Intervortex forces in competing-order superconductors

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(Received 18 May 2020; revised 9 November 2020; accepted 11 December 2020; published 21 January 2021)

The standard Ginzburg-Landau model of competing-order superconductors, applicable to various high T_c cuprates, is studied. It is observed that this model possesses two distinct species of vortex, and consequently has two distinct integer valued topological charges. A simple point particle model of long-range forces between (anti)vortices of any species is developed and compared with numerical simulations of the full field theory, excellent agreement being found. Some of the results are quite counterintuitive. For example, a parameter regime exists where vortices of one species repel both vortices and antivortices of the other.

DOI: 10.1103/PhysRevB.103.014514

I. INTRODUCTION

High T_c cuprate superconductors often exhibit a superconducting ground state that is in close proximity to other ordered ground states. The standard approach models these two phases separately with separate order parameters. However, it has been shown that when in close proximity the superconducting state competes with these other orders, for example antiferromagnetic order [1,2] or charge order [3,4]. In particular there has been considerable recent interest in such models, driven by experimental results, showing the importance of charge order in underdoped cuprates [3,5–7].

If a magnetic field is applied to such a system, vortices form, locally suppressing the superconducting state in the core. This leads to competing correlations in the core, studied both theoretically [1,8,9] and experimentally [2,10–12] in cuprates. In addition it has also been shown that in $YBa_2Cu_3O_y$, vortex cores overlap before H_{c_2} is reached, allowing charge order across the system [4].

A common tool used to study competing phases, is extending the target space to include the competing order parameters. The extended target space comes with additional constraints, such that suppression of the dominant phase is matched by excitation of the competing phase. Historically this was introduced in cuprates to model the competition between the superconducting phase and antiferromagnetic phase as an $SO(5)$ model. The approach considered a coupled complex valued order parameter Δ for the superconducting phase, and a vector valued order parameter $\boldsymbol{m} = (m_1, m_2, m_3)$ for the antiferromagnetic phase, and phase competition introduced through the constraint $|\Delta|^2 + |\boldsymbol{m}|^2 = \text{const}$ [1]. Hence the composite order parameter $(\Delta, \boldsymbol{m})$ takes values in a four-dimensional sphere inside $\mathbb{R}^5$.

Recently it has been proposed that a similar approach using an $SO(3)$ model, where the target space is expanded to a two-sphere $S^2 \subset \mathbb{R}^3$, can be used to model the competition between superconductivity and charge order [13]. Restricting a half-filled attractive Hubbard model to nearest neighbor hopping leads to the superconducting and charge density wave orders becoming degenerate in energy. This suggests an S^2 order parameter [14–18], formed of a superconducting component, written as the complex field Δ, and charge density wave component, written as the real field ρ. These fields are subject to the constraint $|\Delta|^2 + \rho^2 = c^2$, such that $|\rho|$ is maximal where $|\Delta|$ vanishes, and vice versa. Hence, we will assume that the north pole ($\rho = c$) and south pole ($\rho = -c$) of the S^2 target space correspond to two different charge density wave orders (with different dominating sublattices), while the equator ($\rho = 0$) exhibits the $U(1)$ superconducting phase.

Note that there have been studies of such competing phase models in uncharged systems [19–22], but such systems do not admit finite energy vortex solutions. As we are interested in vortices in this paper we will deal entirely with the charged model.

We also note that the effect of competing order is of general interest in superconductivity. It is important to understand such systems and their generalizations, with a focus on their solitonic excitations, from multicomponent systems with density-density couplings [23] (which also exhibits nontrivial vortex interactions) to competition with spin density waves [24,25].

This paper will focus on the continuous effective Ginzburg-Landau (GL) formalism proposed in Refs. [13,26], which is derived from the attractive $SO(3)$ Hubbard model mentioned above. It is similar to other phenomenologically proposed models [27–29], introduced in an attempt to model the experimental observation of competing phases in charged systems. To derive a GL model one must take the Hubbard model and assume that anisotropy near the charge ordered, superconducting transition is negligible. Taking this isotropic limit and assuming a quadratic symmetry breaking term gives fields subject to the free energy density,

$$\mathcal{E} = \frac{\chi}{2}\left|\left(\nabla - \frac{2ie}{\hbar}\boldsymbol{A}\right)\Delta\right|^2 + \frac{1}{8\pi}|\nabla \times \boldsymbol{A}|^2 + \frac{\chi}{2}|\nabla\rho|^2 - |\Delta|^2 - (1-\delta)\rho^2, \tag{1.1}$$

where $\boldsymbol{A}$ is the electromagnetic gauge potential coupled to the charged field Δ, and χ and δ are positive constants, δ

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representing the strength of next-nearest neighbor hopping. As with the Hubbard model, order competition is imposed via the constraint $|\Delta|^2 + \rho^2 = c^2$. In mathematical terms, this is an example of a gauged sigma model, objects of strong intrinsic interest.

The purpose of this paper is to develop a theory of the long-range interactions between vortices in this model within the point vortex formalism. A key observation is that the model supports two distinct species of vortex which we call North vortices, with $\rho = c$ in the vortex core, and South vortices, with $\rho = -c$ in the vortex core, and that each of these has an antivortex counterpart (possessing a quantum of *negative* magnetic flux).

While vortices have been previously studied [13,26,30], there has been no detailed study of the different (anti)vortex interactions in the model. One paper briefly considered the effect that introducing charge order has on the strength of purely superconducting interactions in the Hubbard model [31]. In that paper, vortex interactions were approximated to be that of a strongly type-II single component model, with a numerically motivated exponential correction term, dependent on the value of δ. However, in this paper we demonstrate that to understand the interactions, one cannot separate the superconducting and charge order components and treat them separately. We will also show that the interactions act as Bessel functions. Our detailed study of the interactions in the model will lead to a typology argument, similar to the standard single component Ginzburg-Landau model, but with additional complications due to the multiple species of vortex.

Since the model supports two different species of vortex, it possesses *two* integer-valued topological charges: The total number n of magnetic flux quanta, and the half-degree d of the map $\mathbb{R}^2 \to S^2$ defined by $(x_1, x_2) \mapsto (\mathrm{Re}\Delta(x_1, x_2), \mathrm{Im}\Delta(x_1, x_2), \rho(x_1, x_2))/c$, or, equivalently the net numbers of North vortices k_+ and South antivortices k_-. This pair of integers cannot change under any smooth deformation of the fields Δ, ρ, A_i preserving finite total energy.

We will see that the interaction between (anti)vortices of all types depends crucially on the coupling parameter

$$\mu = \frac{\hbar\delta}{2\sqrt{2\pi}e\chi c}, \tag{1.2}$$

which plays a role analogous to the Ginzburg-Landau parameter in conventional (single component) GL theory. If $\mu > 1$, vortices of any species repel one another, as do antivortices of any species, while vortices always attract antivortices. If $\mu < 1$, the behavior is more surprising: Like vortices attract, as do like antivortices, but *unlike* vortices repel, as do unlike antivortices, *and* unlike vortex-antivortex pairs. The regime of critical coupling $\mu = 1$ is particularly subtle with various combinations of vortices and antivortices experiencing no static interactions at all. The situation is summarized in Table I. This constitutes the equivalent of the familiar typology argument for the standard single component GL model, where the parameter μ is now the GL parameter, determining the interaction type. Hence for $\mu > 1$ we call this a type-II superconductor, for $\mu < 1$ a type-I superconductor, and $\mu = 1$ a critically coupled superconductor.

The rest of this paper is structured as follows. In Sec. II we choose length, energy, and charge units to reduce the GL model to a standard gauged sigma model, review its topological properties, and construct its (anti)vortices, paying particular attention to their asymptotics at spatial infinity. In Sec. III we develop a theory of long-range intervortex interactions by modeling vortices as solutions of the linearization of the sigma model about its vacuum, in the presence of appropriate point sources at the vortex center, chosen to replicate the vortex's large r behavior. This models vortices as composite point particles carrying a scalar monopole charge, inducing a real scalar field of mass μ (roughly, the field ρ) and a magnetic dipole moment inducing a vector field of mass 1 (roughly, A_i). The interaction energy between pairs of such point particles is easily computed, producing the predictions of Table I, as well as precise asymptotic formulas for the interaction energies valid at large separation. In Sec. IV we verify these predictions by numerically computing the interaction energy of (anti)vortex pairs via a gradient descent energy minimization method. Finally, Sec . V presents some concluding remarks.

TABLE I. Summary of interactions between (anti)vortex pairs. N denotes North vortex, S denotes South vortex, and an overbar denotes the corresponding antivortex. The 0 entries in the $\mu = 1$ table indicate (anti)vortex pairs which experience no interaction: Their total energy is independent of their separation.

	$\mu < 1$					$\mu = 1$					$\mu > 1$			
	N	$\bar{N}$	S	$\bar{S}$		N	$\bar{N}$	S	$\bar{S}$		N	$\bar{N}$	S	$\bar{S}$
N	attract	attract	repel	repel	N	0	attract	repel	0	N	repel	attract	repel	attract
$\bar{N}$		attract	repel	repel	$\bar{N}$		0	0	repel	$\bar{N}$		repel	attract	repel
S			attract	attract	S			0	attract	S			repel	attract
$\bar{S}$				attract	$\bar{S}$				0	$\bar{S}$				repel

II. COMPETING-ORDER VORTICES

We first choose scales to minimize the number of parameters in the free energy (1.1). Let

$$\mathcal{E} = \lambda_{\mathcal{E}}\mathcal{E}^{\text{new}} - c^2, \quad x_i = \lambda_x x_i^{\text{new}}, \quad A_i = \lambda_A A_i^{\text{new}},$$

$$(u_1 + iu_2, u_3) = (\Delta/c, \rho/c),$$

$$D_i\boldsymbol{u} = \frac{\partial \boldsymbol{u}}{\partial x_i^{\text{new}}} - A_i^{\text{new}}\boldsymbol{e} \times \boldsymbol{u}, \tag{2.1}$$

where $\boldsymbol{e} = (0, 0, 1)$. Then, with the choices

$$\lambda_{\mathcal{E}} = 4\pi\chi^2 c^4\left(\frac{2e}{\hbar}\right)^2, \quad \lambda_x^2 = \frac{\chi c^2}{\lambda_{\mathcal{E}}}, \quad \lambda_A = \frac{\hbar}{2e\lambda_x}, \tag{2.2}$$

we find that

$$\mathcal{E}^{\text{new}} = \frac{1}{2}D_i\boldsymbol{u}\cdot D_i\boldsymbol{u} + \frac{1}{2}(B^{\text{new}})^2 + \frac{\mu^2}{2}(\boldsymbol{e}\cdot\boldsymbol{u})^2 \tag{2.3}$$

where $B^{\text{new}} = \partial_1^{\text{new}}A_2^{\text{new}} - \partial_2^{\text{new}}A_1^{\text{new}}$ and μ is defined in Eq. (1.2). We henceforth discard the superscript "new."

The total energy of a pair of fields $(\boldsymbol{u}, A)$ is the integral

$$E = \int_{\mathbb{R}^2} \mathcal{E}dx_1dx_2. \tag{2.4}$$

In order for this to be finite, $\boldsymbol{u}$, at large r (where $(x_1, x_2) =: r(\cos\theta, \sin\theta)$), must approach the equator $u_3 = 0$ on S^2. It need not, however, be constant: It may wind around the equator

$$\boldsymbol{u} \sim (\cos n\theta, \sin n\theta, 0) \tag{2.5}$$

some integer n times. Then finite energy also implies $|D\boldsymbol{u}| \sim 0$ as $r \to \infty$, so $\boldsymbol{A} \sim \frac{n}{r}(-\sin\theta, \cos\theta)$, from which, by a standard application of Stokes's theorem one finds that the total magnetic flux of any finite energy configuration is quantized,

$$\int_{\mathbb{R}^2} B dx_1 dx_2 = 2\pi n. \tag{2.6}$$

If $n \neq 0$, there must be points in the plane where $u_1 + iu_2 = 0$. Note, however, that these come in two distinct species since u_3 may take the value $+1$ or -1 at each such point. Consider a point $\boldsymbol{x}^+$ where $\boldsymbol{u}(\boldsymbol{x}^+) = (0, 0, 1)$. This point itself may be assigned a sign $\sigma(\boldsymbol{x}^+)$ according to whether the field $\boldsymbol{u}(\boldsymbol{x})$ is locally an orientation preserving ($\sigma = +1$) or orientation reversing ($\sigma = -1$) map close to $\boldsymbol{x}^+$. The sum of these signs over all points where $\boldsymbol{u} = (0, 0, 1)$ is an integer-valued topological invariant of the field $\boldsymbol{u}$,

$$k_+ = \sum_{\boldsymbol{x}\in\boldsymbol{u}^{-1}(\boldsymbol{e})} \sigma(\boldsymbol{x}), \tag{2.7}$$

which we may interpret as the net excess of North vortices over North antivortices in the field configuration. We may similarly assign a sign $\sigma(\boldsymbol{x}^-)$ to each point $\boldsymbol{x}^-$ in the plane at which $\boldsymbol{u}(\boldsymbol{x}^-) = (0, 0, -1)$. Again, $\sigma(\boldsymbol{x}^-) = +1$ if $\boldsymbol{u}(\boldsymbol{x})$ is locally orientation preserving and $\boldsymbol{u}(\boldsymbol{x}^-) = -1$ if it is locally orientation reversing. One should note, however, that, while (u_1, u_2) is a good oriented local coordinate system for S^2 in a neighborhood of $(0, 0, 1)$, it is *anti-oriented* in a neighborhood of $(0, 0, -1)$, so each point with $\sigma(\boldsymbol{x}^-) = +1$ contributes *negatively* to the winding of the field $\boldsymbol{u}$ about the equator in S^2. Hence, the integer-valued topological invariant associated with the South (anti)vortex positions

$$k_- = \sum_{\boldsymbol{x}\in\boldsymbol{u}^{-1}(-\boldsymbol{e})} \sigma(\boldsymbol{x}) \tag{2.8}$$

represents the net excess of South *antivortices* over South *vortices* in the field configuration. One sees that the winding number at spatial infinity, which determines the total magnetic flux, is determined by k_+, k_- as

$$n = k_+ - k_-. \tag{2.9}$$

Furthermore, the total signed area in S^2 covered by the mapping $\boldsymbol{u}(\boldsymbol{x})$ is $2\pi(k_+ + k_-)$, so we may identify $k_+ + k_-$ has the *half-degree* of the map $\boldsymbol{u} : \mathbb{R}^2 \to S^2$. The four types of (anti)vortex supported by this model are summarized pictorially in Fig. 1. We reiterate that the difference between North and South vortices is the dominant sublattice for the charge density wave order in the core of the vortex.

To understand the (anti)vortices in more detail, we must numerically solve the Euler-Lagrange equations for the functional E,

$$P_{\boldsymbol{u}}(-D_iD_i\boldsymbol{u} + \mu^2(\boldsymbol{e}\cdot\boldsymbol{u})\boldsymbol{e}) = 0, \tag{2.10}$$

$$-\partial_i\partial_i A_j + \partial_j\partial_i A_i - \boldsymbol{e}\cdot(\boldsymbol{u}\times D_i\boldsymbol{u}) = 0, \tag{2.11}$$

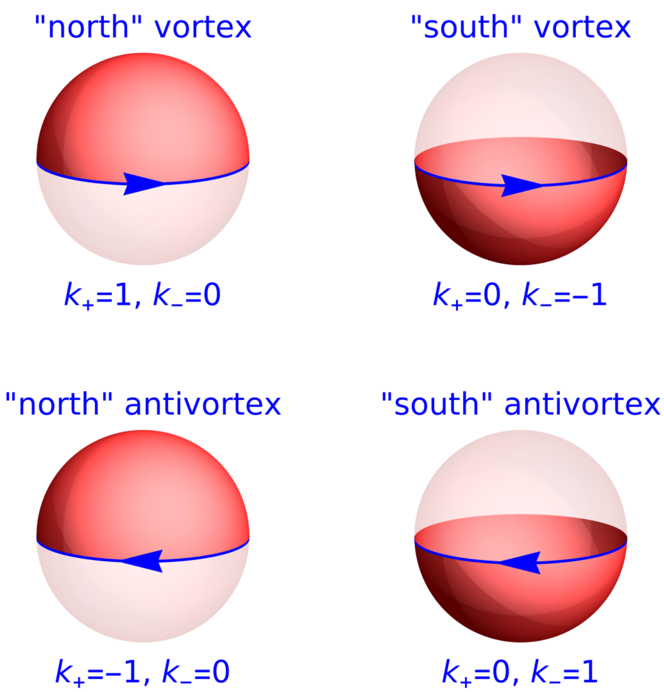

FIG. 1. The field values attained by the four species of (anti)vortex. The field $\boldsymbol{u}(\boldsymbol{x})$ wraps the circle at spatial infinity once around the equator in the direction indicated, anticlockwise for vortices, clockwise for antivortices (viewed from above the North pole). The (anti)vortex interior then covers either the Northern or the Southern hemisphere once. The topological charges k_+, k_- measure the number of times the field assumes the pole values $(0, 0, 1)$ and $(0, 0, -1)$ respectively, counted with orientation and multiplicity. These poles indicate two different sublattices for the charge density wave order in the core of the (anti)vortex.

where $P_{\boldsymbol{u}}$ denotes projection orthogonal to $\boldsymbol{u}$, that is, $P_{\boldsymbol{u}}(\boldsymbol{v}) := \boldsymbol{v} - (\boldsymbol{u}\cdot\boldsymbol{v})\boldsymbol{u}$. These are consistent with the ansatz

$$\boldsymbol{u}^N = (\sin f(r)\cos\theta, \sin f(r)\sin\theta, \cos f(r)), \tag{2.12}$$

$$\boldsymbol{A}^N = \frac{a(r)}{r}(-\sin\theta, \cos\theta), \tag{2.13}$$

where the profile functions f, a, satisfy the coupled ordinary differential equation system

$$f'' + \frac{1}{r}f' - \frac{(1-a)^2}{r^2}\sin f\cos f + \mu^2\sin f\cos f = 0, \tag{2.14}$$

$$a'' - \frac{1}{r}a' + \sin^2 f(1-a) = 0, \tag{2.15}$$

subject to the boundary conditions $f(0) = a(0) = 0$, $f(\infty) = \pi/2$, $a(\infty) = 1$. Having found f and a, we may easily construct the other three species of (anti)vortex,

$$\boldsymbol{u}^S = (\sin f(r)\cos\theta, \sin f(r)\sin\theta, -\cos f(r)),$$

$$\boldsymbol{A}^S = \frac{a(r)}{r}(-\sin\theta, \cos\theta),$$

$$\boldsymbol{u}^{\bar{N}} = (\sin f(r)\cos\theta, -\sin f(r)\sin\theta, \cos f(r)),$$

$$\boldsymbol{A}^{\bar{N}} = \frac{a(r)}{r}(\sin\theta, -\cos\theta),$$

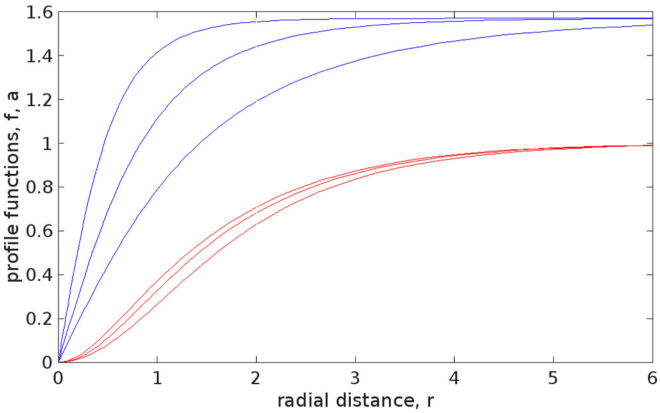

FIG. 2. The profiles functions $f(r)$ (blue curves) and $a(r)$ (red curves) of a North vortex at couplings $\mu = 2$ (top), $\mu = 1$ (middle) and $\mu = 0.5$ (bottom).

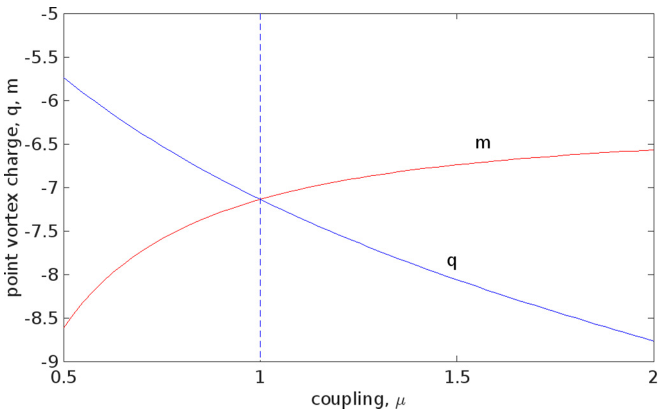

FIG. 3. The large r shooting parameters q, m of the North vortex solution as functions of the coupling μ. These may be interpreted as the scalar monopole charge (q) and magnetic dipole moment (m) of the corresponding point vortex.

$$\boldsymbol{u}^{\bar{S}} = (\sin f(r)\cos\theta, -\sin f(r)\sin\theta, -\cos f(r)),$$
$$\boldsymbol{A}^{\bar{S}} = \frac{a(r)}{r}(\sin\theta, -\cos\theta). \tag{2.16}$$

The system (2.14), (2.15) does not appear to be integrable, so we resort to numerical integration to find f, a. Regularity at the origin requires $f(r) \sim \alpha_1 r$ and $a(r) \sim \alpha_2 r^2$ for some constants α_1, α_2. For large r, $\widehat{f}(r) := f(r) - \pi/2$ and $\widehat{a}(r) := a(r) - 1$, being small, should be asymptotic to decaying solutions of the *linearization* of the system about $(f, a) = (\pi/2, 1)$,

$$\widehat{f}'' + \frac{1}{r}\widehat{f}' - \mu^2\widehat{f} = 0, \tag{2.17}$$

$$\widehat{a}'' - \frac{1}{r}\widehat{a}' - \widehat{a} = 0. \tag{2.18}$$

Hence, at large r,

$$f(r) \sim \frac{\pi}{2} + \frac{q}{2\pi}K_0(\mu r), \quad a(r) \sim 1 + \frac{m}{2\pi}rK_1(r), \tag{2.19}$$

where K_0, K_1 are modified Bessel's functions of the second kind, and q, m are unknown constants. The factors of 2π are included for later convenience. Our numerical strategy is to solve (2.14), (2.15) on $[r_0, R]$, with $r_0 > 0$ small and R large by shooting rightwards from r_0, using (α_1, α_2) as shooting parameters, leftwards from R using (q, m) as shooting parameters, and imposing that f, a and their derivatives match at some interior point r_1 of order 1. The results of this scheme for various values of the coupling μ are depicted in Fig. 2. Of particular interest are the values of the constants (q, m) as functions of μ, depicted in Fig. 3. Note that $q \equiv m$ when $\mu = 1$. This is not a coincidence: The system (2.14), (2.15) reduces to a *first* order system at this critical value of the coupling,

$$f' = \frac{1-a}{r}\sin f, \quad a' = r\cos f, \tag{2.20}$$

from which it follows immediately that $q \equiv m$. This is a symptom of the *self duality* (or Bogomol'nyi Prasad Sommerfield property) enjoyed by the model at $\mu = 1$, whose full consequences are both deep and far ranging [32,33]. In this paper we will concentrate on the case $\mu \neq 1$, however.

III. THE POINT VORTEX MODEL

It is convenient to think of (anti)vortices as static solutions of the Lorentz invariant model on $(2+1)$-dimensional Minkowski space whose static energy is E, that is, the model with Lagrangian density

$$\mathcal{L} = \frac{1}{2}D_\mu\boldsymbol{u}\cdot D^\mu\boldsymbol{u} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\mu^2}{2}(\boldsymbol{e}\cdot\boldsymbol{u})^2, \tag{3.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, spacetime indices μ, ν run over $0, 1, 2$, and the Minkowski metric has signature $+--$. We have merely extended the indices to include time components for all derivatives and the gauge field. We emphasize that this is a mathematical device, introducing second-order Lorentzian dynamics. This allows us to access some techniques and results familiar in the study of topological solitons in high energy physics. We certainly do *not* assert that the time dynamics defined by this relativistic extension is relevant to competing order superconductors.

The key observation is that static vortices, far from their core, are indistinguishable from solutions of the linearization of the model (3.1) about the vacuum [meaning $A_\mu = 0$, $\boldsymbol{u} = (1, 0, 0)$] in the presence of appropriate point sources placed at the vortex center. Since physics is model independent, the forces between well-separated vortices should coincide with those between the corresponding point sources interacting via the fields they induce in the linear theory. These are easily computed, yielding an asymptotic formula for the interaction energy between well-separated vortices. This underlying idea was introduced by Manton to study long-range forces between magnetic monopoles [34], and subsequently applied to nuclear Skyrmions by Schroers [35]. It was adapted to vortices in the conventional Ginzburg-Landau model in Ref. [36], then multicomponent vortices in Refs. [37–39].

Our first task is to identify the point sources that replicate the vortex asymptotics, and to do this we must first rewrite it in the gauge in which, as $r \to \infty$, $\boldsymbol{u} \to (1, 0, 0)$ in every direction, that is, the gauge where $u_2 = 0$ and $u_1 \geqslant 0$. This is accomplished by applying the singular (at $r = 0$) gauge transformation $(u_1 + iu_2, u_3) \mapsto (e^{-i\theta}(u_1 + iu_2), u_3)$. The order parameter takes the form $\boldsymbol{u} = (\cos\Theta, 0, \sin\Theta)$ in this gauge, the vacuum is $\Theta = 0$ and the North vortex has

$$\Theta(r) = f(r) - \frac{\pi}{2} \sim \frac{q}{2\pi}K_0(\mu r),$$
$$(A_0, A_1, A_2) = \frac{a(r)-1}{r}(0, -\sin\theta, \cos\theta) \sim \frac{m}{2\pi}(0, \partial_2, -\partial_1)K_0(r). \quad (3.2)$$

These are precisely [36] the fields induced in the linearized model

$$\mathcal{L}_{lin} = \frac{1}{2}\partial_\mu\Theta\partial^\mu\Theta - \frac{\mu^2}{2}\Theta^2 + \rho\Theta - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}A_\mu A^\mu + j_\mu A^\mu \quad (3.3)$$

by the static sources

$$\rho = q\delta(\boldsymbol{x}), \quad (j_0, j_1, j_2) = m(0, \partial_2, -\partial_1)\delta(\boldsymbol{x}), \quad (3.4)$$

so our linearized model of a North vortex is a composite point source consisting of a scalar monopole of charge $q^N = q$, inducing a real scalar field Θ of mass μ, and a magnetic dipole of moment $m^N = m$ inducing a Proca field A_μ of mass 1. The corresponding sources for the other species of (anti)vortex follow immediately by unwinding Eq. (2.16). All are scalar monopole/magnetic dipole composites, with charges

$$(q^N, m^N) = (q, m), \quad (q^S, m^S) = (-q, m),$$
$$(q^{\bar{N}}, m^{\bar{N}}) = (q, -m), \quad (q^{\bar{S}}, m^{\bar{S}}) = (-q, -m). \quad (3.5)$$

The interaction Lagrangian for a pair of sources $(\rho^{(1)}, j_\mu^{(1)})$, $(\rho^{(1)}, j_\mu^{(1)})$ is

$$L_{\rm int} = \int_{\mathbb{R}^2} \left(\rho^{(1)}\Theta^{(2)} + j_\mu^{(1)}A^\mu_{(2)}\right)dx_1dx_2, \quad (3.6)$$

where $(\Theta^{(2)}, A_\mu^{(2)})$ are the fields induced by the second source. We apply this in the case where the sources are static scalar monopole/magnetic dipole composites of charges (q_1, m_1), (q_2, m_2) located at $\boldsymbol{y}$ and $\boldsymbol{z}$, respectively. The result is a function of $s := |\boldsymbol{y} - \boldsymbol{z}|$, the vortex separation. It may be interpreted as *minus* the interaction energy of the source pair, so

$$E_{\rm int}(s) = -L_{\rm int} = \frac{1}{2\pi}[m_1m_2K_0(s) - q_1q_2K_0(\mu s)]. \quad (3.7)$$

If $\mu > 1$, the first term, representing magnetic interactions, dominates at large s, whereas if $\mu < 1$, the second term, representing scalar interactions dominates. By choosing (q_1, m_1), (q_2, m_2) from the list (3.5), we obtain long-range interaction energies between (anti)vortices of any species. The nature of these interactions is summarized in Table I. The zero entries for critical coupling, $\mu = 1$, follow from the observation that $q = m$ here. Our calculation establishes that the leading order interactions for NN, SS, $N\bar{S}$ and $S\bar{N}$ pairs vanish in this case. In fact, the self-duality structure can be used to prove that the interaction vanishes exactly for these pairs [32]: Static solutions exist with the individual vortices placed at any points in the plane when $\mu = 1$.

Of course, these predicted interaction potentials are based on a leap of faith; that physics is model independent. This particular faith allows, indeed encourages, scepticism in its acolytes. Luckily it also admits a definitive test: We can compute the energy between vortices held at a fixed separation by numerical simulation of the original nonlinear model. This is the subject of the next section.

IV. NUMERICAL RESULTS

How can we compute the interaction energy $E_{\rm int}^{NN}(s)$ between two North vortices held distance s apart? Note that no such *static* solution exists (unless $s = 0$, or $\mu = 1$), precisely because vortices exert forces on one another. The answer is that we solve a *constrained* minimization problem for the energy functional E: We minimize among all fields having $k_+ = 2$ and $k_- = 0$ subject to the constraint that $\boldsymbol{u}(s/2, 0) = \boldsymbol{u}(-s/2, 0) = \boldsymbol{e}$. In practice, we discretize space, replacing spatial derivatives by difference operators on a regular $n_1 \times n_2$ lattice of spacing h (we used $n_1 = n_2 = 251$ and $h = 0.1$). This replaces the continuum energy functional $E(\boldsymbol{u}, \boldsymbol{A})$ by a discrete approximant $E_{\rm dis} : \mathcal{C}_{\rm dis} \to \mathbb{R}$ where $\mathcal{C}_{\rm dis} = (S^2)^{n_1n_2} \times (\mathbb{R}^2)^{n_1n_2}$ is the discretized configuration space. We then construct an appropriate initial guess $\boldsymbol{u}_{i,j}$, $\boldsymbol{A}_{i,j}$ with, around the boundary of the lattice, $\boldsymbol{u}_{i,j} \cdot \boldsymbol{e} = 0$ and winding 2, and

$$\boldsymbol{u}_{\pm i_0, 0} = \boldsymbol{e}, \quad (4.1)$$

where $s = 2i_0h$. We then minimize $E_{\rm dis}$ among all points in $\mathcal{C}_{\rm dis}$ satisfying the constraint (4.1) using arrested Newton flow [40] for the function $E_{\rm dis}$, but *never updating* $\boldsymbol{u}_{\pm i_0, 0}$ (or $\boldsymbol{u}$, $\boldsymbol{A}$ on the boundary of the lattice). This automatically produces a solution of the Euler-Lagrange equations for our energy functional on $\mathbb{R}^2\backslash\{(\pm s/2, 0)\}$ satisfying the constraint (4.1) at the missing points. An alternative to this procedure is to solve the Euler-Lagrange equations on $\mathbb{R}^2\backslash\{(\pm s/2, 0)\}$ directly, an approach exploited for the standard GL model in [41]. Having computed the lowest energy among all $(k_+, k_-) = (2, 0)$ field configurations with $\boldsymbol{u}(\pm s/2, 0) = \boldsymbol{e}$, we then subtract twice the energy of a single North vortex to obtain $E_{\rm int}^{NN}(s)$.

Interaction energies for any other vortex combination can be computed similarly by modifying the constraint (4.1) and boundary behavior of the field configuration appropriately. By symmetry, $NN \equiv \bar{N}\bar{N} \equiv SS \equiv \bar{S}\bar{S}$, $N\bar{N} \equiv S\bar{S}$, $NS \equiv \bar{N}\bar{S}$ and $N\bar{S} \equiv \bar{N}S$, so only four of the ten distinct (anti)vortex pairs need be considered, and we can, without loss of generality, assume that the left vortex is N. The results are depicted in Fig. 4. They match perfectly the predictions of our simple point vortex model.

V. CONCLUDING REMARKS

In this paper we have developed a simple point vortex model of long-range interactions between (anti)vortices in the usual GL model of competing-order superconductors. The model supports two distinct species of vortex, each with a matching antivortex, and hence there are ten different (anti)vortex pairs possible. Symmetries reduce this to four energetically distinct pairs: NN, $N\bar{N}$, NS, and $N\bar{S}$. The point vortex model predicts asymptotic formulas for the interaction energy of each of these pairs, as a function of separation, with considerable success. This allows us to make typologylike arguments similar to those in the standard single component GL model. The qualitative nature of the interactions depends

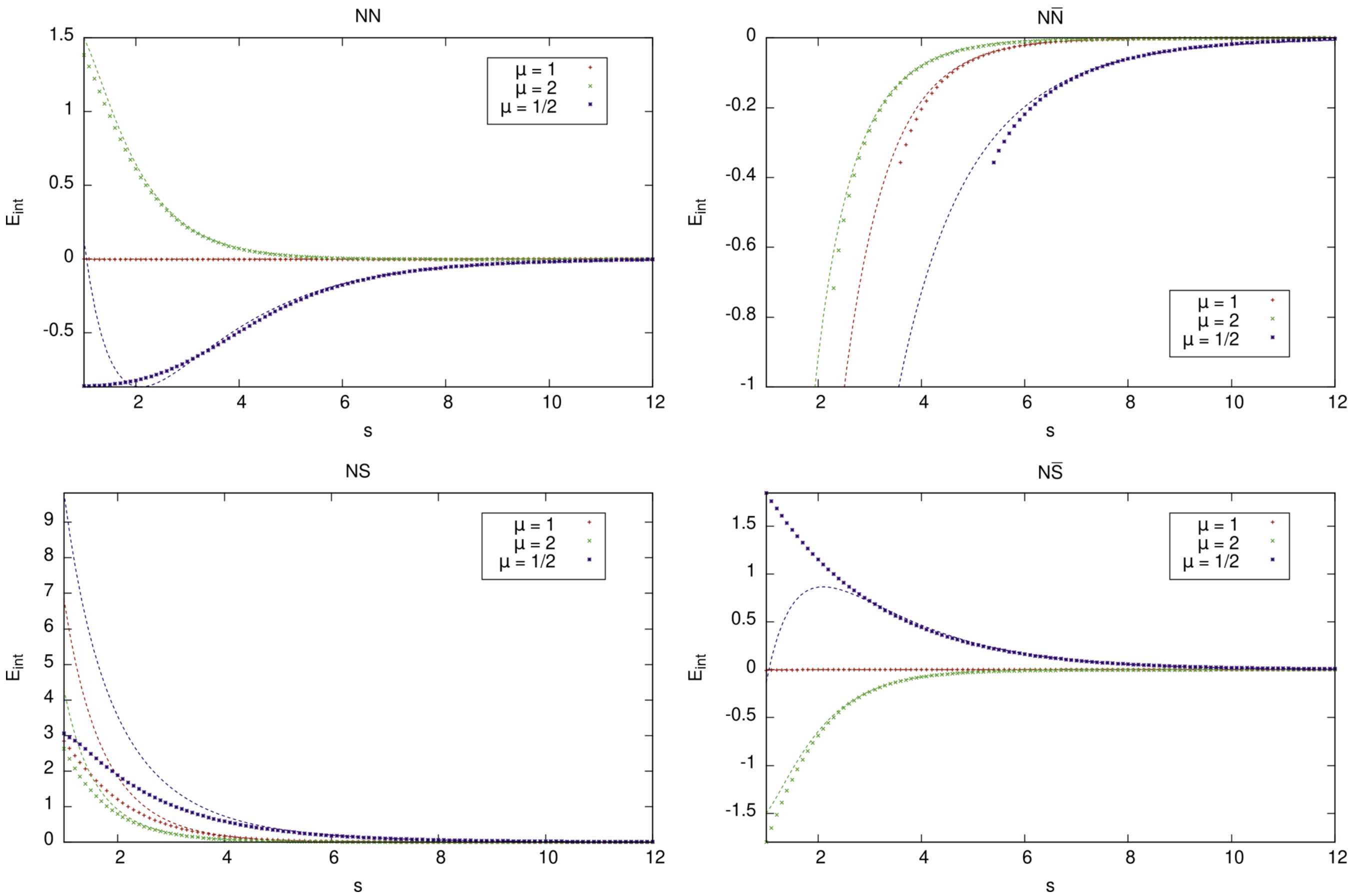

FIG. 4. Plot of the interaction energies for different vortex pairs and separations $E_{\rm int} = E - 2E^1$. The dashed lines are the point vortex approximations given by (3.7). Note that the interactions agree with table I.

on a single parameter μ, the equivalent of the GL parameter in the standard model. If $\mu < 1$ (equivalent of type I) the interactions display some counterintuitive features. For example, the interaction between vortices of one species and antivortices of the other is *repulsive*.

It would be interesting to study vortex lattices in this model in an applied magnetic field. Although, for $\mu > 1$, pure N (or pure S) arrays are energetically favoured over NS mixtures, if the state emerges from disorder, presumably some species mixing is inevitable. Some work on vortex lattices has already been done [26,31], however, there is further understanding to be gleaned here, as even understanding the "type" of the superconductor is subtle. In addition, for $\mu < 1$, while it may be preferable for superconducting domains to form rather than vortices, as in a single band superconductor, these domains can now be N or S domains, which will repel each other, leading to metastable states.

Another possibility is the studying of vortex/antivortex bound states when applying a magnetic field. While there is previous work on vortices in superconductors [13,26,30,31], the importance of antivortices has been completely ignored in the literature until now.

It would also be interesting to consider specific materials such a s YBCO. Note that while it it is challenging to actually determine the parameter μ of a given material, it has been suggested that YBCO [20] exhibits vortices and is type II [26]. In this model this likely means that $\mu \gg 1$ so we have vortex/vortex repulsion for all species.

Finally it would be particularly interesting to consider in detail the effect of adding a small term linear in ρ to the original Ginzburg-Landau theory, breaking the energy degeneracy of the two CDW ground states, the upshot of which is that (after rescaling) the energy density becomes

$$\mathcal{E} = \frac{1}{2}D_i\boldsymbol{u}\cdot D_i\boldsymbol{u} + \frac{1}{2}B^2 + \frac{\mu^2}{2}(\tau - \boldsymbol{e}\cdot\boldsymbol{u})^2, \tag{5.1}$$

where τ is an extra small parameter. This term breaks the symmetry between N and S vortices: If $\tau > 0$ then S vortices are slightly more energetically costly than N vortices (and *vice versa* if $\tau < 0$). Remarkably, when $\mu = 1$, the model still enjoys a self-duality structure, and N vortices exert no net force on S antivortices. The basic point-vortex model of intervortex forces is similar to the one developed here, in that a point vortex still consists of a scalar monopole of some charge q and a magnetic dipole of some moment m, but these sources induce fields of mass $\sqrt{1-\tau^2}\mu$ and $\sqrt{1-\tau^2}$, and there is no symmetry relating q^N with q^S or m^N with m^S. Introducing a linear term has the effect of increasing the range of intervortex forces, therefore, as well as breaking the degeneracy of N and S vortices.

ACKNOWLEDGMENTS

The authors would like to thank E. Babaev for useful conversations. This work was supported by the UK Engineering and Physical Sciences Research Council through Grant No. EP/P024688/1. All simulations were run using the Soliton Solver library developed by T.W. at the University of Leeds.

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