# Heat transport in insulator/ferromagnetic-insulator/insulator heterogeneous nanostructures at low temperatures

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A kinetic approach to the heat transport by phonons and magnons through a ferromagnetic insulator (FI) layer located between two massive insulators ($I_1$ and $I_2$) is analytically considered. The effective transverse heat conductivity of such a layered system with an arbitrary thickness of the FI layer is calculated, and the thickness at which the size effect is manifested in the thermal conductivity is found.

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## I. INTRODUCTION

Magnon propagation in magnetically ordered dielectric crystals can transfer heat in the same way as lattice excitations or phonons [1–4]. Heat transfer in multilayer heterostructures containing a layer of a ferromagnetic dielectric is of interest for both basic and applied research. The scientific aspect of the problem of thermal conductivity of multilayer systems is that Fourier's law cannot be directly applied to the analysis of heat transport in layered nanostructures, where the mean free path of phonons is greater than the layer thickness, and the boundary conditions at the interfaces have a significant effect on heat transport. Thus, the microscopic approach is required to describe the heat flux in $I_1/FI/I_2$ nanostructures, since this allows for the correct consideration of the influence of the interlayer boundaries on the thermal conductivity of the multilayer structure.

In multilayer systems, heat transfer plays an important role in the spin Seebeck effect (SSE) and in the whole field of spin caloritronics, which has been actively developed in recent years. [5–7] Specifically, spin caloritronics considers the problems of generation and control of spin currents by means of heat fluxes. [8] In this area, the longitudinal SSE (LSSE), which consists of generation of a spin current parallel to the heat temperature gradient, is of great interest because it can produce spin current densities that are two orders of magnitude larger than those produced via electronic or resonant excitation. [9,10] The LSSE experimental results allow us to study the kinetics of interacting electrons, phonons, and magnons in multilayer structures. Since the ferromagnetic insulator (FI) plates (or films) deposited on the high heat-conducting dielectric substrates are usually studied in LSSE experiments, for a correct theoretical description, it is also necessary to consider two related problems.

The first problem is to calculate the temperature jump at the $FI/I$ interface, i.e., the thermal resistance of the interface $R_{\text{th}}$ (commonly called the "Kapitza resistance"). It has been previously discussed in detail in Ref. [11].

The second problem, which was not solved microscopically until now is the calculation of the effective transverse heat conductivity of $I_1/FI/I_2$ layered system (with temperatures $T_1 \neq T_2$ for $I_1$ and $I_2$) at an arbitrary thickness of the FI layer, where both magnons and phonons transfer the heat flow.

In Ref. [11] it was shown that, for the ferrodielectric-insulator $FI/I$ interface at low temperatures ($T \ll \theta_{\text{D}}$, where $\theta_{\text{D}}$ is the Debye temperature of the FI layer), there exists a size effect. The latter manifests itself in the dependence of the Kapitza resistance $R_{\text{th}}$ for thin FI plates (films) on the frequency of phonon-magnon collisions, whereas for thick plates, the value of $R_{\text{th}}$ does not contain the magnetic characteristics of a ferrodielectric. To explain the growth of the magnetic contribution with decreasing thickness of the FI layer, we note that the transfer of heat from the heated magnons to the cooler $I$ layer is realized with phonons. If the thickness of the FI layer $d$ is much larger than the average free path of phonons with respect to their scattering on magnons $l_{pm}$, then the magnons and phonons in the FI layer are thermalized and $R_{\text{th}}$ is determined by the acoustic transparency of the $FI/I$ interface. In this case, there is no contribution of magnons to $R_{\text{th}}$.

However, if $d \ll l_{pm}$, then most phonons emitted by magnons in the film leave it without interacting with the magnons, even after several reflections from the boundaries. As a result, in contrast to the case $d \gg l_{pm}$, the Kapitza resistance $R_{\text{th}}$ depends more on the magnon-phonon interaction than on the acoustic transparency of the $FI/I$ boundary.

In our approach, the transverse heat flow through the FI layer, located between two massive insulators with temperatures $T_H$ and $T_B$ is considered ($T_H > T_B$). The analysis of the transverse thermal conductivity of a layered heterostructure is based on the Boltzmann kinetic equation for the phonon distribution function $N_q$, as well as on the assumption that magnons in the FI layer are thermalized due to magnon-magnon collisions and have a temperature $T_m$. This assumption is

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justified because the magnon-magnon relaxation time is usually shorter than the magnon-phonon collision time. [12,13] It is assumed that phonons interact with magnons, however, the phonon temperature is not introduced when describing the phonon-magnon interaction. Since heat is transferred by phonons across the boundary of $FI/I$, an important part of the analysis is ruled by the boundary conditions for the function $N_q$. These conditions account for the phonon reflection from the FI boundaries and the exchange of phonons between the FI layer and the massive $I$ plates with temperatures of $T_H$ and $T_B$. The temperature $T_H$ will be considered sufficiently low, and therefore, collisions of phonons with lattice defects and with each other as well as phonon-magnon Umklapp processes can be ignored. Since we neglect phonon-phonon collisions, the phonon temperature in the FI layer cannot be introduced. Note that the relative simplicity of such a microscopic model allows one to obtain results in an analytic form.

The paper is organized as follows. In Sec. II, a microscopic description of the heat transfer in a multilayer system is presented, and the transverse effective thermal conductivity of the layered $I_1/FI/I_2$ heterostructure is calculated. In Sec. III, the transverse heat transfer is analyzed using the phenomenological two-temperature model (2TM), i.e., in terms of magnon and phonon temperatures. Comparison of the 2TM results with the results of the microscopic approach allows us to find out the conditions under which the use of the phenomenological 2TM is justified. In Sec. IV, the main conclusions are formulated. Appendix A contains the calculation of the phonon distribution function in the FI layer. The numerical value of the criterion for thick and thin layers of yttrium iron garnet (YIG) is found in Appendix B.

## II. KINETIC APPROACH TO HEAT TRANSFER ACROSS $I_1/FI/I_2$ INTERFACES

In this section, we consider the transverse heat transfer in heterostructures containing a layer of FI. Even through our approach can be applied to different layered heterostructures, here, we will consider only a relatively simple heterostructure depicted in Fig. 1. Suppose that insulator $I_1$ has a temperature $T_B$, the temperature of insulator $I_2$ is equal to $T_H$, and $T_H > T_B$.

The phonon contribution to the heat flow through the heterostructure is found under the assumption that magnons are thermalized due to magnon-magnon collisions and have a temperature $T_m$. The condition for thermalizing the magnon subsystem in FI is that the frequency of magnon-magnon collision is higher than the frequency of magnon-phonon collisions. By the way, even when the magnon temperature of the Bose-Einstein distribution can no longer be established on the basis of direct intermagnon collisions, $T_m$ can still be introduced [14]. Namely, the value of $T_m$ is justified in the limit $d \gg l_{pm}$ because of the effective intermagnon collisions via the phonons. These circumstances allow us to reduce the formulated problem to a solution of the stationary kinetic equation for the phonon distribution function and then to determine $\nabla T_m$ as a function of thermal flow $Q$ and the temperatures of the insulators from the heat-balance equation.

In addition, the good transparencies $\alpha_1 \sim 1$ and $\alpha_2 \sim 1$ of the $FI/I$ interfaces will be of special interest, since they

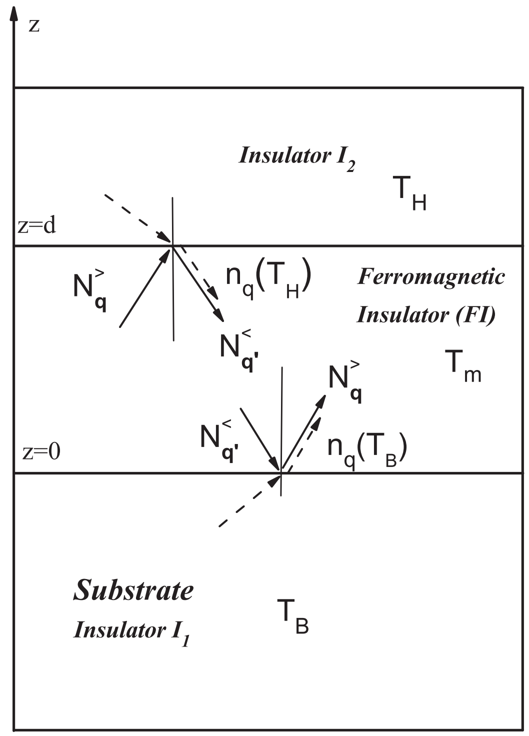


FIG. 1. Reflection and refraction of phonon modes at the boundaries of media in a $I_1/FI/I_2$ layered structure. The filling numbers of phonon states with wave vector $\mathbf{q}$ are denoted by $N_{\mathbf{q}}^{\lessgtr}$. The symbol $>$ represents phonons with a positive $z$ component of the wave vector $\mathbf{q}$, whereas the symbol $<$ denotes phonons with a negative $z$ component of $\mathbf{q}$. Phonons transitioning from the dielectric layer to the FI layer are shown by dashed lines. $T_B$ is the temperature of the massive substrate, which plays the role of the thermostat, $T_m$ is the magnon temperature, and $T_H$ is the temperature of the top dielectric plate ($T_H > T_B$).

allow simple boundary conditions for the phonon distribution function. The ballistic propagation of the phonon emitted by the FI layer not only simplifies the expressions for heat dissipation in the sample but also stipulates the necessary condition for realization of the size effect, described thoughtfully in Ref. [11].

In accordance with the considerations above, we assume that the distribution of magnons is characterized by the temperature $T_m$. At the same time, the distribution function for phonons $N_q(z)$, where $\mathbf{q}$ is the phonon wave vector, should be determined from the kinetic equation.

$$s_z \frac{\partial N_{\mathbf{q}}(z)}{\partial z} = L_{pm}\{N, n\}, \tag{1}$$

with appropriate boundary conditions. In Eq. (1), $s_z$ is the projection of the phonon velocity on the $z$ axis, and $L_{pm}$ is the phonon-magnon collision integral [14], which can be expressed as

$$L_{pm}\{N, n\} = \nu_{pm}[T_m(z), q]\{n[T_m(z)] - N_{\mathbf{q}}(z)\}. \tag{2}$$

Here, $n[T_m(z)] = [\exp(\varepsilon_k/T_m) - 1]^{-1}$ is the equilibrium Bose-Einstein distribution with the $z$-dependent magnon temperature $T_m$ ($k_B = 1$). In the long wave limit $ka \ll 1$, the magnon dispersion law is $\varepsilon_k = \theta_C(ak)^2$, where $a$ is the lattice constant and $\theta_C$ is the temperature, which coincides in order of magnitude with the Curie temperature. In Eq. (2), $\nu_{pm}[T_m(z), q]$ is the frequency of collisions between the phonon of frequency $\omega_q = sq$ and the magnons. The dependence of the frequency $\nu_{pm}$ on the magnon temperature leads to the fact that the mean free path of phonons depends on the transverse coordinate $z$. This feature somewhat complicates the analysis of the $I_1/FI/I_2$ system as compared with the $I_1/N/I_2$ system, where the mean free path of phonons in a normal metal $N$ does not depend on the electron temperature [15].

Adding the solution scheme from Ref. [16] for the kinetic equation, with details placed in the Appendix section of Ref. [11], we denote the phonon reflection coefficients at boundaries 1 and 2 as $\beta_1$ and $\beta_2$, such that $\beta_i = 1 - \alpha_i$, $i = 1, 2$, where $\alpha(\theta)$ is the transparency coefficient. We consider the case of ballistic propagation of the phonons emitted by FI through the $FI/I$ boundary, taking into account the finite transparency of the $FI/I$ interface within the framework of the acoustic-mismatch theory [17]. The notation $N_{\mathbf{q}}^{\gtrless}(z) = N(z, q_x, q_y, q_z \gtrless 0)$ allows us to write the boundary conditions for $N_{\mathbf{q}}(z)$ in Eq. (1) for $z = 0$ and $z = d$ as follows (see Fig. 1):

$$N_{\mathbf{q}}^{>}(0) = \alpha_1 n_q(T_B) + \beta_1 N_{\mathbf{q}'}^{<}(0), \tag{3}$$

$$N_{\mathbf{q}}^{<}(d) = \alpha_2 n_q(T_H) + \beta_2 N_{\mathbf{q}'}^{>}(d). \tag{4}$$

These boundary conditions suggest that phonons emitted from the FI layer to the insulators $I_1$ and $I_2$ no longer return. This assumption is justified when $I_1$ and $I_2$ are single-crystal dielectrics with high thermal conductivity.

In the acoustic mismatch model [17–19], the probability of passage $\alpha$ depends on the angle of incidence of the phonon $\theta$ and acoustic impedances of adjacent media $Z = \rho s$ and $Z' = \rho' s'$:

$$\alpha(\theta) = 4ZZ' \cos\theta \cos\theta' / (Z \cos\theta' + Z' \cos\theta)^2. \tag{5}$$

The condition for the independence of the heat flux from the $z$ coordinate can serve as the equation for $T_m(z)$:

$$Q_z = -k_m \frac{dT_m}{dz} + \int_{q_z>0} \frac{d^3q}{(2\pi)^3} \hbar\omega_q s_z [N_{\mathbf{q}}^{>}(z) - N_{\mathbf{q}'}^{<}(z)]. \tag{6}$$

Here, $k_m$ is the magnon thermal conductivity, and the second term on the right side is the share of the heat flux carried by phonons. It can be seen that substituting Eq. (A9) and Eq. (A10) into Eq. (6) gives the integro-differential equation for $T_m(z)$.

Below, considerable attention is paid to the limit $d \gg l_{pm}$, when the determinant $D \approx 1$, since it is in this limit that the magnons make a significant contribution to the heat flow through the $I_1/FI/I_2$ heterostructure. Note that the calculation of the integral $J_1$ neglects terms that are of the order of $(d^2T_m/dz^2)|_{z=0}$ and the term proportional $dT_m/dz$ falls out, since $(dT_m/dz)|_{z=0}= 0$. As a result, we have $J_1 \approx n_q[T_m(0)]$, and similarly the integral $J_2 \approx n_q[T_m(d)]$.

Expanding $T_m(z')$ with respect to the small temperature gradient looks like $T_m(z') = T_m(z) + (dT_m/dz)|_z(z - z')$. Substituting this expansion into Eq. (A9) gives the phonon distribution function at $d \gg l_{pm}$:

$$N_{\mathbf{q}}^{>}(z) = e^{-r(z)}\{\alpha_1 n_q(T_B) + \beta_1 n_q[T_m(0)]\} + n_q[T_m(z)] \times [1 - e^{-r(z)}] + \frac{dn_q}{dT_m}\frac{dT_m}{dz}\frac{dz}{dr}[-1 + (r+1)e^{-r(z)}], \tag{7}$$

and similarly, from Eq. (A10), we get the expression

$$N_{\mathbf{q}'}^{<}(z) = e^{-r(d)+r(z)}\{\alpha_2 n_q(T_H) + \beta_2 n_q[T_m(d)]\} + n_q[T_m(z)][1 - e^{-r(d)+r(z)}] + \frac{dn_q}{dT_m}\frac{dT_m}{dz}\frac{dz}{dr} \times \{-1 + [r(d) - r(z)]e^{-r(d)+r(z)}\}, \tag{8}$$

If $z \gg l_{pm}$ and $(d - z) \gg l_{pm}$, then terms containing $\exp(z/l_{pm})$ and $\exp[-(d - z)/l_{pm}]$ can be ignored. As such, in the region that is removed from the transition layers, the heat flux can be written as

$$Q_z = -k_m[T_m(z)]\frac{dT_m}{dz} - 2\int_{q_z>0} \frac{d^3q}{(2\pi)^3} \hbar\omega_q \frac{s_z^2}{\nu_{pm}} \frac{dn_q}{dT_m}\frac{dT_m}{dz}. \tag{9}$$

Since at $z \gg l_{pm}$ and at $(d - z) \gg l_{pm}$ the magnons and phonons are thermalized, then Eq. (9) can also be written as

$$Q_z = -k_m[T_m(z)]\frac{dT_m}{dz} - k_p[T_m(z)]\frac{dT_m}{dz}, \tag{10}$$

with

$$k_p(T) = 2\int_{q_z>0} \frac{d^3q}{(2\pi)^3} \hbar\omega_q \frac{s_z^2}{\nu_{pm}(T)} \frac{dn_q}{dT}. \tag{11}$$

(Here and below, $T_p = T_m$ is accounted for.) Note that the value of phonon thermal conductivity is determined by the frequency of phonon-magnon collisions. In pure FIs, the frequency of phonon-magnon collisions $\nu_{pm}(T)$ is given by the following expression (see for example, Ref. [11]):

$$\nu_{pm}(T) = D(T)J_D(T, x, y_0), \tag{12}$$

in which $D(T) = (\theta_C\theta_D/8\pi\hbar\theta_p)(T/\theta_C)^3$, where $\theta_D = \hbar s/a$, $\theta_p = Ms^2$, $s$ is the average sound velocity, $M$ is the magnetic ion mass, and

$$J_D(T, x, y_0) = \int_{y_0}^{\infty} dy\, y(x + y)\left[\frac{1}{e^y - 1} - \frac{1}{e^{x+y} - 1}\right]. \tag{13}$$

Here, $x = \hbar\omega_q/T$, $y = \varepsilon_k/T$, and $y_0 = \theta_D^2/4T\theta_C$. In the integral over the dimensionless magnon energy $y$, the lower integration limit $y_0$ reflects the Cherenkov character of the emission of phonons by magnons. Namely, only magnons whose energy is higher than $\theta_D^2/4\theta_C$ can emit phonons.

At low temperatures $T \ll \theta_D^2/4\theta_C$, the phonon thermal conductivity has the form

$$k_p(T) = \frac{4C}{3\pi}\frac{\theta_C^2\theta_p}{\hbar^2 s\theta_D}\left(\frac{4T\theta_C}{\theta_D^2}\right)^2 \exp\left(\frac{\theta_D^2}{4T\theta_C}\right), \tag{14}$$

where $C = \int_0^\infty x^4 e^{2x}(e^x - 1)^{-3}dx \approx 27.41$. The rapid increase in phonon thermal conductivity with decreasing temperature is a consequence of the Cherenkov character of the emission of phonons by magnons. We note that the formula in Eq. (14) is valid if $l_{pm}$ is less than $d$.

From the general equation for $Q_z(z)$, it is possible to derive an equation for $T_m(0)$ and $T_m(d)$. Let $z=0$, then $(dT_m/dz)|_{z=0}=0$, and considering that $d \gg l_{pm}$,

$$Q_z = -\int_{q_z>0} \frac{d^3q}{(2\pi)^3} \hbar\omega_q s_z \alpha_1 \{n_q[T_m(0)] - n_q(T_B)\} = -\frac{\pi^2}{120} \frac{\langle\alpha_1\rangle}{\hbar^3 s^2} \left[T_m^4(0) - T_B^4\right], \tag{15}$$

where $\langle\alpha_1\rangle = \int_0^{\pi/2} \alpha(\theta)\sin(2\theta)d\theta$.

In the case of small heat fluxes, when $T_m(0) - T_B \ll T_B$,

$$Q_z = -\frac{\pi^2}{30} \frac{\langle\alpha_1\rangle T_B^3}{\hbar^3 s^2} [T_m(0) - T_B].$$

At $z = d$, considering that $(dT_m/dz)|_{z=d}=0$ and $d \gg l_{pm}$,

$$Q_z = -\frac{\pi^2}{120} \frac{\langle\alpha_2\rangle}{\hbar^3 s^2} \left[T_H^4 - T_m^4(d)\right]. \tag{16}$$

In the case of low heat fluxes, when $T_H - T_B \ll T_B$ and $T_m(d) - T_B \ll T_B$,

$$Q_z = -\frac{\pi^2}{30} \frac{\langle\alpha_2\rangle T_B^3}{\hbar^3 s^2} [T_H - T_m(d)].$$

In the FI region removed from boundary layers, the FI temperature gradient, with an accuracy up to terms proportional to $l_{pm}/d$, is determined by the following equation:

$$\frac{dT_m}{dz} = \frac{[T_m(d) - T_m(0)]}{d},$$

At $d \gg l_{pm}$, the effective thermal conductivity of the FI layer $k_{\rm eff} = |Q_z|d/(T_H - T_B)$ can be obtained from the following systems of equations:

$$|Q_z|R_{\rm th,1} + |Q_z|R_{\rm th,2} = T_H - T_m(d) + T_m(0) - T_B, \tag{17}$$

$$|Q_z| = (k_m + k_p)[T_m(d) - T_m(0)]/d, \tag{18}$$

where the magnon and phonon thermal conductivities are taken at $T = T_B$, and the thermal resistance of the boundary between the FI and the insulator is written as

$$R_{{\rm th},i} = \frac{30\hbar^3 s^2}{\pi^2 \langle\alpha_i\rangle T_B^3}. \tag{19}$$

As such, at $d \gg l_{pm}$, we have

$$k_{\rm eff} = \frac{d}{R_{\rm th,1} + R_{\rm th,2} + d/(k_m + k_p)}. \tag{20}$$

Note that the 2TM discussed in Sec. III gives Eq. (36) for the effective thermal conductivity, which coincides exactly with the microscopic calculation result in Eq. (20) at $d \gg l_{pm}$.

We turn to the limiting case of thin FI layers $d \ll l_{pm}$ when the determinant of the system $D = 1 - \beta_1\beta_2$. Since in the linear approximation with respect to $d/l_{pm}$ we have to set $\exp(-z/l_{pm}) \approx 1$ and $\exp[-(d-z)/l_{pm}] \approx 1$, then $J_1 = J_2 = dn_q(\overline{T}_m)/l_{pm}$, where $\overline{T}_m = T_m(d/2)$. Thus, the integrals $J_1$ and $J_2$ are of the order of $d/l_{pm}$ and can be neglected in the zero approximation with respect to $d/l_{pm}$. As a result, in the zero approximation with respect to $d/l_{pm}$, the phonon distribution is given the following equalities:

$$N_{\mathbf{q}}^{>} = \{\alpha_1 n_q[T_B + \beta_1\alpha_2 n_q(T_H)]\}\big/\big(1 - \beta_1\beta_2\big), \tag{21}$$

$$N_{\mathbf{q}}^{<} = \{\alpha_2 n_q[T_H + \beta_1\alpha_1 n_q(T_B)]\}\big/\big(1 - \beta_1\beta_2\big). \tag{22}$$

Neglecting the magnon heat transfer (since in thin FI layers, the phonons do not have time to transfer energy to magnons), we have

$$Q_z = -\frac{\pi^2}{120\hbar^3 s^2}\left\langle \frac{\alpha_1\alpha_2}{1 - \beta_1\beta_2}\right\rangle \left(T_H^4 - T_B^4\right) \approx -\frac{\pi^2 T_B^3}{30\hbar^3 s^2}\left\langle \frac{\alpha_1\alpha_2}{1 - \beta_1\beta_2}\right\rangle (T_H - T_B). \tag{23}$$

From here, we obtain the result

$$k_{\rm eff} = \frac{\pi^2 T_B^3}{30\hbar^3 s^2} d \left\langle \frac{\alpha_1\alpha_2}{1 - \beta_1\beta_2}\right\rangle, \tag{24}$$

where averaging over incidence angles is defined by $\langle f\rangle = \int_0^{\pi/2} \sin 2\theta f(\theta) d\theta$. Note that, in the case of thin FI layers and $\alpha_i \sim 1$, the thermal conductivity $k_{\rm eff}$ is of the order of $c_p s d$; that is, it coincides with the phonon thermal conductivity with an average phonon mean free path of the order of the thickness of the FI layer. [In our model, the phonon specific heat $c_p = (2\pi^2/15)(T_B^3/\hbar^3 s^3)$.]

## III. THERMAL CONDUCTIVITY OF THE $I_1/FI/I_2$ HETEROSTRUCTURE IN THE TWO-TEMPERATURE APPROXIMATION

In the microscopic approach used above, the calculation of the phonon contribution to the transverse thermal conductivity of the $I_1/FI/I_2$ system required solving a kinetic equation. To simplify the thermal conductivity calculations, it is desirable to use a simpler phenomenological approach. As will be shown below, the two-temperature approximation, in which the phonons and magnons are considered to have temperatures $T_p$ and $T_m$, respectively, could be suitable for this task [20,21].

At first, we will not take into account the Kapitza resistance at the $I/FI$ boundary. The simplest formulation is that of the problem with a given heat flux $Q_z$, which receives contribution from the phonons and magnons. The heat flux associated with phonons is determined by the equality $Q_p = -k_p\nabla T_p$, where the temperature of the phonons obeys the following stationary heat equation:

$$-k_p \frac{d^2 T_p}{dz^2} = D(T_m)K(T_m, T_p), \tag{25}$$

where $D(T_m)$ is presented in Eq. (12) and

$$K(T_m, T_p) = \int_0^\infty \frac{u^3 du}{e^u - 1}[J_D(T_m, x = u, y_0) - \mu^4 J_D(T_m, x = u\mu, y_0)],$$

with

$$J_D(T_m, x, y_0) = \sum_{p=1}^{\infty}(1 - e^{-px})e^{-py_0} \times \left[x\left(\frac{y_0}{p} + \frac{1}{p^2}\right)+\left(\frac{y_0^2}{p} + \frac{2y_0}{p^2} + \frac{2}{p^3}\right)\right].$$

Here, $\mu = T_p/T_m$. One sees that, when $\mu = 1$, i.e., the phonon and magnon temperatures are equal, $K = 0$, as expected. In the limiting case of large $y_0$, which corresponds to the limit of low temperatures, $K$ becomes exponentially small: $K \sim e^{-y_0}$. Because the contribution to $J_D(T_m, x, y_0)$ from the term with $p = 2$ is proportional to $e^{-2y_0}$, we can confine our consideration by $p = 1$ in the limit $y_0 \gg 1$, obtaining [11]

$$K(T_m, T_p) = \varphi_1\Gamma(5)\{1 + \mu^5[\zeta(5, 1 + \mu) - \zeta(5)]\} + \varphi_2\Gamma(4)\{1 + \mu^4[\zeta(4, 1 + \mu)] - [\zeta(4)]\}.$$

Here, $\varphi_1 = e^{-y_0}(y_0 + 1), \varphi_2 = e^{-y_0}(y_0^2 + 2y_0 + 2)$, $\Gamma(n)$ is the $\Gamma$ function of $n$, and $\zeta(n, 1 + \mu)$ is the generalized Riemann $\zeta$ function [see their definitions in Eq. (A31) in Ref. [11]].

A similar equation for magnon temperature

$$-k_m\frac{d^2T_m}{dz^2} = -D(T_m)K(T_m, T_p). \quad (26)$$

In the linear approximation, when $T_H - T_B \ll T_B$, the temperatures of the magnons and phonons differ little from the substrate temperature $T_B$. If the magnon-phonon $\tau_{mp}$ and phonon-magnon $\tau_{pm}$ energy relaxation time are introduced according to $c_p/\tau_{pm} = c_m/\tau_{mp}$, then the equations for $T_p$ and $T_m$ look like

$$\frac{d^2T_p}{dz^2} + \frac{c_p}{\tau_{pm}k_p}(T_m - T_p) = 0, \quad (27)$$

$$\frac{d^2T_m}{dz^2} + \frac{c_p}{\tau_{pm}k_m}(T_p - T_m) = 0, \quad (28)$$

To use the symmetry of the problem, in this section, we will assume that the FI layer is located in the region $-d/2 < z < d/2$. In this case, the boundary conditions look like

$$T_p(-d/2) = T_B + |Q_z|R_{\text{th},1}, \quad T_p(d/2) = T_H - |Q_z|R_{\text{th},2}, \quad (29)$$

$$dT_m/dz|_{-d/2} = dT_m/dz|_{d/2} = 0. \quad (30)$$

The last expression reflects the fact that magnons do not transfer heat across the boundaries of the FI layer. Note that, because of symmetry, $T_p(0) = T_m(0)$. The heat flow equation

$$Q_z = -k_p\frac{dT_p}{dz} - k_m\frac{dT_m}{dz} \quad (31)$$

can be integrated from zero to some coordinate $z$. Since the heat flux in the FI layer is constant, we have the equality

$$T_p = \frac{1}{k_p}[-k_mT_m(z) + (k_p + k_m)T_p(0) - Q_zz]. \quad (32)$$

It follows from Eq. (28) that

$$\frac{d^2T_m}{dz^2} - \frac{c_p}{\tau_{pm}k_m}\left\{T_m + \frac{1}{k_p}[k_mT_m - k_TT_p(0) + Q_zz]\right\}= 0, \quad (33)$$

where $k_T = k_m + k_p$. The solution of the equation for $T_m$ that satisfies the boundary condition in Eq. (30) looks like

$$T_m(z) = T_m(0) + \frac{Q_z\lambda}{k_T}\frac{\sinh(z/\lambda)}{\cosh(d/2\lambda)} - \frac{Q_z}{k_T}z. \quad (34)$$

where $\lambda = [\tau_{pm}k_mk_p/c_p(k_m + k_p)]^{1/2}$. Substituting $T_m(z)$ in Eq. (32), we get

$$T_p(z) = T_p(0) - \frac{k_m}{k_p}\frac{Q_z\lambda}{k_T}\frac{\sinh(z/\lambda)}{\cosh(d/2\lambda)} - \frac{Q_z}{k_T}z. \quad (35)$$

Considering Eq. (29) for the effective thermal conductivity of the FI layer, $k_{\text{eff}} \equiv |Q_z|d/(T_H - T_B)$, we have

$$k_{\text{eff}} = k_T\bigg/\left[1 + \frac{k_m}{k_p}\frac{2\lambda}{d}\tanh\frac{d}{2\lambda} + \frac{k_T}{d}(R_{\text{th},1} + R_{\text{th},2})\right]. \quad (36)$$

Unlike Ref. [21], the expression in Eq. (36) takes into account the contribution of the thermal resistance of the boundaries to $k_{\text{eff}}$. This contribution can be significant at low temperatures ($T_B \sim 1$ K), and it must be taken into account when analyzing experimental data.

Note that, at $d \gg l_m$, the expression in Eq. (36), obtained on the basis of the 2TM, coincides exactly with the result of the microscopic calculation in Eq. (20). At the same time, at small thicknesses of the FI layer, $d \ll l_{pm}$, the 2TM gives only a qualitatively correct result.

The physical meaning of the length $\lambda$ becomes clear if we consider the limiting case $k_m \gg k_p$ when $\lambda = (\tau_{pm}k_p/c_p)^{1/2}$. It is well known that, for gas, the thermal conductivity $k \sim cl\overline{v}$, where $c$ is the specific heat of the gas, $l$ is the average mean free path of particles, and $\overline{v}$ is their average thermal velocity. For phonon gas, $\overline{v} = s$, and if we neglect the scattering by impurities, the average phonon mean free path $l = l_{pm}$. As such, for a phonon gas, we have $k_p \sim c_ps^2\tau_{pm}$, and $\lambda \sim s\tau_{pm} = l_{pm}$. Thus, $\lambda$ is the length of the phonon-magnon collisions, that is, the length at which phonons transfer their energy to magnons.

According to Eqs. (20) and (36), for thick FI layers ($d \gg l_{pm}$), the size effect in the thermal conductivity of layered nanostructures begins to manifest itself at FI layer thicknesses $d_{\text{cr}} \sim k_T(R_{th,1} + R_{th,2})$. If $d \gg d_{\text{cr}}$, the thermal conductivity of the thick FI layer is approximately equal to the total thermal conductivity of the magnons and phonons, and if $d \ll d_{\text{cr}}$, the magnons and phonons scattering in the FI layer (i.e., the quality of the FI layer) play a small role, and the traverse thermal conductivity of the layered nanostructures is determined by the acoustic mismatch of the adjacent materials $k_{\text{eff}} = d/(R_{\text{th},1} + R_{\text{th},2})$. In the case of thin FI layers with $d \ll l_{pm}$, the thermal conductivity $k_{\text{eff}}$ is given by Eq. (24). We would like to emphasize that, since the values of $l_{pm}$ and $d_{\text{cr}}$ increase with decreasing thermostat temperature, the role of the size effect in the transverse thermal conductivity of layered structures increases with decreasing temperature.

## IV. CONCLUSIONS

In this paper, the transverse heat transfer in a layered $I_1/FI/I_2$ heterostructure at low temperatures is analyzed when the magnons in the FI layer are thermalized. The analysis is based on the Boltzmann kinetic equation for the phonon distribution function with boundary conditions that account for the reflection and refraction of acoustic waves as they pass through the interlayer interfaces (Sec. II). Effective thermal conductivity perpendicular to the layers is also calculated in the two-temperature approximation, i.e., in terms of magnon and phonon temperatures (Sec. III). Comparison of the results obtained in Secs. II and III shows that a relatively simple two-temperature approximation correctly describes the kinetics of heat transfer in a multilayer system only in the case of thick FI layers, wherein the thickness of the layer is significantly greater than the phonon-magnon free path. For thinner FI layers, the two-temperature approximation gives only a qualitatively correct result for the effective transverse thermal conductivity of the layered structure. (Note that the numerical value of the criterion for thick and thin PI layers is given in Appendix B for an example of a layer of YIG.)

The dependence of the transverse thermal conductivity of the layered $I_1/FI/I_2$ structures on the thickness of the FI layer increases significantly with decreasing temperature; therefore, it is necessary to take into account the size effect when analyzing experimental results for low temperatures.

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## APPENDIX A: PHONON DISTRIBUTION FUNCTION

The equation for $N^>_{\mathbf{q}}(z)$

$$\frac{dN^>_{\mathbf{q}}}{dz} + \frac{1}{l_{pm}(z)} N^>_{\mathbf{q}} = \frac{1}{l_{pm}(z)} n_q[T_m(z)], \tag{A1}$$

has the solution

$$N^>_{\mathbf{q}}(z) = N^>_{\mathbf{q}}(0) e^{-r(z)} + \int_0^z \frac{dz'}{l_{pm}(z')} e^{[r(z')-r(z)]} n_q[T_m(z')], \tag{A2}$$

where the phonon mean free path $l_{pm}(z) = |s_z|/\nu_{pm}(z)$ and

$$r(z) = \int_0^z \frac{dz'}{l_{pm}(z')}. \tag{A3}$$

The solution to the equation for $N^<_{\mathbf{q}}(z)$ looks like

$$N^<_{\mathbf{q}'}(z) = N^<_{\mathbf{q}'}(d) e^{[r(z)-r(d)]} + \int_z^d \frac{dz'}{l_{pm}(z')} e^{[r(z)-r(z')]} n_q[T_m(z')], \tag{A4}$$

The constant $N^>_{\mathbf{q}}(0)$ and $N^<_{\mathbf{q}'}(d)$ are determined by the boundary conditions. Substituting Eqs. (A2) and (A4) into the boundary conditions gives

$$N^>_{\mathbf{q}}(0) = \frac{1}{D}[\alpha_1 n_q(T_B) + \beta_1\alpha_2 e^{-r(d)} n_q(T_H) + \beta_1 J_1 + \beta_1\beta_2 e^{-r(d)} J_2], \tag{A5}$$

$$N^<_{\mathbf{q}'}(d) = \frac{1}{D}[\alpha_2 n_q(T_H) + \beta_2\alpha_1 e^{-r(d)} n_q(T_B) + \beta_2 J_2 + \beta_1\beta_2 e^{-r(d)} J_1], \tag{A6}$$

where the determinant $D = 1 - \beta_1\beta_2 e^{-2r(d)}$,

$$J_1 = \int_0^d \frac{dz'}{l_{pm}(z')} e^{-r(z')} n_q[T_m(z')], \text{ and} \tag{A7}$$

$$J_2 = \int_0^d \frac{dz'}{l_{pm}(z')} e^{-r(d)+r(z')} n_q[T_m(z')]. \tag{A8}$$

Note that, if $d \gg l_{pm}$, then the integral $J_1$ gets its main contribution from the region $z' \lesssim l_{pm} \ll d$, and the integral $J_2$ get its main contribution from $(d - z') \lesssim l_{pm} \ll d$. If $d \ll l_{pm}$, then phonons have almost no interaction with magnons, and the heat transport through the thin FI layer is of a purely phonon nature. At $d \ll l_{pm}$, the integrals $J_1$ and $J_2$ are small, since they are of the order of $d/l_{pm}$.

By substituting $N^>_{\mathbf{q}}(0)$ and $N^<_{\mathbf{q}'}(d)$ into Eqs. (A2) and (A4), we get the following for the phonon distribution functions:

$$N^>_{\mathbf{q}}(z) = \frac{\exp[-r(z)]}{D}[\alpha_1 n_q(T_B) + \beta_1 J_1 + \beta_1\alpha_2 e^{-r(d)} n_q(T_H) + \beta_1\beta_2 e^{-r(d)} J_2] + \int_0^z \frac{dz'}{l_{pm}(z')} e^{-r(z)+r(z')} n_q[T_m(z')], \tag{A9}$$

$$N^<_{\mathbf{q}'}(z) = \frac{\exp[-r(d)+r(z)]}{D}[\alpha_2 n_q(T_H) + \beta_2 J_2 + \beta_2\alpha_1 e^{-r(d)} n_q(T_B) + \beta_1\beta_2 e^{-r(d)} J_1] + \int_z^d \frac{dz'}{l_{pm}(z')} e^{-r(z')+r(z)} n_q[T_m(z')]. \tag{A10}$$

## APPENDIX B: PHONON MEAN FREE PATH

For a given phonon energy, the frequency of phonon-magnon collisions is determined by Eqs. (12) and (13). The phonon energy-averaged frequency of phonon-magnon collisions was calculated in Ref. [22] and has the following form:

$$\overline{\nu}_{pm} = \nu_0 \left(\frac{T}{\theta_C}\right)^3 \int_0^\infty dx \frac{x^4 e^x}{e^x - 1} \int_{y_0}^\infty dy \frac{(y + x) y e^y}{(e^y - 1)(e^{x+y} - 1)}, \tag{B1}$$

where

$$\nu_0 = \frac{15}{32\pi^5} \frac{\theta_C}{Mas}.$$

To compare theory with experiment, one should also take into account the scattering of phonons at the boundaries of FI layer. Since phonon-magnon collisions and phonon scattering at the sample boundaries are statistically independent, the average phonon mean free path $l_p$ can be written as $l_p = (1/l_{pm} + 1/l_d)^{-1}$, where $l_d$ is boundary scattering phonon

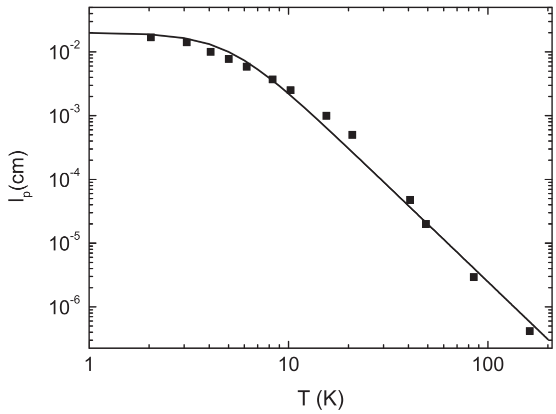


FIG. 2. The temperature dependence of the phonon mean free path $l_p(T) = [1/l_{pm}(T) + 1/l_d]^{-1}$. Experimental data are taken from Ref. [3].

mean free path, and $l_{pm} = s/\overline{\nu}_{pm}$. The temperature dependence of the average phonon free path is shown in Fig. 2. Agreement between theory and experiment [3] was achieved by selecting two adjustable parameters, namely the value of $l_d$ and the numerical coefficient in the expression for $l_{pm}$. At $T \gtrsim 1$ K, the dependence $l_{pm}(T)$ weakly depends on $y_0$, which means that the Cherenkov effect upon emission of phonons by magnons manifests itself at lower temperatures. From a comparison of theory with experiment, it follows that, at temperatures $T < 1$ K, phonons are mainly scattered at the interface between FI and I layers, and at temperatures $T > 10$ K, phonon scattering by magnons dominates, while phonon scattering at the sample boundaries and Umklapp processes play a secondary role.

As seen from Fig. 2, for YIG at $T \approx 10$ K, the phonon-magnon mean free path $l_{pm} \approx 10^{-3}$ cm, from which the criterion of thick and thin YIG layers (at $T \approx 10$ K) immediately follows.


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