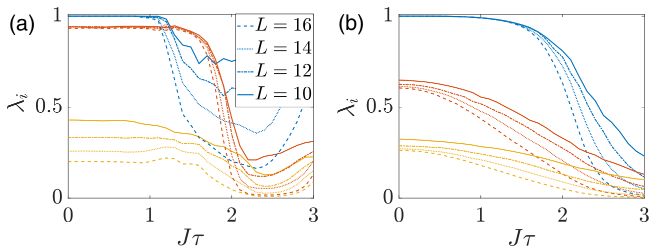

FIG. 2. By considering the matrix Λ obtained for each $J\tau$ Trotter step, we calculate the three largest eigenvalues as a function of $J\tau$ for (a) KDM and (b) ADM. The curve color represents different eigenvalues and the curve style represents different system sizes. From the eigenvalues and their dependence on system size, we see that there are two eigenquasiconserved observables in KDM, while only one in ADM.

and thus not practical. We then want to find a small, local set of observables that approximates the exact eigenobservables and has nonzero eigenvalues, that is, are quasiconserved. We start from a basis set $\{\mathcal{O}_{(\alpha)}\}$ of Hermitian observables that are translationally invariant sums of local operators,

$$\mathcal{O}_{(\alpha)} = \sum_j S^j_{\alpha_1} S^{j+1}_{\alpha_2} \cdots S^{j+r-1}_{\alpha_r}. \tag{1}$$

Here, $(\alpha) \equiv (\alpha_1, \dots, \alpha_r)$ with $\alpha_k \in \{x, y, z, 0\}$, where S^j_0 denotes the identity matrix operating on the jth spin. By imposing $\alpha_1, \alpha_r \neq 0$, we say $\mathcal{O}_{(\alpha)}$ is of the range of r: each term in $\mathcal{O}_{(\alpha)}$ acts nontrivially on most r neighboring spins. Since the number of operators is exponentially large in system size, we restrict our search to the operator subspace spanned by $\mathcal{O}_{(\alpha)}$ whose range $r \leqslant r_c$, which are local and thus experimentally relevant. Starting from an orthonormal operator basis $\{\mathcal{O}_\mu\}$ of this subspace (with $\langle \mathcal{O}_\mu \mathcal{O}_\nu \rangle = \delta_{\mu\nu}$), we construct a matrix from all pair correlations, $\Lambda_{\mu\nu} = \langle \mathcal{O}_\mu(\infty) \mathcal{O}_\nu \rangle$. The matrix Λ is the projection of the infinite-time propagator $\hat{U}_F(t \to \infty)$ onto the r_c-local subspace. The diagonalization of Λ yields the local eigenobservables $\mathcal{E}_k$, and eigenvalues λ_k, satisfying $\langle \mathcal{E}_k(\infty) \mathcal{E}_l \rangle = \lambda_k \delta_{kl}$. Note that since Λ is not ensured to be unitary, its eigenvalues do not have unit amplitude, $\lambda_k \leqslant 1$. We note that the larger the λ_k, the better $\mathcal{E}_k$ approximates an exactly conserved observable. The correlations $\langle \mathcal{O}(\infty) \mathcal{O}' \rangle$ between any two observables whose locality is bounded by r_c can be directly derived by decomposing the observables onto the $\mathcal{E}_\mu$ basis,

$$\langle \mathcal{O}(\infty) \mathcal{O}' \rangle = \sum_\mu \lambda_\mu \langle \mathcal{O} \mathcal{E}_\mu \rangle \langle \mathcal{E}_\mu \mathcal{O}' \rangle. \tag{2}$$

We apply this systematic procedure to the two models under consideration. The infinite-time limit $\mathcal{O}(\infty)$ is taken by considering the diagonal ensemble of $\mathcal{O}$ (that is, keeping only the diagonal matrix elements of $\mathcal{O}$ in the Floquet energy eigenbasis), which gives the same result as averaging $\mathcal{O}$ over a long time. The results for $r_c = 3$ are shown in Fig. 2. At large Trotter steps τ, most eigenvalues go to zero. The upward trends of the eigenvalues when $J\tau = h\tau \to \pi$ (most pronounced for the largest eigenvalue) are due to the fact that $[e^{-iH_1^{(K)}\tau}, e^{-iH_2^{(K)}\tau}] = 0$ at $J\tau = h\tau \to \pi$, making the system equivalent to a time-independent system. Even for small Trotter steps, most eigenvalues are already small and decrease when increasing system size. However, a few eigenvalues are large and show little dependence on system size. This last group comprises the eigenvalues associated with the eigenquasiconserved observables that govern the nontrivial dynamics at long times.

Based on these results, we find that there are two eigenquasiconserved observables for KDM, $\mathcal{E}_1^{(K)}, \mathcal{E}_2^{(K)}$, and one for ADM, $\mathcal{E}_1^{(A)}$. In both models, $\mathcal{E}_1$ is close to their average Hamiltonian, $\overline{H} = H_1 + H_2$ (blue curves in Fig. 2), while $\mathcal{E}_2^{(K)}$ for KDM is close to D_z [red curves in Fig. 2(a)]. Similar additional conserved quantities were predicted in static models [49]. Here we can more carefully analyze these Floquet quasiconserved observables and describe analytically their origin in the limit of small τ in the next section. Even so, we remark that there is an interesting regime at intermediate τ, where $\mathcal{E}_1^{(K)}, \mathcal{E}_2^{(K)}$ are well conserved, since $\lambda_1^{(K)}, \lambda_2^{(K)}$ are still large, but they deviate from their static ($\tau \to 0$) counterparts. This indicates that the quasiconserved observables truly arise from the Floquet dynamics and are not simply a remnant of the approximated, static Hamiltonian.

III. ANALYTICAL DERIVATION OF CONSERVED OBSERVABLES

A. Prethermal Hamiltonian

It is intuitive to expect that a quasiconserved observable might emerge from energy conservation. Indeed, one can always regard the Floquet evolution as arising from an effective static Hamiltonian by setting $U_F = e^{-i\tau H_F}$ for some Hermitian operator H_F. However, in general, this Hamiltonian is highly nonlocal and thus it is not associated to a local quasiconserved observable. Still, when the driving frequency is large compared to local energy scales (here, J, h), the stroboscopic dynamics is given by a time-independent local prethermal Hamiltonian $H_{\rm pre}$ plus a small correction $\delta H(t)$ [46,48], which may be nonlocal. It is this prethermal Hamiltonian $H_{\rm pre}$ that can be associated with a local quasiconserved observable. $H_{\rm pre}$ can be obtained from the Floquet-Magnus expansion [68,69] truncated at an optimal order m^*,

$$H_{\rm pre} = \sum_{m=0}^{m^*} \tau^m \Omega_m, \tag{3}$$

where the zeroth-order term is the average Hamiltonian $\Omega_0 = \overline{H} = 1/\tau \int_0^\tau H(t)dt$ and higher-order terms Ω_m involve m nested commutators. Then, for spin chains with nearest-neighbor couplings, the range of Ω_m grows linearly with m.

The truncation m^* is crucial not only to keep the prethermal Hamiltonian local, but also because the series in Eq. (3) diverges for a generic many-body system [48]. The time-dependent correction δH is, however, exponentially small in $1/J\tau$, leading to an exponentially long time $t_{\rm pre}$ for the system to heat up. Thus, for $t < t_{\rm pre}$, the system effectively prethermalizes to the state $e^{-\beta H_{\rm pre}}$, where β is determined by the initial state energy, making $H_{\rm pre}$ an eigenquasiconserved observable. Although one should investigate the prethermalization process by studying the dynamics of an infinitely large system at long times approaching infinity, numerically we

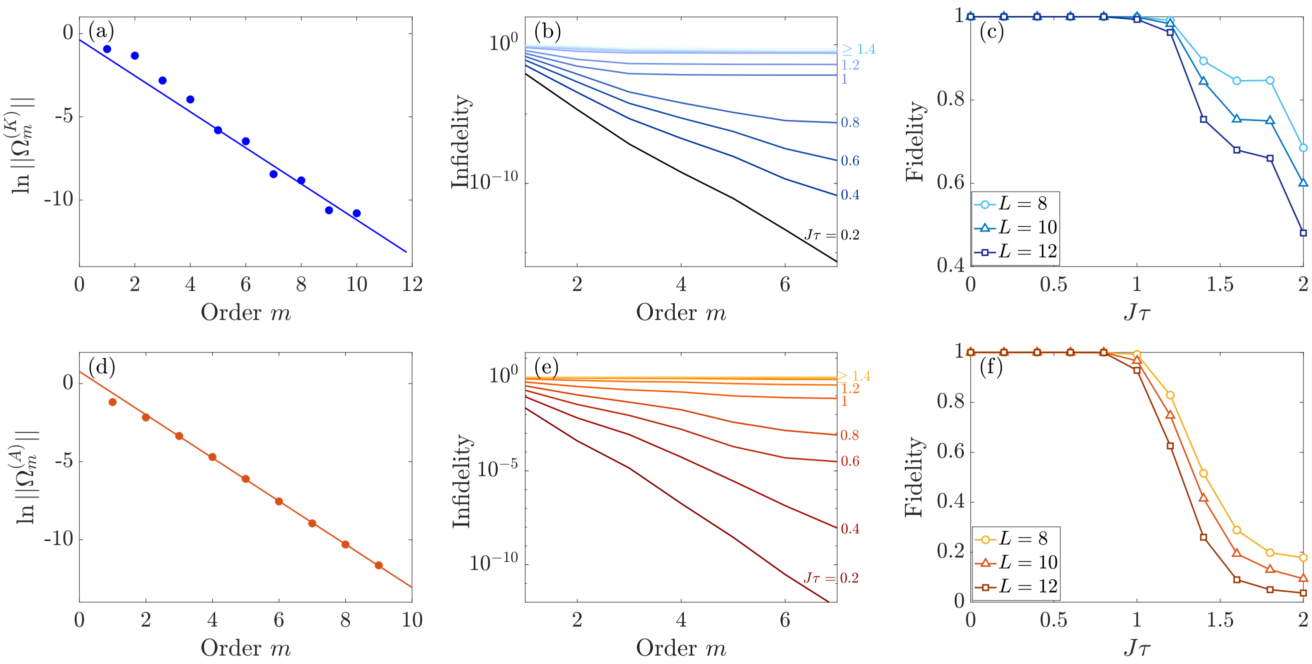

FIG. 3. (a)–(c) The Magnus expansion given by Eq. (3) of KDM; (d)–(f) that of ADM. (a),(d) Circles show the norm of Ω_m (normalized by $L2^L$). The solid line represents the linear fit. (b),(e) Infidelity $1 - \langle H_{\rm pre}(\infty)H_{\rm pre}\rangle$ of infinite-time averaged $H_{\rm pre}$ evaluated up to the mth order. Different curves stand for $J\tau$ from 0.2 to 2, with a step of 0.2. The darker color represents smaller $J\tau$. $L = 12$ is used. (c),(f) Infinite-time autocorrelation of $H_{\rm pre}$ as a function of $J\tau$ for different system sizes. Order $m = 7$.

can only tackle small system sizes, so we take a different approach—we set the time to infinity and study how the observable correlations change when increasing system size. The validity of this approach relies on the fact that for a system size $L < m^*$, the term δH does not appear in the expansion, making $\mathcal{O}_1 = H_{\rm pre}$ exactly conserved even at infinite time for sufficiently small τ. From a physics point of view, this means that the energy $2\pi\hbar/\tau$ is larger than the many-body bandwidth ($\sim JL$), and thus the system cannot absorb energy from the drive if it is faster than $1/JL$. Since the zeroth-order term of $H_{\rm pre}$ is $\overline{H}$, the autocorrelation of $H_{\rm pre}$ provides a bound for that of $\overline{H}$, leading to bounded Trotter error in the Trotter-Suzuki scheme [53].

As further verification, we calculate numerically the Floquet Magnus expansion, given by Eq. (3), up to $m = 10$ and evaluate not only the convergence of the expansion, but also operator conservation. For the first metric, we plot $\|\Omega_m\|$ in Figs. 3(a) and 3(d) for the two models studied. We find that up to the computationally accessible order, the norm of Ω_m decays exponentially, indicating that $H_{\rm pre}$ converges when τ is small. From the slopes in Figs. 3(a) and 3(d), we get radii of convergence $J\tau \approx 3$ for both models. Still, the expansion convergence does not guarantee the resulting $H_{\rm pre}$ is a quasiconserved observable. In Figs. 3(b) and 3(e), we compute the long-time infidelity $(1 - \langle H_{\rm pre}(\infty)H_{\rm pre}\rangle)$ by truncating the expansion in Eq. (3) at increasing orders. When $J\tau$ is small, the autocorrelation exponentially approaches 1 with increasing order, suggesting that the optimal truncation order m^* should be larger than our largest accessible order here, or even absent in the system size we study. Instead, for larger $J\tau$, the correlation stops converging at some order; for even larger $J\tau$ ($J\tau = 1$, for example), the correlation is almost zero for all orders. Therefore, even within the radius of convergence, $J\tau \approx 3$, $H_{\rm pre}$ from Eq. (3) may fail to be quasiconserved. We plot the infinite-time correlation $\langle H_{\rm pre}(\infty)H_{\rm pre}\rangle$ versus $J\tau$ in Figs. 3(c) and 3(f) and show how it changes with system size (here, $H_{\rm pre}$ is evaluated to seventh order). The drop of $\langle H_{\rm pre}(\infty)H_{\rm pre}\rangle$ with increasing system size is evident for $J\tau \gtrsim 1.2$ in both models, suggesting that for the system size we explore, the effective Hamiltonian picture fails in the above parameter space. Note that in the $L \to \infty$ limit, the correlations are expected to be zero for any $\tau > 0$, as will be discussed in Sec. IV.

B. Emergent dipolar order

To search for additional conserved observables in KDM, we develop a method inspired by the existence of discrete time-translation symmetry-protected phases in prethermal Floquet systems [50]. Similar results have been obtained for the static Hamiltonian $\overline{H} = hZ + JD_y$ associated with the (zero-order) KDM. For this model, it has been shown that the polarization Z is quasiconserved and does not reach its thermal equilibrium value until a time that is exponentially long in h/J [49,50,70], even if according to eigenstate thermalization hypothesis (ETH) the system should thermalize.

Since the average Hamiltonian picture breaks down when increasing τ, even though we see from Fig. 2(a) that the second observable is conserved even for larger τ, we must go beyond the static case and work directly on the Floquet system. This kind of system was first studied in [50], where they further focused on the case $h = \pi$ to identify a prethermal Floquet time crystal. Here we generalize their analysis to obtain the novel quasiconserved observable for any h, by following the intuition in [49]: thanks to the integer spectrum of Z, we expect that there exists a frame where the polarization is conserved up to some small, highly nonlocal corrections in the Hamiltonian. Taking into account the Floquet nature of our problem, we find such rotated frame order by order, using not only J/h but also $h\tau$ as a small parameter.

We first transform the Floquet operator by going to a rotated frame as

$$e^S e^{-ih\tau Z} e^{-i\tau H_1} e^{-S} = e^{-ih\tau Z} e^{-i\tau(D+\delta H)}, \tag{4}$$

and demand $[Z, D] = 0$. By appropriately choosing S, D, it will be shown that δH is exponentially small in $\min[O(\frac{h}{J}), O(\frac{1}{h\tau})]$ [71]. Therefore, for small τ and large enough ratio $h/J \gtrsim 0.5$ [70], the operator D approximately commutes with the Floquet unitary in the rotated frame, making $D_{\rm pre} = e^{-S} D e^S$ a prethermal quasiconserved observable in the original frame. We emphasize that the right-hand side of Eq. (4) still describes a Floquet system; therefore we derived the quasiconservation without first transforming to a static Hamiltonian. Note that $Z_{\rm pre} = e^{-S} Z e^S$ is quasiconserved in the same sense as $D_{\rm pre}$. However, whereas $D_{\rm pre}$, is orthogonal to $H_{\rm pre}$ to zeroth order, $Z_{\rm pre} \approx H_{\rm pre} - D_{\rm pre}$ and it cannot thus be considered an eigenquasiconserved observable.

Now we describe in detail how to find the desired S, D. We first write the transformation given by Eq. (4) in an equivalent form,

$$e^{-i\tau(D+\delta H)} = e^{i\epsilon\tilde{Z}} e^S e^{-i\epsilon\tilde{Z}} e^{-i\epsilon^2\tau H_1} e^{-S}. \tag{5}$$

Here we make the shortcut $\tilde{Z} \equiv h\tau Z$, and assume that J/h and $h\tau$ are small parameters of the same order marked by ϵ. S and D can be expressed as a Taylor series of ϵ, $S = \epsilon S_1 + \epsilon^2 S_2 + \cdots$, $D = \epsilon^2 D_2 + \epsilon^3 D_3 + \cdots$. Because S_j are artificial variables, we can choose S_j such that D_{j+1} satisfy the requirement $[Z, D_{j+1}] = 0$. Repeating the process order by order, we have $[Z, D] = 0$ up to a small error term δH. More specifically, one can do Magnus expansion of the right-hand side of Eq. (5) to get

$$-i\tau \sum_{j=2}^{j^*} \epsilon^j D_j = \sum_{j=2}^{j^*} \epsilon^j([i\tilde{Z}, S_{j-1}] + h_j), \tag{6}$$

where we have ignored the high-order δH. Here, h_j is defined recursively as nest commutators of $i\tilde{Z}$, $-i\tau H_1$ and $S_{j'}$ with $j' < j - 1$. For example, the first few orders are

$$h_2 = -i\tau H_1,$$
$$h_3 = [S_1, h_2] + \tfrac{1}{2}([i\tilde{Z}, [i\tilde{Z}, S_1]] - [S_1, [S_1, i\tilde{Z}]]). \tag{7}$$

Recursively, assuming all $S_{j'}$ with $j' < j - 1$ (and thus h_j) are known (which is trivially true for $j = 2$), we determine S_{j-1} and D_j from the jth order of Eq. (6) by requiring that $D_j = [i\tilde{Z}, S_{j-1}] + h_j$ commutes with Z. To do this, we first decompose $h_j = \sum_{q=0,\pm1,\ldots} h_{jq}$ such that $[Z, h_{jq}] = q h_{jq}$ (h_{jq} are called the qth quantum coherence of Z [72–74]). This decomposition is possible as long as the dominant part of the Hamiltonian has integer eigenvalues (up to a common constant), a frequent feature shared by the collective rotation $H_2^{(K)} \propto Z$ in our case. $[Z, D] = 0$ is then satisfied by choosing $-i\tau D_j = h_{j0}$ and $S_{j-1} = i\sum_{q\neq0} h_{jq}/(hq\tau)$. We note that S is a sufficiently local operator, $r(S_j) = j$, for KDM with nearest-neighbor interaction. Similar to the prethermal Hamiltonian given by Eq. (3), the expansion in ϵ generally diverges and should be truncated at some order j^*, leading to the exponentially small nonlocal residual δH; see, e.g., Refs. [46,50].

When τ is small, the S_j operators are dominated by the $(J/h)^j$ term. Therefore, in the $\tau \to 0$ limit, the quasiconserved observable found here for the Floquet model reduces to the prethermal quasiconserved observable of the static Hamiltonian $\overline{H}^{(K)}$ [50,70], where the expansion is a series of J/h and $\delta\tilde{H} \approx \exp[-O(h/J)]$. In this regime, $D_{\rm pre} = -\frac{1}{2}D_z + O[(J/h)^2]$, and the expansion converges for $h/J \gtrsim 0.5$ (up to truncation at exponentially large order) as shown in Ref. [70] (note that here we used $h/J = 1$). Instead, for relatively larger $h\tau$, the S_j operators are dominated by $(h\tau)^j$ and $\delta\tilde{H} \approx \exp[-O(1/h\tau)]$, and thus the system exhibits exponentially slow Floquet heating as expected.

We numerically evaluate the convergence properties of $D_{\rm pre}$ in the KDM [Fig. 4(a)], using the metrics discussed in the previous section, i.e., convergence of the order-by-order expansion terms and infinite-time autocorrelation. We find that the series converges up to order 7 in the $h\tau$ regime that we are interested in. The infinite-time autocorrelation is close to 1 at small τ, as shown in Figs. 4(b) and 4(c), confirming that the local truncation of $D_{\rm pre}$ (as obtained by the first few orders) gives rise to quasiconserved observable $\mathcal{E}_2^{(K)}$. Comparing these results to the prethermal Hamiltonian shown in Figs. 3(b) and 3(c), we find that (i) the normalized autocorrelation of $D_{\rm pre}$ converges to 1 in a larger parameter range ($J\tau \lesssim 1.6$ for $D_{\rm pre}$ and $J\tau \lesssim 1$ for $H_{\rm pre}$), (ii) the autocorrelation shows a significant drop at $J\tau \gtrsim 1.8$ for $D_{\rm pre}$ and $J\tau \gtrsim 1.2$ for $H_{\rm pre}$, with a steeper drop when L is increased from 8 to 12. Both facts suggest that $D_{\rm pre}$ is more robust than $H_{\rm pre}$, in agreement with the experimental results presented in Ref. [51]. This provides evidence that it is possible to realize novel Floquet phases beyond the effective Hamiltonian picture.

IV. TOWARD INFINITE TEMPERATURE: EXPERIMENTAL AND NUMERICAL SIGNATURES

Although it is generally believed that Floquet many-body systems should heat up to infinite temperature, some numerical works [53–56] have found signs of nonthermal behavior in various models. Here we provide evidence of thermalization in the long-time and thermodynamic limit, using numerics and experiments in a NMR quantum simulator [51,70,72], respectively. In simulations, we can access the infinite-time limit using exact diagonalization, but only for small system sizes. Conversely, the system size in NMR experiments is large enough to achieve the thermodynamic limit, but the evolution time cannot be too long due to hardware limitation. Still, by looking at the dynamics for increasingly longer times (experimentally) and larger system sizes (numerically), we can extract insight into the final fate of the Floquet systems.

The experimental system is a single crystal of fluorapatite (FAp) [75]. We study the dynamics of ^{19}F spin-1/2 using NMR techniques. Although the sample is 3D, ^{19}F form a quasi-1D structure because the interaction within the chain is ~40 times larger than the interaction between different chains [76–78]. The average chain length is estimated to be >50

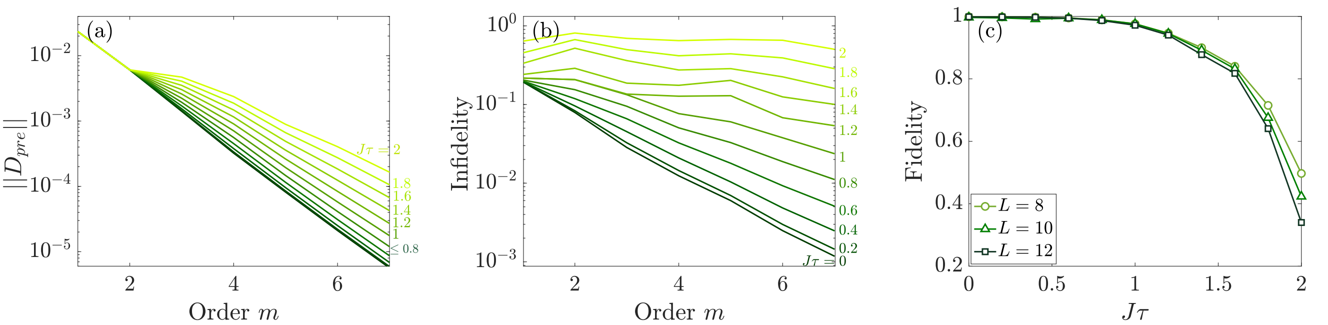

FIG. 4. $D_{\rm pre}$ expansion of KDM. (a) Norm of the mth-order term of the quasiconserved observable $D_{\rm pre}$ (normalized by $L2^L$). Different curves stand for $h\tau = J\tau$ from 0 to 2, in steps of 0.2. The darker color represents smaller $J\tau$. (b) Infidelity $1 - \langle D_{\rm pre}(\infty) D_{\rm pre}\rangle$ of infinite-time averaged $D_{\rm pre}$ evaluated up to the mth order. $L = 12$ is used. (c) Fidelity $\langle D_{\rm pre}(\infty) D_{\rm pre}\rangle$ evaluated to seventh order as a function of $h\tau$ for different system sizes.

and the coherence time of the ^{19}F spins is $T_1 \approx 0.8$ s. The sample is placed in a 7 T magnetic field where the Zeeman interaction dominates, thus reducing the ^{19}F spins interaction to the secular dipolar Hamiltonian $H = J_0 D_z$ with $J_0 = -29.7$ krad/s (we define z as the magnetic field direction). While the corresponding 1D, nearest-neighbor XXZ Hamiltonian is integrable [79–81], the experimental $1/r^3$ Hamiltonian can lead to diffusive [82,83] and chaotic [84] behavior in 3D. In the presence of a transverse field, the system is known to show a quantum phase transition [85]. We use 16 rf pulses [51,70,72,86] to engineer the natural Hamiltonian into $H_1^{(A)} = JD_y$ and $H_2^{(A)} = JD_x$ with tunable J. This enables varying the Floquet steps by tuning J, while keeping τ fixed. Then, experimental imperfections such as decoherence and pulse errors remain the same, and we can faithfully quantify the Floquet heating rate. The initial state is a high-temperature thermal state with small thermal polarization in the magnetic field direction, $\rho(0) \approx (\mathbb{1} - \epsilon Z)/2^L$ with $\epsilon \approx 10^{-5}$, and the observable is the collective magnetization along the x axis, $\mathcal{O} = X$. As the identity part does not change under unitary evolution and does not contribute to the signal, it is convenient to consider only the deviation from the identity $\delta\rho(0) = Z$, which can be rotated to a desired observable $\mathcal{O}'$. Therefore, the NMR signal is equivalent to an infinite-temperature correlation, $\mathrm{Tr}[\delta\rho(t)X] \to \langle \mathcal{O}'(t)\mathcal{O}\rangle_{\beta=0}$.

We experimentally study the heating rates of the quasiconserved observables and their scaling with Floquet period to reveal the prethermal phase and investigate the eventual heating to infinite temperature. In Fig. 5, we show results for ADM (the two quasiconserved observables in KDM show similar behavior as reported elsewhere [51]). To study the autocorrelation of $H_{\rm pre} = \overline{H} + O(\tau)$ in ADM, we measure the average Hamiltonian, $\overline{H}^{(A)} = JD_y + JD_x = -JD_z$, since the higher-order terms in Eq. (3) are not accessible. We use the Jeener-Broekaert pulse pair [87] to evolve the initial state $\delta\rho$ and experimental observable X into $D_z \propto \overline{H}^{(A)}$. Because of the difference $H_{\rm pre} - \overline{H}$, we still expect an initial transient, over a time $\sim \|H_{\rm pre}\|^{-1}$, where the average Hamiltonian thermalizes to the prethermal Hamiltonian. When more Floquet periods are applied, the autocorrelation of D_z slowly decays from its prethermal value.

The decay rate in the prethermalization regime is shown in Fig. 5(b) and can be fitted to an exponential function in $1/(J\tau)$ on top of a constant background decay (which is due to experimental imperfections; see SM [88] for more details). By normalizing the data to the data collected under the fastest drive ($J\tau = 0.35$), the background decay is canceled, and the resulting dynamics only arises from the coherent evolution, as shown in Fig. 5(c). For given n, the normalized correlation decreases when increasing $J\tau$ because $H_{\rm pre} = \overline{H} + O(J\tau)$ and thus $\overline{H}$ that we measure has less overlap with the true quasiconserved observable $H_{\rm pre}$ for larger $J\tau$. The overall drop of the curves when increasing n is instead an indicator of Floquet heating.

To better quantify the final thermalization process, we define a critical value J_c such that when $J\tau > J_c\tau$, the system is thermalized at a given number n of periods in the thermodynamic limit or for a system size L at infinite time. Studying the scaling of J_c as a function of n (experimentally) and L (numerically) provides hints on the long-time, thermodynamic limits.

We numerically obtain the autocorrelations $\langle \mathcal{O}(\infty)\mathcal{O}\rangle$ as a function of $J\tau$ using exact diagonalization. In Fig. 6(a), we show simulation results for $\mathcal{O} = \overline{H}^{(K)}, D_z$ for KDM, and $\mathcal{O} = \overline{H}^{(A)}$ for ADM. (Here we explicitly consider the exact dipolar interaction instead of truncating to nearest neighbors). Note that both observables in KDM show a nonmonotonic behavior. They appear to be quasiconserved until $J\tau = 1$; the decrease in overlap is, however, interrupted by a revival at $J\tau = 1.6$. This is because $\overline{H}^{(K)}$ and D_z are an approximation of $H_{\rm pre}$ and $D_{\rm pre}$ to leading order. Thus, $\overline{H}^{(K)}$ (D_z) still has a small overlap with $D_{\rm pre}$ ($H_{\rm pre}$), giving rise to a second plateau at $J\tau \approx 1.6$ ($J\tau \approx 1$). The experimentally measured autocorrelations of quasiconserved observables in KDM can be found in [51]. For both experiments and simulations, we then find $J_c\tau$ from the point where the curves drop below a threshold value of 0.5 (any other reasonable choice would not qualitatively change the results). We linearly interpolate between data points to get $J_c\tau$ for every quasiconserved observable and plot the $J_c\tau$ in Figs. 6(b) and 6(c). The decrease of numerically calculated $J_c\tau$ with L in Fig. 6(b) indicates that even the correlations of quasiconserved observables decay to zero

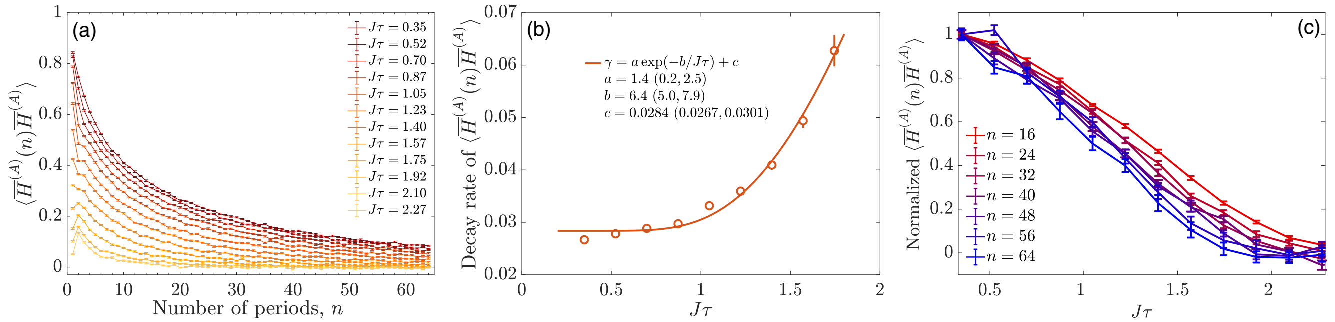

FIG. 5. Autocorrelation of the average Hamiltonian for the alternating dipolar model. (a) Autocorrelation as a function of n. Different curve stands for $J\tau$ from 0.35 to 2.27, with a step of 0.175. The darker color represents smaller $J\tau$ and the lighter color represents larger $J\tau$. We fit the autocorrelations from $n = 20$ to $n = 64$ to exponentially decaying function $\exp(-\gamma n)$ and plot the decay rate γ in (b). The length of the error bars corresponds to two standard deviations of the fitted decay rate. The solid curve indicates the fit to function $\gamma = a\exp(-b/J\tau) + c$. The fitted coefficients a, b, c are shown in the plot with the 95% confidence interval. (c) Autocorrelation vs $J\tau$ for different n. The lighter colors represent smaller n and the darker colors represent larger n. For a given n, the autocorrelation is normalized by $\langle \overline{H}(n)\overline{H} \rangle$ at $J\tau = 0.35$, i.e., the leftmost point is normalized to 1. In (a) and (b), error bars are determined from the noise in the free induction decay [see the Supplemental Material (SM) [88] for details of the experimental scheme].

as the system thermalizes to infinite temperature, suggesting that this nonthermalizing behavior should not persist to the thermodynamic limit. A similar result is also observed from experimentally measured $J_c\tau$, as shown in Fig. 6(c) [89]. Note that although $J_c\tau$ for $\langle \overline{H}^{(K)}(n)\overline{H}^{(K)} \rangle$ shows only a moderate dependence on n [Fig. 6(c)], its decay is still larger than experimental uncertainties.

V. CONCLUSION

As Floquet driving is a promising avenue for quantum simulation, it is crucial to evaluate its robustness, the existence of a long-lived prethermal phase, and the eventual thermalization to infinite temperature. Investigating Floquet heating, which breaks the prethermal regime, is particularly challenging, not only because of inherent limitations in numerical and experimental studies, but also because of the challenge to properly identify all quasiconserved observables in the complex, many-body driven dynamics.

Here we tackle both of these issues by combining analytical, numerical, and experimental tools. First, we provide a systematic strategy to find local, eigenquasiconserved observables in the prethermal regime using infinite-temperature correlations. By systematically searching over local operators, we find that counterintuitive quasiconserved observables might emerge, as we identify two eigenquasiconserved observables: the first, not surprisingly, is associate with energy, $H_{\rm pre}$, under sufficient fast drive; in addition, we find another quasiconserved observable, $D_{\rm pre}$, for the KDM in the presence of a large driving field. Our search protocol would be useful in other settings, such as identifying the underlying Hamiltonian or symmetries from measurements.

We then use numerical and experimental evidence to obtain insight into the inaccessible thermodynamic limit and long-time regime, to show that autocorrelations of quasiconserved observables indeed decrease toward zero due to Floquet heating, suggesting the Floquet system approaches the infinite-temperature state.

Our results not only provide a metric to study thermalization in driven quantum systems, but also open intriguing perspectives into the existence of quasiconserved observ-

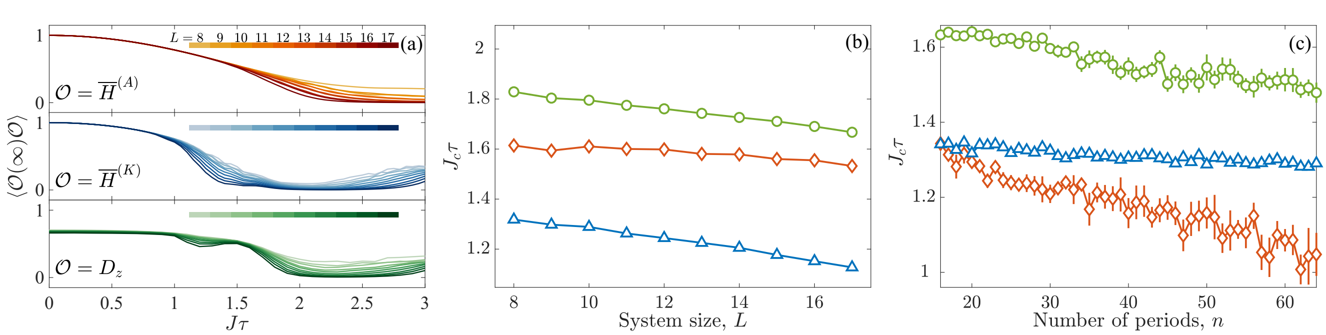

FIG. 6. Scaling of the critical Trotter step for KDM ($\overline{H}^{(K)}$: blue; D_z: green) and ADM ($\overline{H}^{(A)}$: red). (a) Simulated autocorrelations as a function of $J\tau$ for $L = 8, 9, \ldots, 17$ using exact diagonalization. The darker colors represent larger L as shown in the color bar. (b) $J_c\tau$ at which the numerical autocorrelation ($L = 17$) drops to half of the value under infinitely fast driving ($J\tau \to 0$). (c) $J_c\tau$ at which the experimentally measured autocorrelation drops to half of the value under the fastest driving ($J\tau = 0.35$). Error bars are determined from the noise in the free induction decay (see SM [88]).

ables other than the energy. It is an open question when they emerge and how they interact with each other. A better understanding of quasiconserved observables would benefit our understanding of heating in closed driven systems and in designing a robust protocol to slow down thermalization.

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