Geometric criterion for solvability of lattice spin systems

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We present a simple criterion for solvability of lattice spin systems on the basis of graph theory and simplicial homology. The lattice systems satisfy algebras with graphical representations. It is shown that the null spaces of adjacency matrices of the graphs provide conserved quantities of the systems. Furthermore, when the graphs belong to a class of simplicial complexes, the Hamiltonians are found to be mapped to bilinear forms of Majorana fermions, from which the full spectra of the systems are obtained. In the latter situation, we find a relation between conserved quantities and the first homology group of the graph, and the relation enables us to interpret the conserved quantities as flux excitations of the systems. The validity of our theory is confirmed in several known solvable spin systems including the one-dimensional (1D) transverse-field Ising chain, the 2D Kitaev honeycomb model, and the 3D diamond lattice model. We also present new solvable models on a 1D trijunction, 2D and 3D fractal lattices, and the 3D cubic lattice.

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I. INTRODUCTION

Exactly solvable models have played important roles in the understanding of physics in strongly correlated systems. In particular, exactly solvable lattice spin models have revealed many important phenomena. For instance, solving the 2D Ising model exactly, Onsager [1] showed the presence of ferromagnetic phase transition in spin systems for the first time, which is one of the milestones in statistical physics. Since Onsager's work, other lattice spin models were solved exactly, such as the Potts model, the hard-hexagon model, and so on [2–4]. More recently, exactly solvable models also have disclosed exotic quantum phases in strongly correlated systems, such as spin liquid phases with non-Abelian anyon excitations [5].

Quantum solvable lattice spin models are classified into three types. The first one has a Hamiltonian of which terms commute with each other, which includes the two-dimensional (2D) Kitaev toric code [5,6], the X-cube model [7,8], and so on. The second one has special symmetries such as Lie groups or quantum groups. This type includes the 1D Heisenberg model and the *XXZ* model [9]. Then, the last one can be transformed into free-fermion systems. For instance, both the 1D *XY* model [10–13] and the 1D transverse-field Ising model [14–17] can be converted into free-fermion systems by using the Jordan-Wigner transformation [18]. Another example is the Kitaev honeycomb lattice model [5,6,19], which is transformed into a free-fermion system by adapting a redundant representation of spins with Majorana operators. In addition to these, there exist a number of other models [20–34].

In this paper, we present a simple criterion for the third type of solvability of lattice spin systems. Our criterion is based on graph theory and simplicial homology. For a lattice spin system with an algebra with a graphical representation, we show that the null space of the adjacency matrix of the graph provides conserved quantities of the system. Furthermore, when the graph belongs to a class of simplicial complexes, we reveal that the Hamiltonian is mapped to a bilinear form of Majorana fermions, from which the full spectrum of the system is obtained. We also find a relation between the conserved quantities and the first homology group of the graph. Based on the relation, we interpret the conserved quantities as flux excitations. We apply our criterion for several known solvable spin systems including the 1D transverse-field Ising chain, the 1D *XY* model, the 2D Kitaev honeycomb model, and the 3D diamond lattice model. We also present new solvable models on a 1D trijunction, 2D and 3D fractal lattices, and the 3D cubic lattice.

The rest of this paper is organized as follows. In Sec. II, we present the main results. We introduce lattice models which satisfy a class of algebras. Representing the algebra in the form of a graph, we present theorems that give the criterion of solvability in terms of graph theory and simplicial homology. In Sec. III, we illustrate our criterion by applying it to the 1D transverse-field Ising model, the *XY* model, the Kitaev honeycomb model, and so on. We also provide new solvable models in Sec. IV. In Sec. V, we present proofs of the theorems in Sec. II. Finally, we give a discussion in Sec. VI.

II. MAIN RESULTS

First, we present our main results in this paper, which are summarized in three theorems. The proofs of these theorems will be given in Sec. V.

In this paper, we consider a class of Hamiltonians H that satisfy the following properties:

(a) H has the form of $H = \sum_{j=1}^{n} \lambda_j h_j$ with coefficients $\lambda_j \in \mathbb{R}$ and operators h_j ($j = 1, \dots, n$).

(b) The operators h_j obey $h_j^2 = 1$, $h_j^\dagger = h_j$, and $h_j h_k = \epsilon_{jk} h_k h_j$ with $\epsilon_{ij} = \pm 1$.

The second property requires that h_j's commute or anticommute with each other. The operators h_j generate an algebra $\mathcal{A}$ on $\mathbb{C}$, which we call the bond algebra (BA) [30,35,36]. To represent the BA $\mathcal{A}$ visually, we introduce a graph $\mathcal{G}(\mathcal{A})$ as follows:

(a) Put n vertices in general position and place h_i on the ith vertex.

(b) When h_i and h_j anticommute (commute) with each other, we draw (do not draw) a line between the vertices with h_j and h_k.

The resulting graph compactly encodes the information of the commutativity among h_j's. We refer to the graph $\mathcal{G}(\mathcal{A})$ as a commutativity graph (CG) [37] of $\mathcal{A}$. The CG $\mathcal{G}(\mathcal{A})$ has an algebraic representation with an adjacency matrix $M(\mathcal{A})$. The adjacency matrix $M(\mathcal{A})$ is a real symmetric $n \times n$ matrix of which elements indicate whether pairs of vertices are adjacent or not in $\mathcal{G}(\mathcal{A})$: The diagonal elements of $M(\mathcal{A})$ are zero, and the (i, j) component is chosen to be 1 (0) if the ith and jth vertices in $\mathcal{G}(\mathcal{A})$ are connected (not connected) by a line. The multiplication and the addition for $M(\mathcal{A})$ are defined as a matrix on the binary field $\mathbb{F}_2$, i.e., a matrix with entries 0 or 1, which satisfy $0 + 0 = 0$, $0 + 1 = 1$, $1 + 0 = 1$, and $1 + 1 = 0$.

Using $M(\mathcal{A})$, we present our first main result. A product $h_{j_1} h_{j_2} \cdots h_{j_k}$ conserves if it commutes with any h_j in H. We find that such conserved quantities in $\mathcal{A}$ can be counted by using the adjacency matrix $M(\mathcal{A})$. More precisely, we have Theorem 1:

Theorem 1. Let $\mathcal{A}$ be the BA of a Hamiltonian $H = \sum_{j=1}^{n} \lambda_j h_j$, $\mathcal{G}(\mathcal{A})$ be the corresponding CG of $\mathcal{A}$, and $M(\mathcal{A})$ be the adjacency matrix of $\mathcal{G}(\mathcal{A})$. Then, the dimension of the kernel space of $M(\mathcal{A})$ coincides with the total number of conserved quantities in the form of $h_{j_1} \cdots h_{j_k}$.

Here, the kernel space (or null space) of $M(\mathcal{A})$ is defined by

$$\ker M(\mathcal{A}) = \{\boldsymbol{v} \in \mathbb{F}_2^n; M(\mathcal{A})\boldsymbol{v} = \boldsymbol{0}\}. \quad (1)$$

As is shown in Sec. V, we can construct the conserved quantities from an element $\boldsymbol{v}$ of $\ker M(\mathcal{A})$: Let $\boldsymbol{v}(h_j)$ be the unit vector on $\mathbb{F}_2$ having a nonzero element only in the jth component,

$$\boldsymbol{v}(h_j) = (0 \quad \cdots \quad 0 \quad 1 \quad 0 \cdots \quad 0)^T. \quad (2)$$

We can uniquely decompose $\boldsymbol{v} \in \ker M(\mathcal{A})$ in the form of

$$\boldsymbol{v} = \boldsymbol{v}(h_{l_1}) + \boldsymbol{v}(h_{l_2}) + \cdots + \boldsymbol{v}(h_{l_m}). \quad (3)$$

Then, $h_{l_1} h_{l_2} \cdots h_{l_m}$ is a conserved quantity of H.

The CG also enables us to characterize the BA geometrically. For this purpose, we adapt the notion of the simplex: A d-simplex is a d-dimensional polyhedron having the minimal number of vertices, namely, $d + 1$ vertices. For instance, a 0-simplex is a vertex, a 1-simplex is a line, a 2-simplex is a triangle, a 3-simplex is a tetrahedron, and so on. In particular, we consider a special set of simplices, which we call point-connected simplices: Let us consider a set of simplices $S = \{s_1, \dots, s_m\}$ and let V be a set consisting of all vertices of $s_\alpha \in S$ ($\alpha = 1, \dots, m$). Then, we call S point connected if S is connected and any pair of $s_\alpha, s_\beta \in S$ ($\alpha \neq \beta$) having a nonempty intersection shares only a single vertex $v \in V$ (namely, $s_\alpha \cap s_\beta = \{v\}$). Furthermore, we call S single-point connected if any vertex $v \in V$ is shared by at most two different s_α's. Adding all faces of $s_\alpha \in S$ ($\alpha = 1, \dots, m$) into S, we obtain a simplicial complex $K(S)$, which we dub a single-point-connected simplicial complex (SPSC). See Fig. 1. Now we describe Theorem 2.

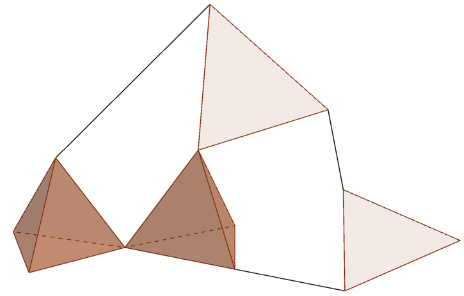

FIG. 1. A single-point-connected simplicial complex. Two 3-simplices (dark brown tetrahedrons), two 2-simplices (light brown triangles), and three 1-simplices (black lines) are connected only by vertices.

Theorem 2. Let $\mathcal{A}$ be the BA of a Hamiltonian $H = \sum_{j=1}^{n} \lambda_j h_j$ and $\mathcal{G}(\mathcal{A})$ be the corresponding CG of $\mathcal{A}$. If $\mathcal{G}(\mathcal{A})$ coincides with a SPSC $K(S)$ with $S = \{s_1, \dots, s_m\}$, then H is written by a bilinear form of m Majorana operators. In particular, h_j is recast into

$$h_j = -i\epsilon_{\alpha\beta}\varphi_\alpha\varphi_\beta, \quad \epsilon_{\alpha\beta} = \pm 1, \quad (4)$$

where φ_α are Majorana operators with the Hermiticity $\varphi_\alpha^\dagger = \varphi_\alpha$ and the anticommutation relation $\{\varphi_\alpha, \varphi_\beta\} = 2\delta_{\alpha,\beta}$.

Remarks are in order. (i) Without loss of generality, we can assume that any vertex v of $s_\alpha \in S$ is shared by another $s_\beta \in S$ ($\beta \neq \alpha$): If not, we can add v itself into S as a 0-simplex to meet the assumption. (ii) Under this assumption, the Majorana operator φ_α in Theorem 2 can be assigned to the simplex $s_\alpha \in S$. Then, φ_α and φ_β in Eq. (4) are given by those on the simplices that share the vertex with h_j. (iii) The sign factors $\epsilon_{\alpha\beta}$ in Eq. (4) are determined as follows. First, we use a sign ambiguity in Majorana operators: We can multiply φ_α by -1 without changing the (anti-)commutation relations between them. Using this gauge transformation, we can change the $m - 1$ relative signs between φ_α, which enables us to erase $m - 1$ $\epsilon_{\alpha\beta}$'s. There still, however, remain $n - m + 1$ $\epsilon_{\alpha\beta}$'s. The following theorem, Theorem 3, tells us that these remaining sign factors are determined by conserved quantities.

Theorem 3. Let $\mathcal{A}$ be the BA obeying the same assumption as in Theorem 2. Then, $K(S)$ has independent $n - m + 1$ noncontractible loops as a simplicial complex on $\mathbb{F}_2$. Correspondingly, there exist $n - m + 1$ conserved quantities that determine the remaining $n - m + 1$ sign factors.

It should be noted here that for each noncontractible loop, there remains a sign factor that cannot be removed by the gauge transformation. To count the number of independent noncontractible loops in $K(S)$, we calculate the homology group $H_q(K(S))$ of $K(S)$. As we shall show in Sec. V, a

TABLE I. Relations between the original model, the commutativity graph (CG), the single-point-connected simplicial complex (SPSC), and the free-fermion representation.

Original model		CG $M(\mathcal{A})$		SPSC $K(S)$		Free-fermion representation
h_j	$\Leftrightarrow$	vertex		$v \in s_\alpha \cap s_\beta$	$\Leftrightarrow$	$-i\epsilon_{\alpha\beta}\varphi_\alpha\varphi_\beta$
$\{h_i, h_j\} = 0$	$\Leftrightarrow$	line				
		clique	$\Leftrightarrow$	$s_\alpha \in K(S)$	$\Leftrightarrow$	Majorana operator φ_α
$[h, H] = 0$	$\Leftrightarrow$	$\ker M(\mathcal{A})$	$\supset$	$H_1(K(S))$	$\Leftrightarrow$	flux ϵ

straightforward calculation shows that $H_{q\geqslant 2}(K(S)) = 0$ and $\dim H_1(K(S)) = n - m + 1$ when $K(S)$ is a SPSC. The latter result implies that $K(S)$ has $n - m + 1$ independent noncontractible loops. We also find that each loop gives a conserved quantity: If we take noncontractible loops that are as small as possible, then the product of all h_j's on each loop gives a conserved quantity. Furthermore, we find that the conserved quantity reduces to the sign factor on the loop by rewriting it in terms of Majorana fermions in Eq. (4).

Theorems 2 and 3 imply that H is solvable as a free Majorana system: We can obtain the full spectrum of H just by diagonalizing the free Majorana Hamiltonian.

We summarize the relation between the original spin model, the CG, the SPSC, and the free-fermion representation in Table I.

III. APPLICATIONS TO KNOWN SOLVABLE MODELS

In this section, we apply our theory to known solvable models, which confirms the validity of our criterion. There are also a lot of solvable lattice models by our method. For example, we have checked our method in models in Refs. [17,19–21,30–34].

A. Transverse-field Ising model and related models

First, we examine a class of spin models obeying the following BA with $n = 2N$:

$$h_j^2 = 1, \quad \{h_j, h_{j+1}\} = 0, \qquad [h_j, h_k] = 0 \quad (j \neq k \pm 1). \tag{5}$$

In the periodic boundary condition $h_{2N+1} = h_1$, the CG of this algebra is a circle in Fig. 2. The corresponding adjacency matrix is given by

$$M(\mathcal{A}) = \begin{pmatrix} 0 & 1 & & & & 1 \\ 1 & 0 & 1 & & 0 & \\ & 1 & \ddots & \ddots & & \\ & & \ddots & \ddots & 1 & \\ & 0 & & 1 & 0 & 1 \\ 1 & & & & 1 & 0 \end{pmatrix}. \tag{6}$$

For $N \geqslant 2$, the kernel space of $M(\mathcal{A})$ has the dimension 2, which is spanned by $(1, 0, 1, 0, \dots)^T$ and $(0, 1, 0, 1, \dots)^T$. Therefore, from Theorem 1, we have two conserved quantities;

$$c_1 = h_1 h_3 \cdots h_{2N-1}, \quad c_2 = h_2 h_4 \cdots h_{2N}. \tag{7}$$

Indeed, we can easily check that c_1 and c_2 commute with any h_j. We also find that the CG in Fig. 2 is a SPSC. Applying Theorem 2, we can rewrite h_j in the form of

$$h_j = -i\epsilon_j \varphi_{j-1}\varphi_j, \tag{8}$$

where φ_j is a Majorana operator and $\epsilon_j = \pm 1$. Then, almost all ϵ_j's can be erased by redefining φ_j as $\varphi_j \to \epsilon_j^{-1}\varphi_j$ ($j = 1, \dots, 2N-1$), and after this, we obtain

$$\begin{aligned} h_j &= -i\varphi_{j-1}\varphi_j \quad (j = 1, \dots, 2N-1), \\ h_{2N} &= -i\epsilon\varphi_{2N-1}\varphi_{2N}. \end{aligned} \tag{9}$$

The remaining ϵ in Eq. (9) is determined by $c_1 c_2$,

$$\epsilon = -c_1 c_2. \tag{10}$$

The sign factor $(-1)^N\epsilon$ corresponds to the π flux through the hole of the CG in Fig. 2 [38].

In the open boundary condition, the CG is a line, and $M(\mathcal{A})$ becomes

$$M(\mathcal{A}) = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 1 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & 1 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{11}$$

of which the kernel is dimension 0 for $n = 2N$. Now no conserved quantity is obtained, and thus $\epsilon = 1$. In particular, in this case, our method naturally reproduces the Jordan-Wigner transformation [16]. We can transform $M(\mathcal{A})$ into the

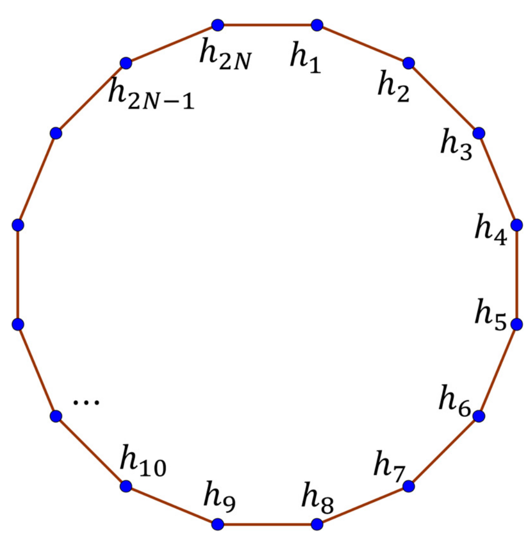

FIG. 2. The CG of Eq. (5). The periodic boundary condition $h_{2N+1} = h_1$ is imposed.

following form:

$$Q^T M(\mathcal{A}) Q = \begin{pmatrix} 0 & 1 & 1 & 1 & \cdots \\ 1 & 0 & 1 & 1 & \cdots \\ 1 & 1 & 0 & 1 & \cdots \\ 1 & 1 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{12}$$

where Q is given by

$$Q = \prod_p P^{[p,p+1]} = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots \\ 0 & 1 & 1 & 1 & \cdots \\ 0 & 0 & 1 & 1 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{13}$$

where $P^{[p,q]}$ is an elementary matrix with the (i, j) component $P^{[p,q]}_{ij} = \delta_{ij} + \delta_{ip}\delta_{jq}$. As we shall show in Sec. V, $P^{[p,q]}$ induces a map

$$\{\dots h_p, \dots, h_q, \dots\} \mapsto \{\dots h_p, \dots, h_p h_q, \dots\}, \tag{14}$$

and thus Q gives a new basis

$$e_j = h_1 h_2 \cdots h_j. \tag{15}$$

The commutation relations in $Q^T M(\mathcal{A})Q$ are $e_i e_j = -e_j e_i$ for all $i \neq j$, those of the Clifford algebra. Introducing the initial operator h_0 that obeys $h_0^2 = -1$, $\{h_0, h_1\} = 0$, and $[h_0, h_j] = 0$ $(j \neq 1)$ and defining φ_j as

$$\varphi_j = i^{j-1} h_0 h_1 h_2 \cdots h_j, \tag{16}$$

we reproduce Eq. (9) with $\epsilon = 1$. Equation (16) is an algebraic generalization of the Jordan-Wigner transformation [16]. Actually, in the case of the transverse Ising chain below, by taking the initial operator as $h_0 = i\sigma_1^x$, Eq (16) reproduces the original Jordan-Wigner transformation.

For simplicity, we only consider the periodic boundary condition below.

1. *Transverse-field Ising chain*

The Hamiltonian of the transverse-field Ising chain is given by

$$H = -J \sum_{j=1}^{N} \sigma_j^x \sigma_{j+1}^x - h \sum_{j=1}^{N} \sigma_j^z, \tag{17}$$

where J is the exchange constant and h is a transverse magnetic field. From Eq. (17), the generator of the BA reads

$$h_{2j-1} = \sigma_j^z, \quad h_{2j} = \sigma_j^x \sigma_{j+1}^x, \tag{18}$$

which satisfies Eq. (5). The conserved quantities in Eq. (7) are given by

$$c_1 = \prod_{j=1}^{N} \sigma_j^z, \quad c_2 = 1, \tag{19}$$

and thus the sign factor in Eq. (10) is

$$\epsilon = -\prod_{j=1}^{N} \sigma_j^z. \tag{20}$$

From Eq. (9), the Hamiltonian is recast into

$$H = h \sum_{j=1}^{N} i\varphi_{2j-2}\varphi_{2j-1} + J \sum_{j=1}^{N-1} i\varphi_{2j-1}\varphi_{2j} + Ji\epsilon\varphi_{2N-1}\varphi_{2N}, \tag{21}$$

which reproduces the result in Ref. [16].

2. *Orbital compass chain*

Another model obeying Eq. (5) is the orbital compass chain [30,39],

$$H = -J_x \sum_{j=1}^{N} \sigma_{2j-1}^x \sigma_{2j}^x - J_y \sum_{j=1}^{N} \sigma_{2j}^y \sigma_{2j+1}^y, \tag{22}$$

where Eq. (5) is obtained by the following identification:

$$h_{2j-1} = \sigma_{2j-1}^x \sigma_{2j}^x, \quad h_{2j} = \sigma_{2j}^y \sigma_{2j+1}^y. \tag{23}$$

The conserved quantities c_1 and c_2 in Eq. (7) become

$$c_1 = \prod_{j=1}^{2N} \sigma_j^x, \quad c_2 = \prod_{j=1}^{2N} \sigma_j^y, \tag{24}$$

and thus ϵ in Eq. (10) is

$$\epsilon = (-1)^{N+1} \prod_{j=1}^{2N} \sigma_j^z. \tag{25}$$

In terms of Majorana operators, H in Eq. (22) is given by

$$H = J_x \sum_{j=1}^{N} i\varphi_{2j-2}\varphi_{2j-1} + J_y \sum_{j=1}^{N-1} i\varphi_{2j-1}\varphi_{2j} + J_y i\epsilon\varphi_{2N-1}\varphi_{2N}, \tag{26}$$

which coincides with Eq. (21) if we identify J_x and J_y with h and J. Therefore there is a one-to-one correspondence between the spectrum of the orbital compass chain and that of the transverse-field Ising chain.

On the other hand, there exist additional degeneracies in the orbital compass chain. First, c_2 in Eq. (24) can be ± 1, which gives twofold degeneracy of each state. Moreover, we also have additional 2^N-fold degeneracy. This originates from the mismatch between the original spin degrees of freedom and the transformed Majorana degrees of freedom: The original spin space is 2^{2N} dimensional, while the space of Majorana fermions is 2^N dimensional. Correspondingly, there are additional conserved quantities d_j $(j = 1, \dots, 2N)$ which cannot be written by h_j,

$$d_{2j-1} = \sigma_{2j-1}^y \sigma_{2j}^y, \quad d_{2j} = \sigma_{2j}^x \sigma_{2j+1}^x. \tag{27}$$

They satisfy the same BA as h_j;

$$d_j^\dagger = d_j, \quad d_j^2 = 1, \quad \{d_j, d_{j+1}\} = 0, \quad [d_j, d_k] = 0 \quad (j \neq k \pm 1), \tag{28}$$

and thus these operators are equivalent to $2N$ Majorana fermions. As a result, they generate additional 2^N-fold degeneracy.

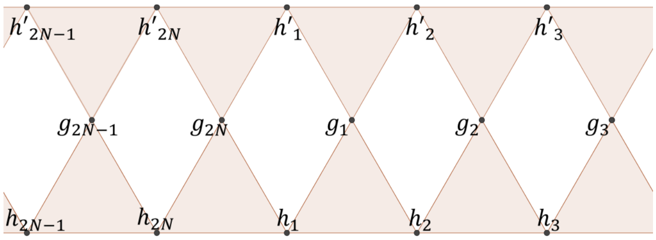

FIG. 3. The CG of Eq. (29).

B. *XY* model and related models

Let h_j, h'_j, and g_j ($j=1,\dots,2N$) be Hermitian operators obeying

$$\begin{aligned}&h_j^2=(h'_j)^2=g_j^2=1,\quad \{h_j,h_{j+1}\}=\{h'_j,h'_{j+1}\}=0,\\&\{h_j,g_j\}=\{h'_j,g_j\}=\{h_{j+1},g_j\}=\{h'_{j+1},g_j\}=0,\end{aligned}\tag{29}$$

where the other relations are commutative and the periodic boundary condition is assumed,

$$h_{i+2N}=h_i,\quad h'_{i+2N}=h'_i,\quad g_{i+2N}=g_i.\tag{30}$$

This algebra defines a class of models with the CG in Fig. 3. The dimension of the kernel space of the adjacency matrix is $2N+2$, and we have $2N+2$ conserved quantities:

$$\begin{aligned}&c_h=h_1\cdots h_{2N},\quad c_{h'}=h'_1\cdots h'_{2N},\quad c_g=g_1\cdots g_{2N},\\&c_j=g_{j-1}h'_jg_jh_j\quad (j=1,\dots,2N),\end{aligned}\tag{31}$$

which satisfy

$$c_hc_{h'}c_1\cdots c_{2N}=1.\tag{32}$$

Since the CG in Fig. 3 is a SPSC, the operators in Eq. (29) can be written by Majorana operators. Using the sign ambiguity (gauge degrees of freedom) of Majorana operators, we have

$$\begin{aligned}h_j&=-i\varphi_{j-1}\varphi_j,\quad h'_j=-i\varphi'_{j-1}\varphi'_j,\\g_j&=-i\epsilon_j\varphi_j\varphi'_j\quad (j=1,\dots,2N-1),\\h_{2N}&=-i\epsilon\varphi_{2N-1}\varphi_{2N},\quad h'_{2N}=-i\epsilon'\varphi'_{2N-1}\varphi'_{2N},\\g_{2N}&=-i\varphi_{2N}\varphi'_{2N},\end{aligned}\tag{33}$$

where φ_i and φ'_i are Majorana operators. The sign factors ϵ_j, ϵ, and ϵ' are determined by the conserved quantities in Eq. (31),

$$\epsilon_j=\prod_{k=1}^{j}c_k,\quad \epsilon=(-1)^Nc_h,\quad \epsilon'=(-1)^Nc_{h'}.\tag{34}$$

1. *XY* model

As a prime example of models with the CG in Fig. 3, we consider the *XY* model,

$$H=-J\sum_{i=1}^{2N}\{(1+\gamma)\sigma_i^x\sigma_{i+1}^x+(1-\gamma)\sigma_i^y\sigma_{i+1}^y\}-h\sum_{i=1}^{2N}\sigma_i^z,\tag{35}$$

where J is the exchange constant, γ is the asymmetric parameter, and h is a magnetic field. Actually, with the identification

$$\begin{aligned}&h_{2j-1}=\sigma_{2j-1}^x\sigma_{2j}^x,\quad h_{2j}=\sigma_{2j}^y\sigma_{2j+1}^y,\\&h'_{2j-1}=\sigma_{2j-1}^y\sigma_{2j}^y,\quad h'_{2j}=\sigma_{2j}^x\sigma_{2j+1}^x,\\&g_j=\sigma_{j+1}^z,\end{aligned}\tag{36}$$

we reproduce the BA in Eq. (29). In this model, the conserved quantities obey

$$c_1=\cdots=c_{2N}=1,\quad c_h=c'_h=-c_g=-\prod_{j=1}^{2N}\sigma_j^z,\tag{37}$$

and thus we have

$$\epsilon_j=1,\quad \epsilon=\epsilon'=(-1)^{N+1}\prod_{j=1}^{2N}\sigma_j^z.\tag{38}$$

Therefore Eq. (33) leads to

$$\begin{aligned}H=&\,iJ\sum_{j=1}^{N}\{(1+\gamma)(\varphi_{2j-2}\varphi_{2j-1}+\varphi'_{2j-1}\varphi'_{2j})\}\\&+iJ\sum_{j=1}^{N}\{(1-\gamma)(\varphi_{2j-1}\varphi_{2j}+\varphi'_{2j-2}\varphi'_{2j-1})\}\\&+ih\sum_{j=1}^{2N}\varphi_j\varphi'_j\\&-iJ(1-\epsilon)\{(1+\gamma)\varphi_{2N-1}\varphi_{2N}+(1-\gamma)\varphi'_{2N-1}\varphi'_{2N}\}.\end{aligned}\tag{39}$$

Equation (39) reproduces the known fermion representation of the *XY* model: Introducing the fermion operators a_j as

$$\begin{aligned}\varphi_{2j-1}&=u_{2j-1}(a_{2j-1}+a_{2j-1}^\dagger),\\\varphi'_{2j-1}&=iu_{2j-1}(a_{2j-1}-a_{2j-1}^\dagger),\\\varphi_{2j}&=-iu_{2j}(a_{2j}-a_{2j}^\dagger),\\\varphi'_{2j}&=u_{2j}(a_{2j}+a_{2j}^\dagger),\end{aligned}\tag{40}$$

with $u_j=(-1)^{j(j-1)/2}$, we obtain

$$\begin{aligned}H=&-2J\sum_{j=1}^{2N-1}[(a_j^\dagger a_{j+1}+a_{j+1}^\dagger a_j)+\gamma(a_j^\dagger a_{j+1}^\dagger+a_{j+1}a_j)]\\&-2h\sum_{j=1}^{2N}\left(a_j^\dagger a_j-\frac{1}{2}\right)\\&+2Jc_g[(a_j^\dagger a_{j+1}+a_{j+1}^\dagger a_j)+\gamma(a_j^\dagger a_{j+1}^\dagger+a_{j+1}a_j)],\end{aligned}\tag{41}$$

which is the same fermion representation as in Ref. [12].

We note that Ref. [35] discussed a bond-algebraic map in the *XY* model to show the self-duality, but our bond algebra in Eq. (29) is different from that in Ref. [35] and contains more generators.

2. *Ladder model*

The second example is the ladder model [40],

$$H = -J_{\mathrm{t}}\sum_{j=1}^{N}\left(\sigma_{2j-1}^x\sigma_{2j}^x+\sigma_{2j}^y\sigma_{2j+1}^y\right) - J_{\mathrm{b}}\sum_{j=1}^{N}\left(\tau_{2j-1}^x\tau_{2j}^x+\tau_{2j}^y\tau_{2j+1}^y\right) - J_{\perp}\sum_{j=1}^{2N}\left(\sigma_j^z\tau_j^z\right), \tag{42}$$

where J_{t} (J_{b}) is the intraexchange constant between top (bottom) spin chains and $J_\perp$ is the interexchange constant between top and bottom chains. This model gives

$$\begin{aligned} h_{2j-1} &= \sigma_{2j-1}^x\sigma_{2j}^x, \quad h_{2j} = \sigma_{2j}^y\sigma_{2j+1}^y, \\ h'_{2j-1} &= \tau_{2j-1}^x\tau_{2j}^x, \quad h'_{2j} = \tau_{2j}^y\tau_{2j+1}^y, \\ g_j &= \sigma_j^z\tau_j^z, \end{aligned} \tag{43}$$

which satisfy Eq. (29). In this model, we have

$$\begin{aligned} c_h &= -\prod_{j=1}^{2N}\sigma_j^z, \quad c_{h'} = -\prod_{j=1}^{2N}\tau_j^z, \quad c_g = c_h c_{h'}, \\ c_{2j-1} &= -\sigma_{2j-1}^y\sigma_{2j}^y\tau_{2j-1}^y\tau_{2j}^y, \\ c_{2j} &= -\sigma_{2j}^x\sigma_{2j+1}^x\tau_{2j}^x\tau_{2j+1}^x, \end{aligned} \tag{44}$$

which lead to

$$\begin{aligned} \epsilon_{2j-1} &= -\sigma_1^y\tau_1^y\left(\prod_{k=2}^{2j-1}\sigma_k^z\tau_k^z\right)\sigma_{2j}^y\tau_{2j}^y, \\ \epsilon_{2j} &= -\sigma_1^y\tau_1^y\left(\prod_{k=2}^{2j}\sigma_k^z\tau_k^z\right)\sigma_{2j+1}^x\tau_{2j+1}^x, \\ \epsilon' &= (-1)^{N+1}\prod_{j=1}^{2N}\sigma_j^z, \quad \epsilon = (-1)^{N+1}\prod_{j=1}^{2N}\tau_j^z, \end{aligned} \tag{45}$$

where $\prod_{k=2}^{1}\sigma_k^z\tau_k^z \equiv 1$. The Hamiltonian is equivalent to

$$\begin{aligned} H &= iJ_{\mathrm{t}}\sum_{j=1}^{2N-1}\varphi_{j-1}\varphi_j + iJ_{\mathrm{t}}\epsilon\varphi_{2N-1}\varphi_{2N} \\ &\quad + iJ_{\mathrm{b}}\sum_{j=1}^{2N-1}\varphi'_{j-1}\varphi'_j + iJ_{\mathrm{b}}\epsilon'\varphi'_{2N-1}\varphi'_{2N} \\ &\quad + iJ_\perp\sum_{j=1}^{2N-1}\epsilon_j\varphi_j\varphi'_j + iJ_\perp\varphi_{2N}\varphi'_{2N}. \end{aligned} \tag{46}$$

3. *Double spin–Majorana coupled model*

The third example is the double spin–Majorana coupled model,

$$H = -ig\sum_{j=1}^{2N}\left(\gamma_j\sigma_j^x\gamma_{j+1}+\gamma'_j\tau_j^x\gamma'_{j+1}\right) - J\sum_{j=1}^{2N}\sigma_j^z\sigma_{j+1}^z\tau_j^z\tau_{j+1}^z, \tag{47}$$

where g and J are real parameters and γ_j's are Majorana operators. The BA of this model reads

$$\begin{aligned} h_j &= i\gamma_j\sigma_j^x\gamma_{j+1}, \quad h'_j = i\gamma'_j\tau_j^x\gamma'_{j+1}, \\ g_j &= \sigma_j^z\sigma_{j+1}^z\tau_j^z\tau_{j+1}^z, \end{aligned} \tag{48}$$

which reproduces Eq. (29), and we obtain

$$\begin{aligned} c_h &= (-1)^N\prod_{j=1}^{2N}\sigma_j^x, \quad c_{h'} = (-1)^N\prod_{j=1}^{2N}\tau_j^x, \quad c_g = 1, \\ c_j &= -\sigma_{j-1}^z\tau_{j-1}^z\sigma_j^x\tau_j^x\sigma_{j+1}^z\tau_{j+1}^z\gamma_j\gamma'_j\gamma_{j+1}\gamma'_{j+1}. \end{aligned} \tag{49}$$

Therefore

$$\begin{aligned} \epsilon_1 &= -\sigma_{2N}^z\tau_{2N}^z\sigma_1^x\tau_1^x\sigma_2^z\tau_2^z\gamma_1\gamma'_1\gamma_2\gamma'_2, \\ \epsilon_j &= -\sigma_{2N}^z\tau_{2N}^z\sigma_1^y\tau_1^y\left(\prod_{k=2}^{j-1}\sigma_k^z\tau_k^z\right) \\ &\quad \times\sigma_j^y\tau_j^y\sigma_{j+1}^z\tau_{j+1}^z\gamma_1\gamma'_1\gamma_{j+1}\gamma'_{j+1} \quad (j = 2,\dots,2N-1), \\ \epsilon &= \prod_{j=1}^{2N}\sigma_j^z, \quad \epsilon' = \prod_{j=1}^{2N}\tau_j^z, \end{aligned} \tag{50}$$

where $\prod_{k=2}^{1}\sigma_k^z\tau_k^z \equiv 1$. The Hamiltonian is recast into

$$\begin{aligned} H &= ig\sum_{j=1}^{2N-1}(\varphi_{j-1}\varphi_j + \varphi'_{j-1}\varphi'_j) \\ &\quad + ig(\epsilon\varphi_{2N-1}\varphi_{2N} + \epsilon'\varphi'_{2N-1}\varphi'_{2N}) \\ &\quad + iJ\sum_{j=1}^{2N-1}\epsilon_j\varphi_j\varphi'_j + iJ\varphi_{2N}\varphi'_{2N}. \end{aligned} \tag{51}$$

In a manner similar to the orbital compass chain in Sec. III A, this model hosts additional degeneracies originating from the mismatch between the original degrees of freedom and the transformed Majorana ones: It is found that the operators d_j and d'_j ($j = 1,\dots,2N$) commute with h_j, h'_j, g_j,

$$d_j = \sigma_{j-1}^z\gamma_j\sigma_j^z, \quad d'_j = \tau_{j-1}^z\gamma'_j\tau_j^z, \tag{52}$$

which satisfies

$$\{d_j, d_k\} = \{d'_j, d'_k\} = 2\delta_{j,k}, \quad \{d_j, d'_k\} = 0. \tag{53}$$

Thus each state of this model has 2^{2N}-fold degeneracy.

C. Kitaev honeycomb lattice model

The Kitaev honeycomb lattice is described by the following Hamiltonian with the nearest-neighbor spin couplings:

$$H = -J_x\sum_{x\ \mathrm{links}}\sigma_j^x\sigma_k^x - J_y\sum_{y\ \mathrm{links}}\sigma_j^y\sigma_k^y - J_z\sum_{z\ \mathrm{links}}\sigma_j^z\sigma_k^z, \tag{54}$$

where the orientation of the x, y, and z links is indicated in Fig. 4. Each term of Eq. (54) anticommutes or commutes with each of the others, and thus it defines the BA. The CG of this model is the kagome lattice in Fig. 5. The kagome lattice is

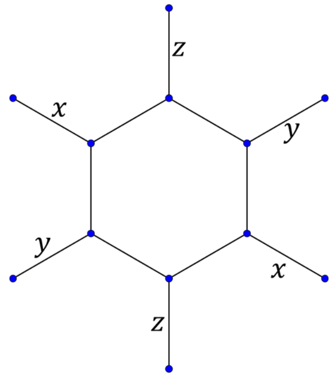

FIG. 4. x, y, and z links in honeycomb lattice.

dual to the original honeycomb lattice, and each vertex in the kagome lattice corresponds to a link in the honeycomb lattice. We assign an operator

$$h_{j,k} = \sigma_j^{\mu(j,k)}\sigma_k^{\mu(j,k)} \tag{55}$$

in the BA to each vertex of the kagome lattice, where $\mu(j,k) = x, y, z$ is the spin orientation at the corresponding (j,k) link in the honeycomb lattice. The conserved quantities are

$$c_p = \prod_{(j,k)\in\partial p} h_{j,k}, \quad c_z = \prod_{(j,k):z\text{link}} h_{j,k}, \tag{56}$$

where p is a hexagon in Fig. 5.

Regarding triangles in Fig. 5 as 2-simplices, the CG can be identified with a SPSC. Therefore we can apply Theorems 2 and 3 to the Kitaev honeycomb lattice model. The operator $h_{j,k}$ is converted into a Majorana-bilinear form

$$h_{j,k} = -i\epsilon_{jk}\varphi_j\varphi_k, \tag{57}$$

so the Hamiltonian is equivalent to

$$H = \sum_{\langle j,k\rangle} iJ_{\mu(j,k)}\epsilon_{jk}\varphi_j\varphi_k, \tag{58}$$

where ϵ_{jk}'s are determined by the conserved quantities in Eq. (56). This result reproduces that in Ref. [6] and is consistent with the bond-algebraic mapping in Ref. [30], although our derivation is much simpler than these approaches.

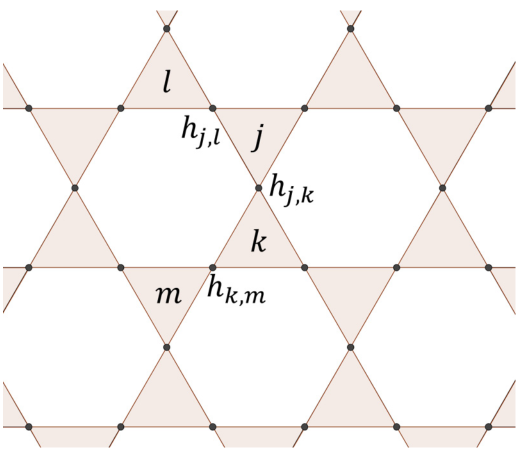

FIG. 5. The CG of the Kitaev honeycomb lattice model.

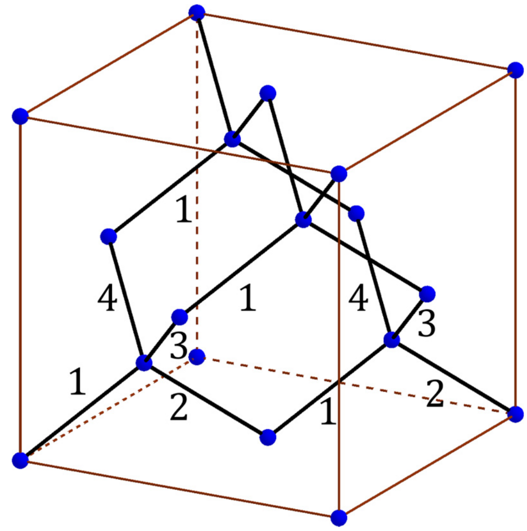

FIG. 6. Diamond lattice. The number at the link indicates the orientation μ of the gamma matrix in the diamond lattice model.

D. Diamond lattice model

The diamond lattice is a three-dimensional analog of the honeycomb lattice [41,42]. We can generalize the Kitaev honeycomb lattice model in three dimensions. The Hamiltonian is given by

$$H = -\sum_{\langle j,k\rangle} J_{jk}\left[\alpha_j^{\mu(j,k)}\alpha_k^{\mu(j,k)} + \zeta_j^{\mu(j,k)}\zeta_k^{\mu(j,k)}\right], \tag{59}$$

where α_j^μ and ζ_j^μ ($\mu = 1,2,3,4$) are two sets of Dirac matrices,

$$\begin{aligned} \alpha_j^a &= \sigma_j^a \otimes \tau_j^x, \quad \alpha_j^4 = \sigma_j^0 \otimes \tau_j^z, \\ \zeta_j^a &= -\sigma_j^a \otimes \tau_j^z, \quad \zeta_j^4 = \sigma_j^0 \otimes \tau_j^x, \end{aligned} \tag{60}$$

where $a = 1,2,3$, j is the site index and $\mu(j,k) = 1,2,3,4$ indicates the orientation of the gamma matrix at the (j,k) link, as illustrated in Fig. 6. We assign the operators $h_{j,k}$ and $h'_{j,k}$ as

$$h_{j,k} = \alpha_j^{\mu(j,k)}\alpha_k^{\mu(j,k)}, \quad h'_{j,k} = \zeta_j^{\mu(j,k)}\zeta_k^{\mu(j,k)}, \tag{61}$$

which satisfy

$$[h_{j,k}, h'_{l,m}] = 0. \tag{62}$$

The CGs of $h_{j,k}$ and $h'_{j,k}$ are two identical pyrochlore lattices in Fig. 7. By regarding tetrahedrons as 3-simplices, the pyrochlore lattice is identified with a SPSC. From straightforward calculation, we also find that the conserved quantities in the two CGs are the same. Therefore we can transform $h_{j,k}$'s and $h'_{j,k}$'s into Majorana-bilinear forms,

$$h_{j,k} = -i\epsilon_{j,k}\varphi_j\varphi_k, \quad h'_{j,k} = -i\epsilon_{j,k}\varphi'_j\varphi'_k. \tag{63}$$

Consequently, the Hamiltonian is converted into

$$H = i\sum_{\langle j,k\rangle} J_{jk}\epsilon_{j,k}(\varphi_j\varphi_k + \varphi'_j\varphi'_k), \tag{64}$$

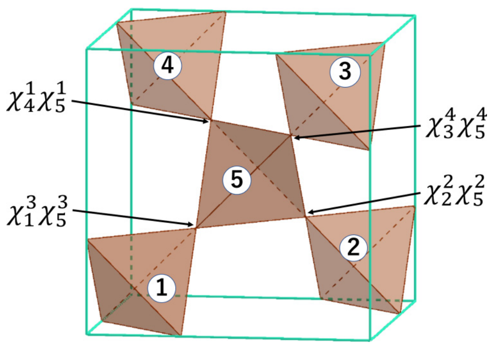

FIG. 7. The CG of the diamond lattice model with $\chi = \alpha, \zeta$.

which reproduces that in Ref. [41].

IV. NEW SOLVABLE MODELS

So far, we have applied our method to known solvable models. Our approach also provides a powerful method to construct new solvable models in a variety of lattices. In this section, we present such new solvable models.

A. Trijunction model

We first consider the transverse-field Ising chains with the trijunction [43–47]. The Hamiltonian is given by

$$H = -\sum_{a=1}^{3}\left(J_a\sum_{j=1}^{N-1}\sigma_{a,j}^z\sigma_{a,j+1}^z + h_a\sum_{j=2}^{N}\sigma_{a,j}^x\right) - t_{12}\sigma_{1,1}^x\sigma_{2,1}^z - t_{23}\sigma_{2,1}^x\sigma_{3,1}^z - t_{31}\sigma_{3,1}^x\sigma_{1,1}^z, \quad (65)$$

where J_a and h_a are the exchange constant and a magnetic field of the ath chain and t_{ab} are the coupling between the ath and bth chains. The CG of this model is Fig. 8, where $h_{a,j}$ $(j = 1, \ldots, 2N-1)$ is defined by

$$h_{a,1} = \sigma_{a,1}^x\sigma_{a+1,1}^z, \quad h_{a,2l} = \sigma_{a,l}^z\sigma_{a,l+1}^z, \quad h_{a,2l+1} = \sigma_{a,l+1}^x. \quad (66)$$

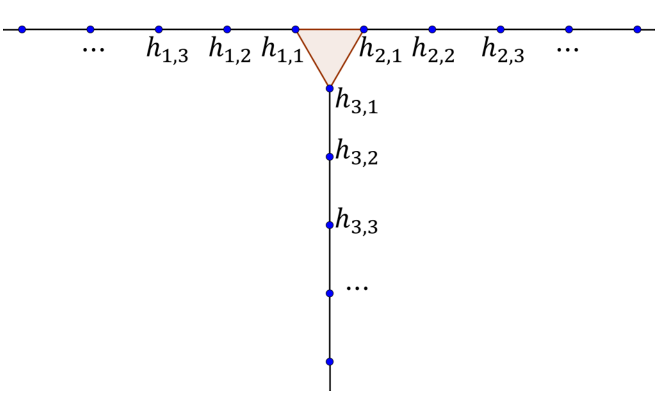

FIG. 8. The CG of the trijunction model.

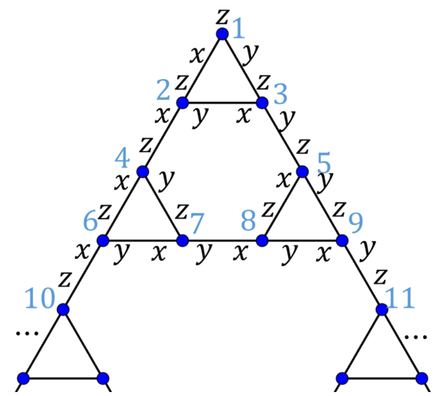

FIG. 9. Hanoi graph. x, y, and z on each site denote the spin orientation of the exchange interaction.

From the adjacency matrix of the CG, we find a conserved quantity

$$\begin{aligned} c &= -i\prod_{a=1}^{3}\prod_{j=1}^{N} h_{a,2j-1} \\ &= \left(\prod_{a=1}^{3}\sigma_{a,1}^y\right)\left(\prod_{a=1}^{3}\prod_{j=2}^{N}\sigma_{a,j}^x\right). \end{aligned} \quad (67)$$

The CG in Fig. 8 can be identified with a SPSC consisting of lines and a triangle. Therefore, applying Theorem 2 to this model, we have

$$\begin{aligned} h_{a,1} &= -i\varphi_{a,1}\varphi, \\ h_{a,j} &= -i\varphi_{a,j-1}\varphi_{a,j} \quad (j = 2, \ldots, N). \end{aligned} \quad (68)$$

By using this, the Hamiltonian is recast into the bilinear form of Majorana operators,

$$H = i\sum_{a=1}^{3}\left(J_a\sum_{j=1}^{N}\varphi_{a,2j-1}\varphi_{a,2j} + ih_a\sum_{j=1}^{N}\varphi_{a,2j}\varphi_{a,2j+1}\right) + (t_{12}\varphi_{1,1} + t_{23}\varphi_{2,1} + t_{31}\varphi_{3,1})\varphi. \quad (69)$$

This model hosts implicit conserved quantities that are not obtained by h_j,

$$c_a = \sigma_{a-1,1}^z\prod_{j=1}^{N}\sigma_{a,j}^x \quad (a = 1, 2, 3), \quad (70)$$

which satisfy

$$[c_a, h_{b,j}] = 0, \quad \{c_a, c_b\} = 2\delta_{a,b}, \quad ic_1c_2c_3 = c. \quad (71)$$

These operators induce additional twofold degeneracy.

By the same method, we can construct an n-junction model whose junction is an $(n-1)$-simplex. We can also design treelike models by junctions.

B. Hanoi graph model

We can construct solvable models in 2D and 3D fractal lattices. Let us consider the Hanoi graph in Fig. 9 and place a spin operator on each site of the Hanoi graph. Then, we

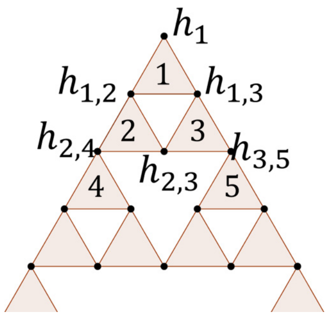

FIG. 10. Sierpiński gasket.

consider the Hamiltonian

$$\begin{aligned} H =& - J_1\sigma_1^z \\ & - J_{12}\sigma_1^x\sigma_2^z - J_{13}\sigma_1^y\sigma_3^z \\ & - J_{23}\sigma_2^y\sigma_3^x - J_{24}\sigma_2^x\sigma_4^z - J_{35}\sigma_3^y\sigma_5^z \\ & - \cdots , \end{aligned} \tag{72}$$

where σ_i^μ is the μth Pauli matrix at the ith site in Fig. 9 and J_{ij} is the exchange constant. The spin orientation of the exchange interaction is determined as illustrated in Fig. 9: In the case of the (1,2) link, for instance, we take σ^x and σ^z from site 1 and site 2, respectively.

The CG of this model is the Sierpiński gasket in Fig. 10, where the operators at vertices are given by

$$\begin{aligned} h_1 &= \sigma_1^z, \\ h_{1,2} &= \sigma_1^x\sigma_2^z, \quad h_{1,3} = \sigma_1^y\sigma_3^z, \\ h_{2,3} &= \sigma_2^y\sigma_3^x, \quad h_{2,4} = \sigma_2^x\sigma_4^z, \quad h_{3,5} = \sigma_3^y\sigma_5^z, \\ & \cdots . \end{aligned} \tag{73}$$

Since the Sierpiński gasket is a SPSC generated by 2-simplices, the Hamiltonian (72) can be transformed into a Majorana-bilinear form. Note that the Sierpiński gasket is dual to the Hanoi graph.

This model has 3D generalization. Instead of the Hanoi graph, we use the dual lattice of the Sierpiński tetrahedron in Fig. 11. Placing a spin-4 generator at each site, we can construct the Hamiltonian of which the CG is the Sierpiński tetrahedron. In the same way as the Hanoi graph, this model can be transformed into a Majorana-bilinear form.

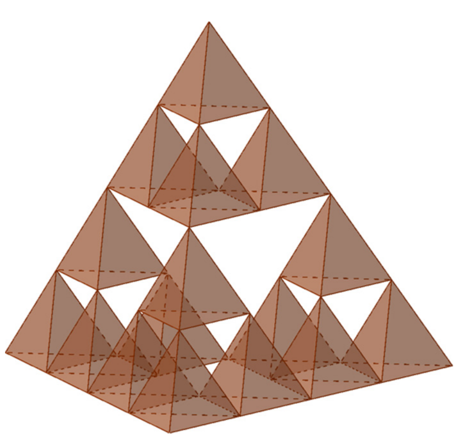

FIG. 11. Sierpiński tetrahedron.

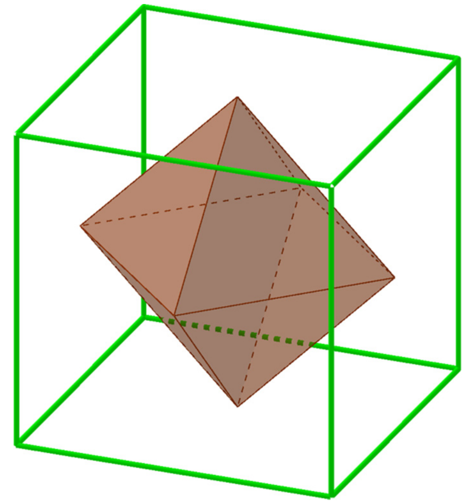

FIG. 12. Octahedron.

C. Octahedron model

The dimension of simplices in a SPSC can be higher than the space dimension. To illustrate this, we consider a spin model in the cubic lattice. We place an SO(6) spin [i.e., a spin-6 generator] on each site of the cubic lattice and consider the nearest-neighbor interaction

$$H = -\frac{1}{2}\sum_{\boldsymbol{j}}\sum_{\mu=1}^{3} J_\mu \gamma_{\boldsymbol{j}}^\mu \gamma_{\boldsymbol{j}+\boldsymbol{e}_\mu}^{\mu+3} - g\sum_{\boldsymbol{j}}\gamma_{\boldsymbol{j}}^7, \tag{74}$$

where J_μ is the exchange constant, $\gamma_{\boldsymbol{j}}^\mu$ is the SO(6) gamma matrix at the site $\boldsymbol{j}$, and $\boldsymbol{e}_\mu$ is the unit vector in the μth direction. We assign operators

$$h_{\boldsymbol{j}}^\mu = \gamma_{\boldsymbol{j}}^\mu\gamma_{\boldsymbol{j}+\boldsymbol{e}_\mu}^{\mu+3}, \quad h'_{\boldsymbol{j}} = \gamma_{\boldsymbol{j}}^7. \tag{75}$$

The conserved quantities are

$$c_{\boldsymbol{j}}^{\mu,\nu} = h_{\boldsymbol{j}}^\nu h_{\boldsymbol{j}+\boldsymbol{e}_\nu}^\mu h_{\boldsymbol{j}+\boldsymbol{e}_\mu}^\nu h_{\boldsymbol{j}}^\mu. \tag{76}$$

The CG of this model is vertex-sharing octahedra with central vertex, shown in Fig. 12. It is a SPSC since an octahedron with central vertex is a 6-simplex. Thus we can transform these operators into

$$h_{\boldsymbol{j}}^\mu = -i\epsilon_{\boldsymbol{j}}^\mu\varphi_{\boldsymbol{j}}\varphi_{\boldsymbol{j}+\boldsymbol{e}_\mu}, \quad h'_{\boldsymbol{j}} = -i\varphi_{\boldsymbol{j}}\varphi'_{\boldsymbol{j}} \tag{77}$$

and conserved quantities into

$$c_{\boldsymbol{j}}^{\mu,\nu} = \epsilon_{\boldsymbol{j}}^\nu\epsilon_{\boldsymbol{j}+\boldsymbol{e}_\nu}^\mu\epsilon_{\boldsymbol{j}+\boldsymbol{e}_\mu}^\nu\epsilon_{\boldsymbol{j}}^\mu. \tag{78}$$

Therefore the Hamiltonian is recast into

$$H = \frac{i}{2}\sum_{\boldsymbol{j}}\sum_{\mu=1}^{3} J_\mu\epsilon_{\boldsymbol{j}}^\mu\varphi_{\boldsymbol{j}}\varphi_{\boldsymbol{j}+\boldsymbol{e}_\mu} + ig\sum_{\boldsymbol{j}}\varphi_{\boldsymbol{j}}\varphi'_{\boldsymbol{j}}. \tag{79}$$

In the following discussion, we take $g = 0$ for simplicity. In this case, $i\varphi'_{\boldsymbol{j}}\varphi'_{\boldsymbol{j}+\boldsymbol{e}_\mu}$ conserves, which induces additional $2^{N/2}$-fold degeneracy with the number of vertices N. From Lieb's theorem [48], the ground state is realized when

$$c_{\boldsymbol{j}}^{\mu,\nu}=-1. \tag{80}$$

To accomplish this condition, we set

$$\epsilon_{\boldsymbol{j}}^{\mu}=\begin{cases}(-1)^{j_2+j_3} & (\mu=1)\\ (-1)^{j_3} & (\mu=2)\\ 1 & (\mu=3).\end{cases} \tag{81}$$

In this case, the Hamiltonian becomes

$$\begin{aligned}H=\frac{i}{2}\sum_{\boldsymbol{j}}[&J_1(-1)^{j_2+j_3}\varphi_{\boldsymbol{j}}\varphi_{\boldsymbol{j}+\boldsymbol{e}_1}\\ &+J_2(-1)^{j_3}\varphi_{\boldsymbol{j}}\varphi_{\boldsymbol{j}+\boldsymbol{e}_2}\\ &+J_3\varphi_{\boldsymbol{j}}\varphi_{\boldsymbol{j}+\boldsymbol{e}_3}].\end{aligned} \tag{82}$$

To this equation, we perform the Fourier transformation,

$$\varphi_{\boldsymbol{j}}=\sqrt{\frac{2}{(2\pi)^3}}\int_{-\pi}^{\pi}d^3pe^{-i\boldsymbol{p}\cdot\boldsymbol{j}}a_{\boldsymbol{p}}, \tag{83}$$

where

$$a_{\boldsymbol{p}}^{\dagger}=a_{-\boldsymbol{p}},\quad \{a_{\boldsymbol{p}},a_{\boldsymbol{q}}\}=\delta(\boldsymbol{p}+\boldsymbol{q}). \tag{84}$$

Then we have

$$H=\int_0^{\pi}d^3p\boldsymbol{A}_{\boldsymbol{p}}^{\dagger}\mathcal{H}_{\boldsymbol{p}}\boldsymbol{A}_{\boldsymbol{p}}, \tag{85}$$

where

$$\boldsymbol{A}_{\boldsymbol{p}}=\begin{pmatrix}a_{\boldsymbol{p}}\\ a_{\boldsymbol{p}-\pi\boldsymbol{e}_2}\\ a_{\boldsymbol{p}-\pi\boldsymbol{e}_3}\\ a_{\boldsymbol{p}-\pi\boldsymbol{e}_2-\pi\boldsymbol{e}_3}\end{pmatrix}, \tag{86}$$

$$\mathcal{H}_{\boldsymbol{p}}=2\begin{pmatrix}J_3\sin p_3 & 0 & J_2\sin p_2 & J_1\sin p_1\\ 0 & J_3\sin p_3 & J_1\sin p_1 & -J_2\sin p_2\\ J_2\sin p_2 & J_1\sin p_1 & -J_3\sin p_3 & 0\\ J_1\sin p_1 & -J_2\sin p_2 & 0 & -J_3\sin p_3\end{pmatrix}. \tag{87}$$

By diagonalizing $\mathcal{H}_{\boldsymbol{p}}$, the quasiparticle spectrum $\varepsilon_{\boldsymbol{p}}$ is obtained as

$$\varepsilon_{\boldsymbol{p}}=2\sqrt{J_1^2\sin^2p_1+J_2^2\sin^2p_2+J_3^2\sin^2p_3}, \tag{88}$$

where the negative-energy states are occupied in the ground state.

V. PROOFS

Now we prove our main results, Theorems 1–3, given in Sec. II. To prove Theorem 1, we examine the basic properties of the CG. Let us consider a transformation of the operators

$$\{\dots h_p,\dots,h_q,\dots\}\mapsto\{\dots h_p,\dots,h_ph_q,\dots\}. \tag{89}$$

Corresponding to this transformation, the CG is modified as follows:

(i) Draw new lines from h_ph_q to all the h_k's that satisfy $h_ph_k=-h_kh_p$.

(ii) If there exist two lines from h_ph_q to h_k, these lines should be eliminated, and there remains no line between h_ph_q and h_k.

Here, rule (ii) corresponds to the fact that when h_p and h_q anticommutate with h_k, then the product h_ph_q commutes with h_k.

We represent modifications (i) and (ii) in terms of the adjacency matrix on $\mathbb{F}_2$: Let $M(\mathcal{A})$ be the adjacency matrix of the CG $\mathcal{G}(\mathcal{A})$, i.e.,

$$M(\mathcal{A})_{ij}=\begin{cases}0 & (h_ih_j=h_jh_i)\\ 1 & (h_ih_j=-h_jh_i).\end{cases} \tag{90}$$

$M(\mathcal{A})$ is symmetric, and its diagonal elements are all 0. The multiplication of h_p to h_q corresponds to the row and column additions of $M(\mathcal{A})$; that is, the qth row is replaced by the sum of the qth and pth rows, and the qth column is replaced by the sum of the qth and pth columns. The row and column additions are given by

$$M(\mathcal{A})\mapsto P^{[p,q]T}M(\mathcal{A})P^{[p,q]}, \tag{91}$$

where $P^{[p,q]}$ is an elementary matrix with the (i,j) component $P_{ij}^{[p,q]}=\delta_{ij}+\delta_{ip}\delta_{jq}$. Here, the rule $1+1=0$ in the matrix corresponds to rule (ii) above.

We can also represent Eq. (89) using the same elementary matrix $P^{[p,q]}$: Let $\boldsymbol{v}(h_j)$ be the unit vector on $\mathbb{F}_2$ having a nonzero element only in the jth component,

$$\boldsymbol{v}(h_j)=(0\quad\cdots\quad 0\quad 1\quad 0\cdots\quad 0)^T. \tag{92}$$

Then, we have

$$P^{[p,q]}\boldsymbol{v}(h_j)=\begin{cases}\boldsymbol{v}(h_p)+\boldsymbol{v}(h_q) & \text{for } j=q\\ \boldsymbol{v}(h_j) & \text{for } j\neq q,\end{cases} \tag{93}$$

which reproduces Eq. (89) by regarding the addition $\boldsymbol{v}(h_p)+\boldsymbol{v}(h_q)$ as the product h_ph_q.

Now consider the following operations on the CG: If there are vertices h_i and h_j that are connected to each other with a line, then multiply h_i to all the vertices h_k that satisfy $h_kh_j=-h_jh_k$ and multiply h_j to all the vertices h_k that satisfy $h_kh_i=-h_ih_k$. Then there remains no line beginning from h_i and h_j, except a line between h_i and h_j. As a result, we obtain a graph consisting only of h_i and h_j, and a graph with other vertices. Repeating the same procedure for the latter graph, we inductively obtain graphs composed of only pairs and those with isolated vertices.

This modification leads to Theorem 1: After the modification of the CG, $M(\mathcal{A})$ is block diagonalized with $r/2$ number of blocks with the form $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ and $n-r$ number of blocks with 0 [49] (r is even). Here, $r/2$ is the number of the pairs and $n-r$ is the number of the isolated vertices in the above. Since r coincides with rank $M(\mathcal{A})$, the number of the pairs is unique. When h belongs to the kernel of $M(\mathcal{A})$, it is evident that h commutes with all the h_i's, and hence $[H,h]=0$. Conversely, assume that $h=h_{j_1}h_{j_2}\cdots h_{j_k}$ satisfies $[H,h]=0$. Then, we find $hh_i=\epsilon_ih_ih$, $\epsilon_i=+1$ or -1, for all h_i. If h is a constant, h generates an isolated vertex and belongs to the kernel. Otherwise, from the condition $[H,h]=0$ and the independence of h_j's, it is easy to derive that h commutes with all h_1,..., h_n, and hence h belongs to the kernel of $M(\mathcal{A})$. Therefore Theorem 1 holds.

By noting that the $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ block and the 0 block correspond to the Clifford algebras Cl_2 and Cl_1, respectively, the above modification process also implies the following proposition:

Proposition 1. Let $\mathcal{A}(X)$ be the BA generated from the set of independent operators X and $M(\mathcal{A}(X))$ be its adjacency matrix. Then, we find $\mathcal{A}(X) \simeq (Cl_2)^{r/2} \otimes (Cl_1)^{n-r}$, and $\mathcal{A}(X) \simeq \mathcal{A}(X')$ if and only if $\mathrm{rank} M(\mathcal{A}(X)) = \mathrm{rank} M(\mathcal{A}(X'))$.

In particular, when $\mathcal{A}$ gives the complete graph with n vertices, i.e., a graph in which all vertices are connected to each other, and when we separate a pair of operators in a manner similar to the above, it is easy to convince ourselves that the remaining graph with $n-2$ vertices becomes again a complete graph. Iterating this procedure, we finally obtain $n/2$ pairs when n is even and obtain $(n-1)/2$ pairs and an isolated vertex when n is odd. The inverse of this modification is always possible. Since the complete graph with n vertices represents the Clifford algebra with n operators Cl_n, the rank of the adjacency matrix of the Clifford algebra with n operators is n when n is even and $n-1$ when n is odd. This corresponds to the known fact that $Cl_{2n} \simeq Cl_2^{\otimes n}$ and $Cl_{2n+1} \simeq Cl_2^{\otimes n} \otimes Cl_1$. Therefore Proposition 1 implies that a BA $\mathcal{A}$ with n operators coincides with the Clifford algebra if $\mathrm{rank} M(\mathcal{A}) = n$ [$\mathrm{rank} M(\mathcal{A}) = n-1$] for even (odd) n.

Theorem 2 follows from the fact that h_j in Eq. (4) reproduces the BA of the CG that coincides with a SPSC: Let $K(S)$ with $S = \{s_1, \dots, s_m\}$ be the SPSC for the BA, and assign a Majorana operator φ_α on each simplex $s_\alpha \in S$. As we mentioned in remark (i) in Sec. II, without loss of generality, we can assume that any vertex v of $s_\alpha \in S$ is shared by another $s_\beta \in S$ ($\beta \neq \alpha$). Moreover, only these simplices share v since S is single-point connected. Under this assumption, we consider $h_j^0 \equiv -i\epsilon_{\alpha\beta}\varphi_\alpha\varphi_\beta$ for the vertex v_j with h_j, where φ_α and φ_β are located on the simplices that share v_j. Then, we find that $\{h_i^0, h_j^0\} = 0$ ($[h_i^0, h_j^0] = 0$) if v_i and v_j are (not) vertices of the same simplex. These relations reproduce the BA of the SPSC, and thus we can identify h_j^0 with h_j.

Finally, we prove Theorem 3. For preparation, we first show the following lemma:

Lemma 1. Let $K(S)$ with $S = \{s_1, \dots, s_m\}$ be a SPSC. Then we have

$$C_q(K(S)) = C_q(K(s_1)) \oplus \cdots \oplus C_q(K(s_m)) \quad (q \geqslant 1), \tag{94}$$

where C_q is the q-chain on $\mathbb{F}_2$ and $\oplus$ is the direct sum [i.e., $C_q(K(s_\alpha)) \cap C_q(K(s_\beta)) = \{0\}$ for $\alpha \neq \beta$]. We also have

$$H_q(K(S)) = 0 \quad (q \geqslant 2). \tag{95}$$

The proof is as follows: Since $K(S)$ consists of all faces of $s_1, \dots, s_m$, we have

$$C_q(K(S)) = C_q(K(s_1)) + \cdots + C_q(K(s_m)) \quad (q \geqslant 1). \tag{96}$$

Furthermore, it holds that $C_q(K(s_\alpha)) \cap C_q(K(s_\beta)) = \{0\}$ for $\alpha \neq \beta$ and $q \geqslant 1$ since $K(S)$ is a SPSC. Thus Eq. (94) holds. Equation (95) immediately follows from Eq. (94): Since the boundary operator ∂ maps a q-chain to $(q-1)$-chain as

$$\partial : C_q(K(s_\alpha)) \to C_{q-1}(K(s_\alpha)), \tag{97}$$

we obtain

$$H_q(K(S)) = H_q(K(s_1)) \oplus \cdots \oplus H_q(K(s_m)) \quad (q \geqslant 2), \tag{98}$$

which turns to be zero because $H_q(K(s_\alpha)) = 0$ ($q \geqslant 1$).

Now we can show that $K(S)$ has $n - m + 1$ independent noncontractible loops. Let h_j ($j = 1, \dots, n$) be the generators of a BA and $S = \{s_1, \dots, s_m\}$ be a set of simplices of which $K(S)$ is a SPSC of the BA. Consider the Euler characteristic of $\chi(K(S))$,

$$\chi(K(S)) = \sum_{q=0}^{\dim K(S)} (-1)^q [\text{the number of } q\text{-faces in } K(S)], \tag{99}$$

where a q-face is a q-simplex included in $K(S)$ [namely, a 0-face is a vertex of $K(S)$, a 1-face is a hinge of $K(S)$, and so on]. In terms of homology groups, $\chi(K(S))$ is also written as [50]

$$\chi(K(S)) = \sum_{q=0}^{\dim K(S)} (-1)^q \dim H_q(K(S)). \tag{100}$$

Since $K(S)$ is connected, we have

$$\dim H_0(K(S)) = 1, \tag{101}$$

and from Lemma 1, it holds that

$$\dim H_{q\geqslant 2}(K(S)) = 0. \tag{102}$$

Thus $\dim H_1(K(S))$ is evaluated as

$$\begin{aligned} &\dim H_1(K(S)) \\ &\quad = 1 - \chi(K(S)) \\ &\quad = 1 - \sum_{q=0}^{\dim K(S)} (-1)^q [\text{the number of } q\text{-faces in } K(S)]. \end{aligned} \tag{103}$$

We compare this with the Euler characteristic of $K(s_\alpha)$ defined by

$$\chi(K(s_\alpha)) = \sum_{q=0}^{\dim s_\alpha} (-1)^q (\text{the number of } q\text{-faces in } s_\alpha). \tag{104}$$

As s_α is a simplex, we have

$$\chi(K(s_\alpha)) = 1, \tag{105}$$

and thus, summing both sides of Eq. (104) for all $s_\alpha \in S$, we obtain

$$m = \sum_{\alpha=1}^{m} \sum_{q=0}^{\dim s_\alpha} (-1)^q (\text{the number of } q\text{-faces in } s_\alpha). \tag{106}$$

On the other hand, as $K(S)$ is a SPSC, we have

$$\begin{aligned} &\sum_{q=0}^{\dim K(S)} (-1)^q [\text{the number of } q\text{-faces in } K(S)] \\ &\quad = \sum_{\alpha=1}^{m} \sum_{q=0}^{\dim s_\alpha} (-1)^q (\text{the number of } q\text{-faces in } s_\alpha) - n. \end{aligned} \tag{107}$$

Combining Eqs. (106) and (107) with Eq. (103), we get

$$\dim H_1(K(S)) = n - m + 1, \tag{108}$$

which implies that there exist $n - m + 1$ noncontractible loops in $K(S)$.

The $n - m + 1$ noncontractible loops give $n - m + 1$ conserved quantities: For each noncontractible loop, we may consider a product of h_j on all vertices of the loop. Obviously, such a product will reduce to a constant if we rewrite it in terms of Majorana fermions of Theorem 2. Thus this product is conserved, and Theorem 3 holds.

VI. DISCUSSION

In this paper, we present a simple criterion for solvability of lattice spin systems on the basis of graph theory and simplicial homology. When the lattice systems obey a class of algebras with the graphical representations, the spin systems can be converted into free Majorana fermion systems. We illustrate the validity of our criterion in a variety of spin systems.

Our method may reveal interesting aspects of lattice spin systems. After the conversion to Majorana-bilinear forms, the lattice spin systems exhibit particle-hole symmetry, in a manner similar to superconductors, because of the self-conjugate property of Majorana fermions. Hence they can be a kind of topological superconductor [51], although the origin of particle-hole symmetry is completely different. The Kitaev honeycomb lattice, for instance, exhibits a 2D non-Abelian topological phase analog to chiral superconductors, in the presence of time-reversal breaking perturbation [5]. Our approach provides a systematic way to explore other interesting topological superconducting phases in spin systems: 3D non-Abelian topological phase [52,53], gapless topological phases [54–57], and topological crystalline superconductors [58,59]. Searching such interesting phases is left for future work.

Note added. Recently, we became aware of a related work (Ref. [60]).

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