Heat transport in overdamped quantum systems

Sadeq S. Kadijani, Thomas L. Schmidt, Massimiliano Esposito, and Nahuel Freitas
Department of Physics and Materials Science, University of Luxembourg, L-1511 Luxembourg, Luxembourg

(Received 28 September 2020; revised 30 November 2020; accepted 1 December 2020; published 21 December 2020)

We obtain an analytical expression for the heat current between two overdamped quantum oscillators interacting with local thermal baths at different temperatures. The total heat current is split into classical and quantum contributions. We show how to evaluate both contributions by taking advantage of the timescale separation associated with the overdamped regime and without assuming the usual weak-coupling and Markovian approximations. We find that nontrivial quantum corrections survive even when the temperatures are high compared to the frequency scale relevant for the overdamped dynamics of the system.

DOI: 10.1103/PhysRevB.102.235422

I. INTRODUCTION

In classical and statistical physics, the overdamped limit is an extremely useful approximation that allows us to simplify problems where the dynamics of a system is dominated by the friction due to its interaction with an environment. This can be understood based on the canonical example of a Brownian particle, where the limit of strong friction induces a timescale separation in which the momentum degree of freedom relaxes much faster than the position. In such a case, the Fokker-Planck equation describing the stochastic evolution of both degrees of freedom can be reduced to the Smoluchowski equation for the evolution of the probability density of the position alone [1].

For quantum systems, an analogous procedure proves to be more demanding. This is because, in general, tractable descriptions for the reduced dynamics of open quantum systems can be obtained only for weak coupling between the system and the environment. In contrast, by definition, the overdamped limit is a strong-coupling regime (however, this does not prevent weak-coupling master equations from providing approximate descriptions of overdamped dynamics under some conditions [2]). In spite of this, a quantum version of the Smoluchowski equation was first obtained by Ankerhold and collaborators [3,4] using path integral techniques. Those results, as well as later extensions to time-dependent systems [5], consider only equilibrium environments; that is, the system in question interacts only with a single thermal bath.

More recently, some efforts in stochastic and quantum thermodynamics have also focused on understanding the impact of strong-coupling effects in both in-equilibrium and out-of-equilibrium settings [6–18]. In this paper we explore the overdamped limit of a quantum system in contact with a nonequilibrium environment; that is, we consider a situation in which the system simultaneously interacts with two thermal baths at different temperatures. Specifically, we consider an electrical circuit composed of two parallel *RLC* circuits coupled by a mutual inductance (see Fig. 1). Here, the resistors represent the thermal baths into which energy can be dissipated. If they are at different temperatures, then the system will reach a nonequilibrium stationary state in which heat flows from the hot to the cold resistor. We are interested in studying the properties of this heat current in the overdamped limit where dissipation dominates, which in this case is achieved for $C_iR_i^2 \ll L_i$. For this purpose, we will exploit the fact that for linear systems like this one an exact integral expression for the heat currents can be obtained, and in some cases it can be evaluated analytically [19,20]. In this way, we are able to split the heat current into classical and quantum contributions and to analyze their behavior in different regimes. We obtain analytical expressions for both contributions that fully take into account non-Markovian effects. Interestingly, we show that the quantum corrections to the heat current do not necessarily vanish in the limit where both temperatures are high with respect to the slow frequency scale (the only one relevant for the dynamics of the circuit in the overdamped regime). The surviving quantum corrections are nontrivial and depend logarithmically on the temperatures. We show that these results are, indeed, accurate by comparing them to exact numerical computations.

This paper is organized as follows. In Sec. II, we describe our model of quantum circuits and introduce the expression for the steady-state heat currents for general harmonic networks. Next, in Sec. III we give the result for the classical and quantum contributions to the steady-state heat current in terms of the circuit parameters. The evaluation of the heat currents in the overdamped regime is then done in Sec. IV together with the analysis of different regimes.

II. THE MODEL AND ITS SOLUTION

We begin by building a quantum model of the circuit in Fig. 1. For this, we will represent each resistor using the usual Caldeira-Leggett model. In this model, a resistor is considered an infinite array of independent *LC* circuits or harmonic modes. In this way, using the usual procedure for canonical quantization [21,22], it is possible to obtain the following

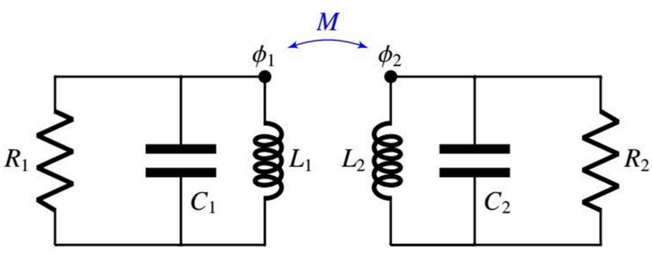

FIG. 1. Two magnetically coupled *RLC* circuits.

Hamiltonian for the full system (see Appendix A for more details):

$$H = \frac{q_1^2}{2C_1} + \frac{q_2^2}{2C_2} + \frac{L_1L_2}{L_1L_2 - M^2}\left(\frac{\phi_1^2}{2L_1} + \frac{\phi_2^2}{2L_2} - \frac{M}{L_1L_2}\phi_1\phi_2\right) + \sum_{m_1}\left(\frac{q_{m_1}^2}{2C_{m_1}} + \frac{(\phi_{m_1}-\phi_1)^2}{2L_{m_1}}\right) + \sum_{m_2}\left(\frac{q_{m_2}^2}{2C_{m_2}} + \frac{(\phi_{m_2}-\phi_2)^2}{2L_{m_2}}\right). \quad (1)$$

Here, q_1 and q_2 are quantum-mechanical operators associated with the charge on the capacitors, while ϕ_1 and ϕ_2 are operators associated with the total magnetic flux through the inductors. They satisfy the usual commutation relations $[q_j, \phi_k] = i\hbar\delta_{j,k}$. In a similar way, $\{q_{m_1}, \phi_{m_1}\}$ and $\{q_{m_2}, \phi_{m_2}\}$ are sets of conjugate operators associated with each of the individual modes in the Caldeira-Leggett model of each resistor. These individual modes are characterized by capacitances C_{m_k} and inductances L_{m_k}, which are, in principle, arbitrary. They enter in the definition of the spectral density associated with the resistors, defined below.

In the following it will be convenient to write the different terms of the previous Hamiltonian in matrix form. In fact, we can write

$$H = H_{\rm sys} + \sum_\alpha (H_{{\rm env},\alpha} + H_{{\rm int},\alpha}), \quad (2)$$

with

$$H_{\rm sys} = q^T\frac{C^{-1}}{2}q + \phi^T\frac{L^{-1}}{2}\phi, \quad (3)$$

$$H_{{\rm env},\alpha} = q_\alpha^T\frac{C_\alpha^{-1}}{2}q_\alpha + \phi_\alpha^T\frac{L_\alpha^{-1}}{2}\phi_\alpha, \quad (4)$$

$$H_{{\rm int},\alpha} = -\phi^T\bar{L}_\alpha^{-1}\phi_\alpha. \quad (5)$$

Here, $H_{\rm sys}$, $H_{{\rm env},\alpha}$, and $H_{{\rm int},\alpha}$ are the Hamiltonians of the system, the thermal baths, and the interaction between the system and the baths, respectively. The index $\alpha \in \{1, 2\}$ identifies a resistor or thermal bath in the environment. Moreover, $q = (q_1, q_2)^T$ and $\phi = (\phi_1, \phi_2)^T$ are column vectors of the charge and flux operators of the system, respectively, and the matrices appearing in $H_{\rm sys}$ are defined by

$$C = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix}, \quad L_0 = \begin{pmatrix} L_1 & -M \\ -M & L_2 \end{pmatrix}, \quad (6)$$

and

$$L^{-1} = L_0^{-1} + \begin{pmatrix} \sum_{m_1} L_{m_1}^{-1} & 0 \\ 0 & \sum_{m_2} L_{m_2}^{-1} \end{pmatrix}. \quad (7)$$

In a similar way, q_α and ϕ_α are column vectors formed by the charge and flux operators of the αth bath, and C_α and L_α are diagonal matrices containing the capacitances and inductances of each bath. Finally, the matrices $\bar{L}_\alpha^{-1}$ are given by

$$\bar{L}_1^{-1} = \begin{pmatrix} L_{11} & L_{12} & \cdots & L_{1N} \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \quad (8)$$

$$\bar{L}_2^{-1} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ L_{21} & L_{22} & \cdots & L_{2N} \end{pmatrix}. \quad (9)$$

The system described so far is a particular case of an open harmonic network. The nonequilibrium thermodynamics of these systems has been extensively studied before [7,8,13,19,20,23–26] since owing to their linearity exact analytical results can be obtained. The central quantities in this study will be the heat currents associated with each thermal bath, i.e., the average rates at which energy is interchanged between the system and each bath. They can be defined as (heat currents are considered to be positive when they enter the system)

$$\dot{Q}_\alpha = -\frac{1}{i\hbar}\langle[H_{{\rm env},\alpha}, H_{{\rm int},\alpha}]\rangle, \quad (10)$$

where the mean value is taken with respect to the instantaneous global state. Given an initial state, the heat currents $\dot{Q}_\alpha$ will depend nontrivially on time during relaxation, after which they will reach stationary values. Typically, one assumes an uncorrelated initial state $\rho_0 = \rho_{\rm sys} \otimes \rho_{\rm env}$ in which each of the baths in the environment is in a thermal state $\rho_\alpha^{\rm th}$ at inverse temperature $\beta_\alpha = (k_bT_\alpha)^{-1}$, i.e., $\rho_{\rm env} = \otimes_\alpha \rho_\alpha^{\rm th}$. Under this assumption, it can be shown that in the long-time limit the average heat currents can be expressed as (see Appendix B)

$$\dot{Q}_\alpha = \frac{\hbar}{2}\sum_{\alpha'\neq\alpha}\int_0^\infty d\omega\,\omega\, f_{\alpha\alpha'}(\omega) \times [\coth(\beta_\alpha\hbar\omega/2) - \coth(\beta_{\alpha'}\hbar\omega/2)], \quad (11)$$

where $f_{\alpha\alpha'}(\omega)$ is the heat transfer matrix element and reads

$$f_{\alpha\alpha'}(\omega) = \frac{\pi}{2}{\rm Tr}[I_\alpha(\omega)g(i\omega)I_{\alpha'}(\omega)g^\dagger(i\omega)]. \quad (12)$$

In the previous expression, $I_\alpha(\omega)$ is the spectral density of the αth bath. It is a 2×2 matrix with elements

$$[I_\alpha(\omega)]_{kl} = \sum_n (\bar{L}_\alpha^{-1})_{kn}(\bar{L}_\alpha^{-1})_{ln}(\omega_\alpha C_\alpha)_{nn}^{-1}\delta[\omega - (\omega_\alpha)_{nn}], \quad (13)$$

where $\omega_\alpha^2 = L_\alpha^{-1}C_\alpha^{-1}$ is a diagonal matrix with the squared natural frequencies of the modes in the αth bath. Also, $g(s)$ in Eq. (12) is the Laplace transform of the circuit Green's function,

$$g(s)^{-1} = Cs^2 + \gamma(s)s + L_0^{-1}, \quad (14)$$

where $\gamma(s)$ is the Laplace transform of the dissipation kernel. It is given by

$$\gamma(s) = \int_0^\infty \frac{I(\omega)}{\omega}\frac{s}{s^2+\omega^2}d\omega \tag{15}$$

in terms of the total spectral density $I(\omega) = \sum_\alpha I_\alpha(\omega)$. Note that, according to these definitions, in the long-time limit we always have $\sum_\alpha \dot{Q}_\alpha = 0$. This expresses nothing more than the conservation of energy, and it follows from Eq. (11) and the fact that the heat transfer matrix $f_{\alpha\alpha'}(\omega)$ is symmetric upon $\alpha \leftrightarrow \alpha'$.

Even if the model is fully quantum, the fact that the system is linear makes its dynamics equivalent to the classical one [27]. Thus, the Green's function of Eq. (14), which appears in the heat transfer matrix of Eq. (12), is just the one solving the classical equations of motion and has no dependence on $\hbar$, which appears only in the noise spectrum $2\hbar\omega\coth(\beta_\alpha\hbar\omega/2)$. In fact, an expression similar to Eq. (11) could be obtained by just computing the classical transfer function of the circuit between the two ports to which the resistors are connected and considering the resistors to be sources of Johnson-Nyquist noise [28,29]. An important difference, however, is that even if they are classical, the exact equations of motion for the system are actually nonlocal in time due to the non-Markovian effects induced by the environment [see Eq. (B1) in Appendix B]. This fact is missed by a classical circuit treatment but is fully taken into account here, as we explain below.

The frequency integral in Eq. (11) can be solved analytically in certain cases. As shown in Ref. [20], when the spectral densities of all baths are of the Lorentz-Drude form, the integral can be evaluated in terms of the eigenvalues and eigenvectors of a cubic eigenvalue problem. Thus, we will assume the following spectral density for the baths:

$$I_\alpha(\omega) = \frac{2}{\pi}\frac{1}{R_\alpha}\frac{\omega\,\omega_c^2}{\omega^2+\omega_c^2}P_\alpha, \tag{16}$$

with

$$P_1 = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \quad P_2 = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}. \tag{17}$$

Note that by Eq. (16), in a continuous limit of Eq. (13) the parameters $\bar{L}_\alpha$ and C_α are determined by the values R_α of the resistances and also the frequency cutoff ω_c. Interestingly, the previous results are valid for any value of ω_c, which controls the autocorrelation time of the environment (the Markovian approximation corresponds to the limit $\omega_c \to \infty$). Thus, our results will automatically include non-Markovian effects. Finally, with the previous choice for the spectral densities, the function $\gamma(s)$ in Eq. (15) becomes

$$\gamma(s) = \left(\frac{P_1}{R_1}+\frac{P_2}{R_2}\right)\frac{\omega_c}{s+\omega_c}. \tag{18}$$

III. CLASSICAL AND QUANTUM CONTRIBUTIONS TO THE HEAT CURRENT

The previous ingredients enable us to find the heat current in terms of the circuit parameters. By plugging the definitions of the spectral densities into Eq. (12) we find

$$f_{1,2}(\omega) = \frac{2}{\pi}\left(\frac{1}{R}\frac{\omega\,\omega_c^2}{\omega^2+\omega_c^2}\right)^2|g_{12}(i\omega)|^2, \tag{19}$$

where $g_{1,2}(s)$ is the off-diagonal element of $g(s)$. For simplicity we will consider the case of a symmetric circuit, i.e., $R_1 = R_2 = R$, $C_1 = C_2 = C$, and $L_1 = L_2 = L$. Then, we obtain, from Eq. (14),

$$g_{12}(s) = \frac{M}{A}\left[\left(Cs^2+\frac{L}{A}+\frac{1}{R}\frac{s\,\omega_c}{s+\omega_c}\right)^2 - \left(\frac{M}{A}\right)^2\right]^{-1}, \tag{20}$$

where $A = L^2 - M^2$. As a consequence, the transfer function can be finally written as

$$f_{1,2}(\omega) = \frac{2}{\pi}\,\omega^2\omega_c^4\left(\frac{RM}{A}\right)^2\frac{1}{|u_+(i\omega)\,u_-(i\omega)|^2}, \tag{21}$$

with

$$u_\pm(s) = (s^3+\omega_c s^2)RC + \left(\frac{R}{L\pm M}+\omega_c\right)s + \frac{R}{L\pm M}\omega_c. \tag{22}$$

We can already see how an exact expression for the heat current can be obtained. Since the transfer function $f_{1,2}(\omega)$ was expressed as a rational function, the frequency integral in Eq. (11) can be evaluated via the residue theorem in terms of the poles of $f_{1,2}(\omega)$. In order to deal with the poles of the functions $\coth(\beta_\alpha\hbar\omega/2)$ at the Matsubara frequencies, it is convenient to write them in terms of digamma functions [30] (see Appendix C):

$$\pi\coth\left(\frac{\beta_\alpha\hbar\omega}{2}\right) = \frac{2\pi}{\beta_\alpha\hbar\omega} - i\psi\left(1-\frac{i\beta_\alpha\hbar\omega}{2\pi}\right) + i\psi\left(1+\frac{i\beta_\alpha\hbar\omega}{2\pi}\right). \tag{23}$$

We note that the terms containing digamma functions vanish in the high-temperature limit. Thus, this decomposition induces a splitting of the heat current into a high-temperature contribution and a low-temperature correction, which we denote as classical and quantum contributions, respectively. Therefore, we have

$$\dot{Q}_1 = \dot{Q}_1^{\rm cl} + \dot{Q}_1^{\rm q}, \tag{24}$$

where

$$\dot{Q}_1^{\rm cl} = \left(\frac{1}{\beta_1}-\frac{1}{\beta_2}\right)\int_0^\infty d\omega f_{12}(\omega), \tag{25}$$

$$\dot{Q}_1^{\rm q} = \frac{i\hbar}{2\pi}\int_{-\infty}^\infty d\omega\omega f_{12}(\omega)\left[\psi\left(1-\frac{i\beta_2\hbar\omega}{2\pi}\right) - \psi\left(1-\frac{i\beta_1\hbar\omega}{2\pi}\right)\right]. \tag{26}$$

Although the previous integrals could, in principle, be evaluated exactly [20], the procedure and the final result are greatly simplified in the overdamped limit in which we are interested. Thus, we now discuss this approximation and the frequency scales involved.

IV. EVALUATION OF THE HEAT CURRENT IN THE OVERDAMPED LIMIT

The classical equation of motion for a single parallel *RLC* circuit is

$$\ddot{\phi} + \gamma\dot{\phi} + \omega_0^2\phi = 0, \tag{27}$$

where ϕ is the flux variable in the inductor and the relevant frequency scales are given by the damping rate $\gamma = 1/RC$ and the natural frequency $\omega_0 = 1/\sqrt{LC}$. The corresponding characteristic equation has roots $\Gamma_\pm = -(\gamma/2) \pm (\gamma^2/4 - \omega_0^2)^{1/2}$. The overdamped limit corresponds to $\gamma \gg \omega_0$, and it can be reached, for instance, by reducing the value of the capacitance so that $C \ll L/R^2$. In that regime we have $\Gamma_+ \simeq -\omega_0^2/\gamma$ and $\Gamma_- \simeq -\gamma + \omega_0^2/\gamma$, which in absolute value are the damping rates of the magnetic flux ϕ and charge q, respectively, and therefore, $|\Gamma_+| \ll |\Gamma_-|$. This is the timescale separation characteristic for overdamped systems, which means in this case that the charge relaxes much faster than the flux. Also, note that the flux damping rate $\omega_d \simeq \omega_0^2/\gamma = R/L$ becomes independent of C. A similar analysis holds for each normal mode of the two coupled *RLC* circuits by just replacing L by $L \pm M$. We can express the functions $u_\pm(s)$ in Eq. (22) in terms of γ and $\omega_\pm = \omega_d/(1 \pm M/L)$,

$$u_\pm(s) = (s^3 + \omega_c s^2)/\gamma + (\omega_\pm + \omega_c)s + \omega_\pm\omega_c. \tag{28}$$

We see that the overdamped limit tends to reduce the weight of the cubic and quadratic terms, although they will always dominate for high frequencies. However, we also note that in the frequency integral of Eq. (11), the factor $\coth(\beta_\alpha\hbar\omega/2) - \coth(\beta_{\alpha'}\hbar\omega/2)$ will cut off frequencies higher than $\omega_{\rm th} = k_b \max_\alpha\{T_\alpha\}/\hbar$. From this it follows that the cubic and quadratic terms can be disregarded with respect to the other two whenever

$$\omega_{\rm th} \ll \gamma, (\gamma\omega_\pm)^{1/2}, (\gamma\omega_\pm\omega_c)^{1/3}. \tag{29}$$

Thus, under those conditions, we can consider

$$u_\pm(s) \simeq (\omega_\pm + \omega_c)s + \omega_\pm\omega_c, \tag{30}$$

where the only remaining relevant frequency scales are $\omega_\pm$ and ω_c. We note that the conditions in Eq. (29) can always be fulfilled by increasing γ and that they do not restrict in any way the ratios between $\omega_{\rm th}$, $\omega_\pm$, and ω_c. However, they pose a restriction on the maximum value of the temperatures, and Table I specifies some temperature ranges relevant in the overdamped regime. This will become important later when we show that quantum effects survive even when the temperatures are high with respect to $\hbar\omega_\pm/k_b$. Using the approximation of Eq. (30), the integrals in Eqs. (25) and (26) can be readily evaluated. For the classical contribution to the heat current, we obtain

$$\dot{Q}_1^{\rm cl} = \frac{k_b}{2}(T_1 - T_2)\left(\frac{M}{L}\right)^2 \frac{\omega_c}{\omega_c + \omega_d}\frac{\lambda_+\lambda_-}{\omega_d}, \tag{31}$$

where $\lambda_\pm$ is the only root of $u_\pm(s)$,

$$\lambda_\pm = -\frac{\omega_c\,\omega_\pm}{\omega_c + \omega_\pm}. \tag{32}$$

The evaluation of the quantum contribution is not as straightforward, and its details are explained in Appendix C. The final result is

$$\dot{Q}_1^{\rm q} = \frac{\hbar}{\pi}\left(\frac{M}{L}\right)^2\left(\frac{\lambda_+\lambda_-}{\omega_d}\right)^2 \ln\left(\frac{T_2}{T_1}\right) + \frac{\hbar}{4\pi}\frac{\omega_c}{\omega_c + \omega_d}\frac{M}{L}\left\{\lambda_+^2\left[\psi\left(1 - \frac{\beta_1\hbar\lambda_+}{2\pi}\right) - \psi\left(1 - \frac{\beta_2\hbar\lambda_+}{2\pi}\right)\right] - \lambda_-^2\left[\psi\left(1 - \frac{\beta_1\hbar\lambda_-}{2\pi}\right) - \psi\left(1 - \frac{\beta_2\hbar\lambda_-}{2\pi}\right)\right]\right\}. \tag{33}$$

TABLE I. Different temperature ranges in the overdamped regime. We consider that the thermal frequency $\omega_{\rm th}$ characterizes the temperatures of both baths; that is, both the temperatures are of the same order. High temperatures are in the range addressed by the classical Smoluchowski equation or overdamped Langevin equations. For intermediate temperatures, the bath temperatures sit in between the frequency gap associated with the overdamped regime, while low temperatures are low compared to the lowest frequency scale of the system. Other ranges can be considered, for example, mixed conditions in which one of the bath temperatures is low while the other is high or taking into account values of $\omega_{\rm th}$ comparable to γ.

	Range
High temperatures	$\omega_\pm \ll \gamma \ll \omega_{\rm th}$
Intermediate temperatures	$\omega_\pm < \omega_{\rm th} \ll \gamma$
Low temperatures	$\omega_{\rm th} < \omega_\pm \ll \gamma$

Equations (31) and (33) are the central results of this work. They make it possible to compute the heat current in the overdamped regime without assuming the weak-coupling or Markovian approximations and thus complement previous results in similar systems that are either numerical or limited by the mentioned approximations [23–26]. We observe that the classical contribution is proportional to the temperature difference $\Delta T = T_1 - T_2$, whereas the quantum contribution depends on T_1 and T_2 in a nonalgebraic way, as expected. In Fig. 2 we compare the exact heat current obtained by numerical integration of Eq. (11) with the one obtained by using Eqs. (31) and (33) for different values of T_1 and T_2 as a function of γ/ω_d ($M/L = 1/2$ and $\omega_c = 5\omega_d$). We see that the two results match as γ/ω_d is increased.

We will now take some relevant limits in order to simplify the previous expressions. The Markovian limit ($\omega_c \to \infty$) can be easily obtained by replacing the factors $\omega_c/(\omega_c + \omega_d)$ by 1 in Eqs. (31) and (33) and noting that the roots $\lambda_\pm$ satisfy

$$\lim_{\omega_c\to\infty} \lambda_\pm = -\omega_\pm. \tag{34}$$

From Eq. (32) we see that the effect of a finite cutoff is equivalent to reducing the values of the frequencies $\omega_\pm$ or, correspondingly, ω_d.

To analyze the low-temperature regime we consider the limit $|\lambda_\pm|/\omega_{\rm th} \gg 1$ (note that this condition implies $\omega_{\rm th} \ll \omega_\pm, \omega_c$). We use the following expansion of the

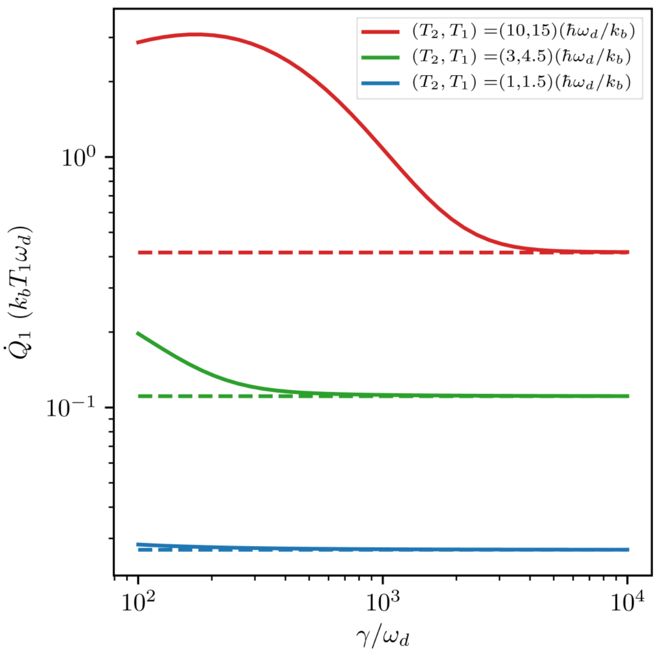

FIG. 2. Heat current with respect to γ/ω_d. Solid lines show the exact heat current in terms of the different values of baths temperatures T_1 and T_2. The dashed lines are the heat current in the overdamped limit. This plot is sketched for $M = 1, L = 2, \omega_c = 5\omega_d$, $\omega_d = 1$.

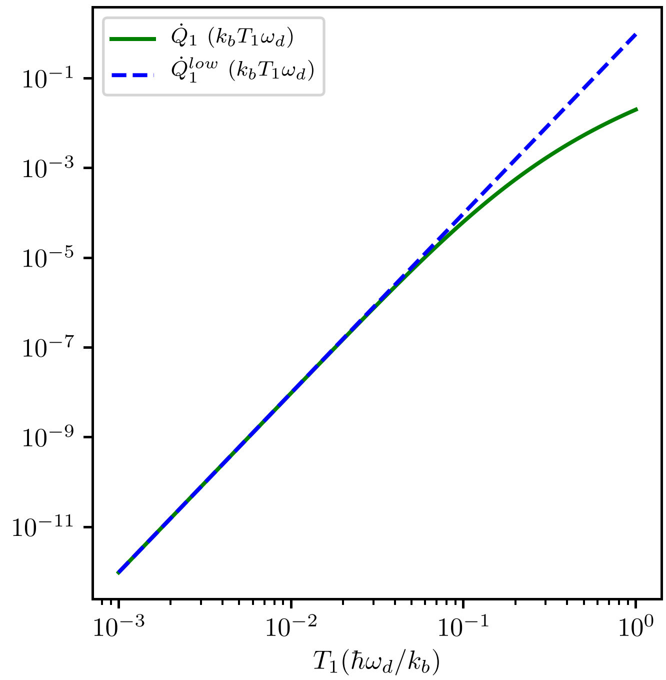

FIG. 3. Comparison between total heat current (green solid line) and the low-temperature regime expression (blue dashed line) with respect to the different values of T_1. Here, we set $T_2 = T_1/2$; thus, we lower the two temperatures at the same time with a constant ratio. We can observe that, when the temperatures are lowered, the two expressions will coincide. $M = 1, L = 2, \omega_c = 5\omega_d, \omega_d = 1$.

digamma function for large x:

$$\psi(x) \approx \ln x - \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} + O(x^5), \qquad (35)$$

The contribution of the first logarithmic term cancels the first term in Eq. (33). Moreover, the contribution of the second term cancels the classical part of the heat current, while those coming from the third term vanish. Thus, the only remaining contributions originate from the term $\propto 1/x^4$, so the final result for the low-temperature heat current is

$$\dot{Q}_1^{\rm low} = \frac{2}{15}\left(\frac{\pi}{\hbar}\right)^3\left(\frac{M}{L}\right)^2\frac{k_b^4}{\omega_d^2}\left(T_1^4 - T_2^4\right) + O\left(T_{1/2}^6\right). \qquad (36)$$

Considering $T_{1/2} = T \pm \Delta T/2$ and to lower order in ΔT, we can express this result as $\dot{Q}_1^{\rm low} = \mathcal{T}G_Q\Delta T$, where $G_Q = \pi k_b^2 T/3\hbar$ is the fundamental unit of thermal conductance [28,31,32] and $\mathcal{T} = (4\pi^2/5)(M/L)^2(k_bT/\hbar\omega_d)^2$ is a temperature-dependent transmission coefficient. Also, we would like to point out that Eq. (36) is independent of the cutoff frequency. This is natural since for low temperatures only low-frequency modes contribute to the heat current, while the cutoff frequency controls the high-frequency region of the spectral densities. It is worth mentioning that one would also obtain the exact same expression for the heat currents at the low-temperature limit in the weak-coupling regime, i.e., $\gamma \ll \omega_\pm$ (see Appendix D). This again is due to the fact that at low temperatures only the low-frequency response of the system is relevant. In Fig. 3 we show the behavior of the total heat current when the bath temperatures are decreased. For low temperatures, we can see that the heat current is, indeed, well approximated by Eq. (36).

Turning to the regime of intermediate temperatures, where $|\lambda_\pm|/\omega_{\rm th} \ll 1$, we employ the following expansion of the digamma function for small values of x,

$$\psi(1 + x) = -\eta + \frac{\pi^2 x}{6} + O(x^2), \qquad (37)$$

where η is the Euler-Mascheroni constant. We then find the following high-temperature expansion of the quantum contribution:

$$\dot{Q}_1^{\rm q} = \frac{\hbar}{\pi}\left(\frac{M}{L}\right)^2\left(\frac{\lambda_+\lambda_-}{\omega_d}\right)^2\ln\left(\frac{T_2}{T_1}\right) + \frac{\hbar^2}{48}\frac{\omega_c}{\omega_c + \omega_d} \times\frac{M}{L}\left(\lambda_+^3 - \lambda_-^3\right)\left(\frac{1}{T_2} - \frac{1}{T_1}\right) + O\left(T_{1/2}^{-2}\right). \qquad (38)$$

Surprisingly, we see that the dominant term does not necessarily vanish for $|\lambda_\pm|/\omega_{\rm th} \to 0$. The reason for this is that under the constraints given in Eq. (29), one can assume that the temperature is high compared to the slow frequency scale ω_d, but it must remain low compared to the fast frequency scale γ. In other words, the temperature sits in the middle of the timescale separation associated with the overdamped regime. Thus, to the first nontrivial order the total heat current for high temperatures is

$$\dot{Q}_1^{\rm high} = \dot{Q}_1^{\rm cl} + \frac{\hbar}{\pi}\left(\frac{M}{L}\right)^2\left(\frac{\lambda_+\lambda_-}{\omega_d}\right)^2\ln\left(\frac{T_2}{T_1}\right). \qquad (39)$$

Figure 4 shows the behavior of the quantum contribution with respect to the growth of the temperature. When both bath

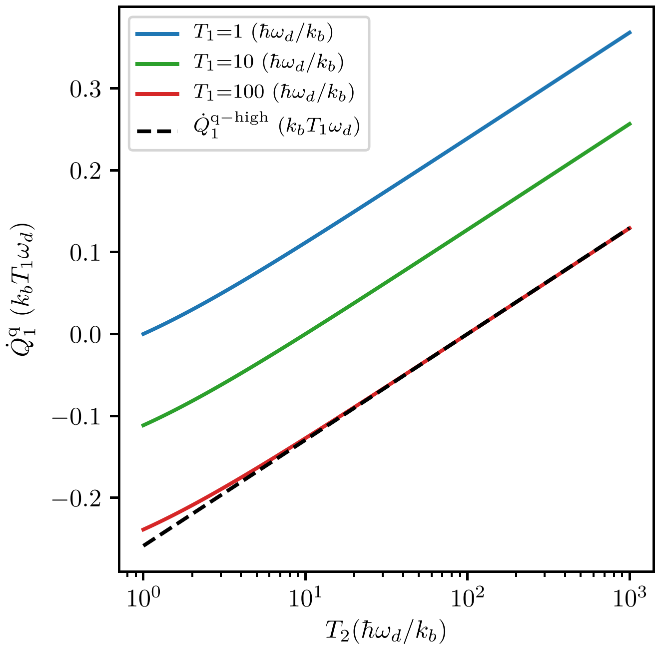

FIG. 4. The quantum contribution to the heat currents for different values of T_1 is sketched with respect to T_2. We can see that, for high values of T_1 and T_2, the quantum correction is not vanishing and it will coincide with a nontrivial logarithmic expression (black dashed line). This plot is sketched for $M = 1, L = 2, \omega_c = 5\omega_d$, $\omega_d = 1$.

temperatures are increased, we can still observe a nonzero quantum correction to the heat currents.

V. CONCLUSIONS

We have investigated the heat current between two overdamped quantum harmonic oscillators interacting with local thermal baths without invoking the weak-coupling and Markovian approximations. Exploiting the timescale separation associated with the overdamped regime, we were able to obtain closed analytical expressions for the heat current, identifying quantum and classical contributions. These analytical results might offer a useful benchmark to test Markovian embedding schemes or other approximate methods, for example, the one developed in [33]. Although our results are valid for general harmonic systems, we have explicitly considered an electronic implementation. This is justified by the fact that low-temperature electronic circuits are a promising platform to study quantum energy transport [29,34–38]. We found that in the overdamped regime a range of intermediate temperatures opens up between the low-temperature and high-temperature regimes usually considered. Our results indicate that in this intermediate range there are significant quantum corrections to the classical heat current, which survive even if the temperatures are high compared to the only relevant frequency scale of the system dynamics.

ACKNOWLEDGMENTS

S.S.K. and T.L.S. acknowledge financial support from the National Research Fund, Luxembourg under the Grants CORE QUTHERM C18/MS/12704391 and ATTRACT A14/MS/7556175/MoMeSys. M.E. and N.F. are funded by the European Research Council, project NanoThermo (ERC2015-CoG Agreement No. 681456). The authors also thank Joachim Ankerhold for useful discussions.

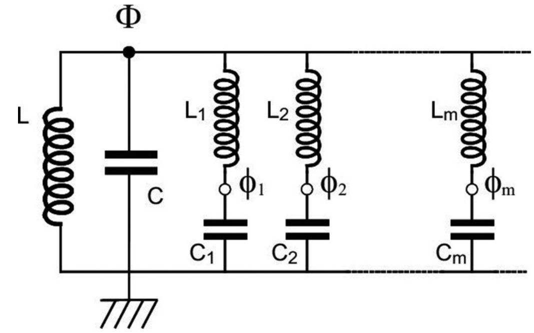

FIG. 5. A damped *LC* circuit.

APPENDIX A: COUPLED *RLC* CIRCUIT HAMILTONIAN

The Hamiltonian of a *LC* circuit can be written such that

$$H = \frac{\phi^2}{2L} + \frac{q^2}{2C}. \tag{A1}$$

In this notation, q and ϕ will play the role of momentum and position conjugate variables, respectively. To quantize the *LC* circuit, we need to replace the classical variables of the Hamiltonian (A1) with their quantum counterparts. In the other words, the Poisson bracket of the flux and charge in the circuit would be

$$\{\phi, q\} = \frac{\partial\phi}{\partial\phi}\frac{\partial q}{\partial q} - \frac{\partial q}{\partial\phi}\frac{\partial\phi}{\partial q} = 1. \tag{A2}$$

As shown by Dirac the value of a classical Poisson bracket imposes its corresponding quantum commutator

$$\{\phi, q\} \rightarrow \frac{1}{i\hbar}[\hat{\phi}, \hat{q}]. \tag{A3}$$

Thus, we see that transforming the classical Hamiltonian into its quantum version will also be backed by the uncertainty relation between flux and charge variables as they play the role of position and momentum, respectively.

The dissipative part would be the resistor attached to the *LC* circuit. However, adding the Hamiltonian of this part is not trivial. To write the full Hamiltonian of an *RLC* circuit, we will employ the Caldeira-Leggett model for the Brownian motion. The resistor can be considered a circuit consisting of an infinite array of independent *LC* circuits, each playing the role of harmonic oscillators of the bath (Fig. 5). Considering the *RLC* circuit in Fig. 5, we can write the full Hamiltonian describing the circuit such that

$$H = \frac{\Phi^2}{2L} + \frac{q^2}{2C} + \sum_m \frac{q_m^2}{2C_m} + \frac{(\phi_m - \Phi)^2}{2L_m}. \tag{A4}$$

In the above expression, the flux variables Φ and ϕ_m correspond to the *node* fluxes. The node flux is defined as the time integral of the voltage along the path connecting the node and the ground. q and q_m are also the charges in capacitors

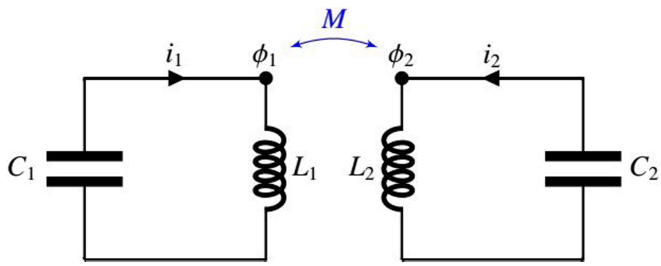

FIG. 6. Two magnetically coupled circuits.

C and C_m. The last term in the Hamiltonian can also be realized as the normalizing term to ensure that there will be no inconsistency in the minimum of the potential energy.

Next, we will magnetically couple two quantum *RLC* circuits by putting them in proximity to each other. Indeed, the coupling occurs due to the presence of a flux running in one circuit which is caused by the other inductor. This leads to the mutual inductance between the two inductors of the two circuits. Before considering two coupled *RLC* circuits, we first look at two simple coupled circuits in Fig. 6. We denote the total flux passing through the lth circuit by ϕ_l, with $l = \{1, 2\}$. The total flux is the sum over the flux ϕ_{ll} produced by the inductor L_l and the mutual flux ϕ_{lk} between the circuits with $k = \{1, 2\}$. We may write this such that

$$\phi_l = \sum_k \phi_{lk}. \tag{A5}$$

To find the Hamiltonian of the coupled circuits, we use Kirchhoff's law for voltages to obtain

$$v_1 = \dot{\phi}_1 = L_1\frac{di_1}{dt} + \dot{\phi}_{12}, \tag{A6}$$

$$v_2 = \dot{\phi}_2 = L_2\frac{di_2}{dt} + \dot{\phi}_{21}, \tag{A7}$$

where v_1 and v_2 are the voltages associated with the two capacitors and i_1 and i_2 are the currents for each circuits. For mutual flux we can write

$$\begin{aligned}\dot{\phi}_{12} &= \frac{d\phi_{12}}{di_2}\frac{di_2}{dt} = M_{12}\frac{di_2}{dt},\\ \dot{\phi}_{21} &= \frac{d\phi_{21}}{di_1}\frac{di_1}{dt} = M_{21}\frac{di_1}{dt},\end{aligned} \tag{A8}$$

where $M_{12/21} = \frac{d\phi_{12/21}}{di_{2/1}}$ is the mutual inductance between the two circuits and it can be proved that $M_{12} = M_{21} = M$.

To calculate the energy stored in the two coupled circuits, we first assume that $i_2 = 0$ and i_1 is increased up to an arbitrary value I_1. Then the power stored in the left circuit is

$$p_1 = v_1 i_1 = i_1 L_1 \frac{di_1}{dt}. \tag{A9}$$

Then the total energy will be

$$E_1 = \int p_1 dt = \int_0^{I_1} i_1 di_1 = \frac{1}{2}L_1 I_1^2. \tag{A10}$$

Now, we assume that $i_1 = I_1$ is constant, and we change i_2 from zero to I_2. Since i_2 is changing, the mutual voltage induced in the left circuit is $M di_2/dt$, and therefore, the total power will become

$$p_2 = v_2 i_2 + I_1 M\frac{di_2}{dt} = i_2 L_2\frac{di_2}{dt} + I_1 M\frac{di_2}{dt}; \tag{A11}$$

thus, the energy stored in the circuit can be written as

$$\begin{aligned}E_2 &= \int p_2 dt = L_2\int_0^{I_2} i_2 di_2 + I_1\int_0^{I_2} M di_2\\ &= \frac{1}{2}L_2 I_2^2 + M I_1 I_2.\end{aligned} \tag{A12}$$

We can write the total energy of the circuits as the sum over E_1 and E_2 such that

$$E_1 + E_2 = \tfrac{1}{2}L_1 I_1^2 + \tfrac{1}{2}L_2 I_2^2 + M I_1 I_2. \tag{A13}$$

Adding the energy with respect to the capacitors to this energy, we can write the Hamiltonian as

$$H = \frac{q_1^2}{2C_1} + \frac{q_2^2}{2C_2} + \frac{1}{2}L_1 i_1^2 + \frac{1}{2}L_2 i_2^2 + M i_1 i_2. \tag{A14}$$

Above, we replaced arbitrary currents I_1 and I_2 by i_1 and i_2. We can see that M is the coupling constant between the two circuits.

To find the Hamiltonian of the two *RLC* circuits, like what we did in Eq. (A4), we attach two resistors to both ends of the coupled *LC* circuits. We can replace currents in Eq. (A14) with their flux variables by using the relation $\phi_{ll} = L_l i_l$. By doing so the Hamiltonian of this model will then become

$$\begin{aligned}H = &\frac{q_1^2}{2C_1} + \frac{q_2^2}{2C_2} + \frac{\phi_{11}^2}{2L_1} + \frac{\phi_{22}^2}{2L_2} + \frac{M}{L_1L_2}\phi_{11}\phi_{22} + \sum_{m_1}\frac{q_{m_1}^2}{2C_{m_1}}\\ &+ \frac{(\phi_{m_1} - \phi_1)^2}{2L_{m_1}} + \sum_{m_2}\frac{q_{m_2}^2}{2C_{m_2}} + \frac{(\phi_{m_2} - \phi_2)^2}{2L_{m_2}}.\end{aligned} \tag{A15}$$

Next, we will eliminate the flux terms ϕ_{ll} to write it in terms of the total flux ϕ_l. To do so we use the following relations between the fluxes:

$$\phi_{12} = M_{12} i_2 = \frac{M}{L_2}\phi_{22}, \tag{A16}$$

$$\phi_{21} = M_{21} i_1 = \frac{M}{L_1}\phi_{11}. \tag{A17}$$

These relations together with Eq. (A5) give

$$\begin{aligned}\phi_{11} &= \frac{L_1L_2}{M^2 - L_1L_2}\left(\frac{M}{L_2}\phi_2 - \phi_1\right),\\ \phi_{22} &= \frac{L_1L_2}{M^2 - L_1L_2}\left(\frac{M}{L_1}\phi_1 - \phi_2\right).\end{aligned} \tag{A18}$$

We can now replace these transformations in Eq. (A15) to find the Hamiltonian of our model such that

$$\begin{aligned}H = &\frac{q_1^2}{2C_1} + \frac{q_2^2}{2C_2} + \frac{L_1L_2}{L_1L_2 - M^2}\left[\frac{\phi_1^2}{2L_1} + \frac{\phi_2^2}{2L_2} - \frac{M}{L_1L_2}\phi_1\phi_2\right]\\ &+ \sum_{m_1}\left(\frac{q_{m_1}^2}{2C_{m_1}} + \frac{(\phi_{m_1} - \phi_1)^2}{2L_{m_1}}\right)\\ &+ \sum_{m_2}\left(\frac{q_{m_2}^2}{2C_{m_2}} + \frac{(\phi_{m_2} - \phi_2)^2}{2L_{m_2}}\right).\end{aligned} \tag{A19}$$

APPENDIX B: STEADY-STATE HEAT CURRENTS

Using the Heisenberg equations of motion, we can find the integro-differential equation for each variable of the system, and then they can be solved using the Green's function matrix of the system $g(t,t')$, which satisfies the integro-differential equation,

$$C\frac{\partial^2}{\partial t^2}g(t,t') + L_0^{-1}g(t,t') + \int_0^t \gamma(t-\tau)\frac{\partial}{\partial\tau}g(\tau,t')d\tau = \delta(t-t'), \tag{B1}$$

with the initial condition $g(0,t')=0$. Note that the term involving the dissipation kernel $\gamma(t)$ is nonlocal in time and takes into account the non-Markovian effects induced by the environment. This kernel is given by

$$\gamma(t) = \sum_\alpha \int_0^\infty d\omega \frac{I_\alpha(\omega)}{\omega}\cos\omega t, \tag{B2}$$

where, for the Markovian Ohmic dissipation, this kernel will become a δ function which is time local.

Assuming that the Green's function $g(t,t')$ is an exponentially decaying function with respect to t, the correlation functions between the system variables, i.e., ϕ and q, will be independent of the initial state of the total system for large t. To capture the correlation functions, we use the covariance matrix σ such that

$$\sigma = \begin{bmatrix}\sigma^{(\phi,\phi)} & \sigma^{(\phi,q)}\\ \sigma^{(q,\phi)} & \sigma^{(q,q)}\end{bmatrix}. \tag{B3}$$

In terms of the Green's function $g(t,t')$, one can obtain

$$\sigma^{(n,m)}(t) = \frac{\hbar}{2}\int_0^t\int_0^t g^{(n)}(t,t_1)\nu_\alpha(t_1-t_2)g^{(m)}(t,t_2)^T dt_1dt_2, \tag{B4}$$

where

$$\nu_\alpha(t) = \int_0^\infty d\omega I_\alpha(\omega)\cos(\omega t)\coth\left(\frac{\hbar\beta_\alpha\omega}{2}\right) \tag{B5}$$

denotes the noise kernel. Also, $\sigma^{(0,0)}=\sigma^{(\phi,\phi)}$, $\sigma^{(0,1)}=\sigma^{(\phi,q)}$, $\sigma^{(1,1)}=\sigma^{(q,q)}$, and $g^{(n)}$ is the nth derivative of g. Considering a situation in which the spectral density is a continuous function of ω, we can write the covariance matrix for the steady-state limit, i.e., $t\to\infty$, such that

$$\sigma^{(n,m)} = \mathrm{Re}\int_0^\infty \frac{\hbar}{2}\omega^{n+m}i^{n-m}g(i\omega)\nu_\alpha(\omega)g(-i\omega)^T C d\omega, \tag{B6}$$

where $\sigma^{(n,m)}$ is the covariance matrix in the asymptotic state, $\nu_\alpha(\omega)$ is the Fourier transform of the noise kernel, and $g(s)$ is the Laplace transform of the Green's function, which can be obtained using Eq. (B1) such that

$$g(s)^{-1} = Cs^2 + \gamma(s)s + L_0^{-1}, \tag{B7}$$

where $\gamma(s)$ is the Laplace transform of $\gamma(t)$.

Now we turn to analyze the heat flow through the system. Since there exists no external drive in the system, the heat current is directly related to the change in the mean value of the energy of the system, and for the steady state one can write

$$\dot{Q}_\alpha = \mathrm{Tr}[P_\alpha L^{-1}\sigma^{(\phi,q)}(t)C^{-1}]. \tag{B8}$$

To calculate the local heat current, we first write $\sigma^{(\phi,q)}(t)$ by using

$$\nu_\alpha(t_1-t_2) = \mathrm{Re}\int_0^\infty \nu_\alpha(\omega)e^{-i\omega(t_1-t_2)}d\omega, \tag{B9}$$

where $\nu_\alpha(\omega) = I_\alpha(\omega)\coth(\frac{\hbar\beta_\alpha\omega}{2})$. Replacing this in Eq. (B4), we can see in the limit $t\to\infty$ we can write

$$\int_0^\infty g(t,t_1)e^{-i\omega t_1}dt_1 = g(i\omega); \tag{B10}$$

thus, we have

$$\sigma^{\phi q}(t) = -\,\mathrm{Re}\int_0^\infty \frac{\hbar}{2}g(i\omega)\nu_\alpha(\omega)i\omega g(-i\omega)^T C d\omega. \tag{B11}$$

Inserting this equation into Eq. (B8), we have the local heat current expression for the steady-state limit,

$$\dot{Q}_\alpha = \frac{\hbar}{2}\sum_{\alpha'}\int_0^\infty \omega f_{\alpha\alpha'}(\omega)\coth\left(\frac{\hbar\beta_{\alpha'}\omega}{2}\right)d\omega, \tag{B12}$$

where we have used the fact that $\mathrm{Re}(-iX) = \mathrm{Im}(X)$. The heat transfer matrix element $f_{\alpha\alpha'}$ is written such that

$$f_{\alpha\alpha'}(\omega) = \mathrm{Im}\mathrm{Tr}[P_\alpha L^{-1}g(i\omega)I_{\alpha'}(\omega)g(-i\omega)^T]. \tag{B13}$$

Here, we have $P_\alpha L^{-1} = P_\alpha L_0^{-1} + L_\alpha^{-1}$. Placing this relation into the above equation, we can see that $\mathrm{Tr}[P_\alpha L_\alpha^{-1}g(i\omega)I_{\alpha'}(\omega)g(-i\omega)^T]=0$ because L_α^{-1} is a symmetric matrix and $g(i\omega)I_{\alpha'}(\omega)g(-i\omega)^T$ is antisymmetric, and the trace of their product will be vanishing. That said, we can write the heat transfer matrix as

$$f_{\alpha\alpha'}(\omega) = \mathrm{Im}\mathrm{Tr}\big[P_\alpha L_0^{-1}g(i\omega)I_{\alpha'}(\omega)g(i\omega)^\dagger\big]. \tag{B14}$$

To expand the above relation a bit further, we first take the Laplace transform of Eq. (B1) such that

$$g(s)^{-1} = Cs^2 + \gamma(s)s + L_0^{-1}. \tag{B15}$$

Writing L_0^{-1} in terms of $g(s)^{-1}$, we have

$$L_0^{-1} = g(s)^{-1} - Cs^2 + \gamma(s)s. \tag{B16}$$

Placing the above equation into Eq. (B14) with $s=i\omega$, we have

$$\begin{aligned} f_{\alpha\alpha'}(\omega) = {} & \mathrm{Im}\mathrm{Tr}[P_\alpha I_{\alpha'}(\omega)g(i\omega)^\dagger] \\ & + \mathrm{Im}\,\omega^2\mathrm{Tr}[CP_\alpha g(i\omega)I_{\alpha'}(\omega)g(i\omega)^\dagger] \\ & + \mathrm{Im}\,i\omega\mathrm{Tr}[P_\alpha\gamma(i\omega)g(i\omega)I_{\alpha'}(\omega)g(i\omega)^\dagger]. \end{aligned} \tag{B17}$$

The first term vanishes because $P_\alpha I_{\alpha'}(\omega)=0$ for $\alpha\neq\alpha'$. The second will also be vanishing because it is a product of two symmetric and antisymmetric matrices. In the third term, the matrix $g(i\omega)I_{\alpha'}(\omega)g(i\omega)^\dagger$ is Hermitian, so we have to calculate only $\mathrm{Im}[i\omega\gamma(i\omega)] = \mathrm{Re}[\omega\gamma(i\omega)] = \frac{\pi}{2}I(\omega)$. Thus, we have

$$f_{\alpha\alpha'}(\omega) = \frac{\pi}{2}\mathrm{Tr}[I_\alpha(\omega)g(i\omega)I_{\alpha'}(\omega)g(i\omega)^\dagger]. \tag{B18}$$

Inserting this matrix back into Eq. (B12), we have

$$\dot{Q}_\alpha = \frac{\hbar}{2}\sum_{\alpha'\neq\alpha}\int_0^\infty \omega d\omega f_{\alpha\alpha'}(\omega)\left[\coth\left(\frac{\hbar\beta_\alpha\omega}{2}\right) - \coth\left(\frac{\hbar\beta_{\alpha'}\omega}{2}\right)\right]. \tag{B19}$$

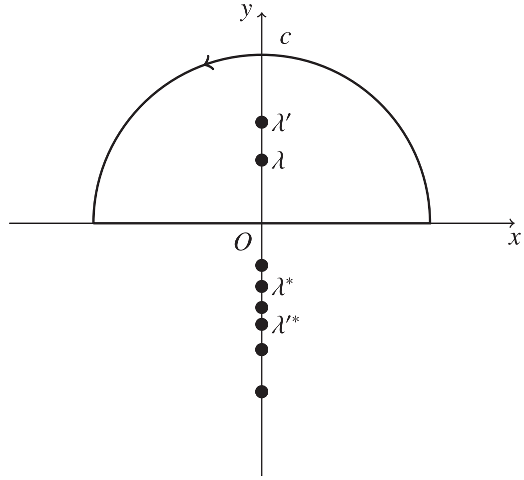

FIG. 7. Upper half plane contour. The dots are the poles of the digamma function.

APPENDIX C: QUANTUM CORRECTION TO THE HEAT CURRENT

To calculate the quantum contribution to the heat currents we will analytically solve the integral in Eq. (26). To do so, we will have to take into account the poles of the digamma function in addition to the poles of $f_{12}(\omega)$. In fact the poles of the function $\psi(1-ix)$ are all located on the lower half of the imaginary axis, i.e., $x=-i,-2i,-3i,\dots$. The poles of the heat transfer matrix element $\lambda_\pm$ and their conjugates $\lambda_\pm^*$ are on the imaginary axis. However, since we would like to exclude the contribution from the digamma function poles, we choose the integration contour to run on the upper half plane, which covers only $\lambda_\pm$. Thus, we can write the integral such that

$$\dot{Q}_1^{\mathrm{q}}=\frac{i\hbar}{2\pi}\int_c d\omega\,\omega\,f_{12}(\omega)\left[\psi\left(1-\frac{i\beta_2\hbar\omega}{2\pi}\right)-\psi\left(1-\frac{i\beta_1\hbar\omega}{2\pi}\right)\right]+\frac{i\hbar}{2\pi}\int_\infty d\omega\,\omega\,f_{12}(\omega)\left[\psi\left(1-\frac{i\beta_2\hbar\omega}{2\pi}\right)-\psi\left(1-\frac{i\beta_1\hbar\omega}{2\pi}\right)\right]. \tag{C1}$$

The first integral is done over the contour c in Fig. 7 by using the residue theorem. The second integral is the contribution for $\omega\to\infty$. In this limit we need to expand the digamma function using

$$\psi(1\pm ix)\simeq\ln(\pm ix)\mp\frac{i}{x} \tag{C2}$$

for $x\to\infty$. Since the integrand is vanishing as $1/\omega$, we need to keep only the logarithmic term in the asymptotic digamma functions. Placing this expansion into the second integral in Eq. (C1), we have

$$\frac{i\hbar}{2\pi}\int_\infty d\omega\,\omega\,f_{12}(\omega)\left[\psi\left(1-\frac{i\beta_2\hbar\omega}{2\pi}\right)-\psi\left(1-\frac{i\beta_1\hbar\omega}{2\pi}\right)\right]=-\frac{i\hbar}{\pi}\left(\frac{M}{L}\right)^2\left(\frac{\lambda_+\lambda_-}{\omega_d}\right)^2\int d\omega\frac{1}{\omega}\ln\left(\frac{\beta_1}{\beta_2}\right). \tag{C3}$$

We change the variable $\omega=\Lambda e^{i\theta}$, and we integrate over the semicircle on the upper half plane for $0\leqslant\theta\leqslant\pi$ and $\Lambda\to\infty$; thus, we have

$$\frac{i\hbar}{2\pi}\int_\infty d\omega\,\omega\,f_{12}(\omega)\left[\psi\left(1-\frac{i\beta_2\hbar\omega}{2\pi}\right)-\psi\left(1-\frac{i\beta_1\hbar\omega}{2\pi}\right)\right]=\frac{\hbar}{\pi}\left(\frac{M}{L}\right)^2\left(\frac{\lambda_+\lambda_-}{\omega_d}\right)^2\ln\left(\frac{T_2}{T_1}\right). \tag{C4}$$

Hence, by adding the above result and the integral over the contour we obtain Eq. (26).

APPENDIX D: WEAK-COUPLING LIMIT

To study the weak-coupling limit of the problem at low temperatures, we still use the Ohmic dissipation but in the Markovian limit ($\omega_c\to\infty$). In that case the spectral density and the dissipation kernel simplify into

$$I_\alpha(\omega)=\frac{2}{\pi}\frac{\omega}{R_\alpha}P_\alpha, \tag{D1}$$

$$\gamma(s)=\frac{1}{R}(P_1+P_2). \tag{D2}$$

Placing the above equations into the heat transfer matrix element $f_{\alpha\alpha'}(\omega)$, we have

$$f_{1,2}(\omega) = \frac{2}{\pi}\left(\frac{M}{L}\right)^2 \frac{\omega^2}{\omega_d^2}(\omega_+\omega_-)^2 \frac{1}{|u_+(i\omega)\, u_-(i\omega)|^2}, \tag{D3}$$

where in the weak-coupling regime we have

$$u_\pm(\omega) = -\frac{1}{\gamma}\left(\omega - \frac{i\gamma}{2} + \sqrt{\gamma\omega_\pm}\right)\left(\omega - \frac{i\gamma}{2} - \sqrt{\gamma\omega_\pm}\right). \tag{D4}$$

Employing the heat transfer matrix element, we may calculate the classical and quantum contributions to the heat currents in Eqs. (25) and (26) using the same method as in Appendix C. Nevertheless, we are interested in the low-temperature limit of the heat currents. Therefore, the classical contribution will be vanishing, and the heat current up to the first nonvanishing order of the quantum correction can be written as

$$\dot{Q}_1^{\text{low}} = \frac{512}{15}\left(\frac{\pi}{\hbar}\right)^3 \left(\frac{M}{L}\right)^2 \frac{k_b^4}{\omega_d^2}\left(\frac{\omega_+\omega_-}{(\gamma+4\omega_+)(\gamma+4\omega_-)}\right)^2 \left(T_1^4 - T_2^4\right) + O\left(T_{1/2}^6\right), \tag{D5}$$

where in the weak-coupling limit of $\gamma \ll \omega_\pm$ the above expression is the same as (36).

[1] R. Hannes and F. Till, *The Fokker-Planck Equation* (Springer, Berlin, 1996).
[2] M. Esposito and F. Haake, Phys. Rev. A **72**, 063808 (2005).
[3] P. Pechukas, J. Ankerhold, and H. Grabert, Ann. Phys. (Berlin, Ger.) **9**, 794 (2000).
[4] J. Ankerhold, P. Pechukas, and H. Grabert, Phys. Rev. Lett. **87**, 086802 (2001).
[5] R. Dillenschneider and E. Lutz, Phys. Rev. E **80**, 042101 (2009).
[6] M. Esposito, M. A. Ochoa, and M. Galperin, Phys. Rev. Lett. **114**, 080602 (2015).
[7] M. Esposito, K. Lindenberg, and C. Van den Broeck, New J. Phys. **12**, 013013 (2010).
[8] M. Esposito, M. A. Ochoa, and M. Galperin, Phys. Rev. B **92**, 235440 (2015).
[9] N. Bergmann and M. Galperin, arXiv:2004.05175.
[10] A. Bruch, M. Thomas, S. Viola Kusminskiy, F. von Oppen, and A. Nitzan, Phys. Rev. B **93**, 115318 (2016).
[11] G. Katz and R. Kosloff, Entropy **18**, 186 (2016).
[12] U. Seifert, Phys. Rev. Lett. **116**, 020601 (2016).
[13] N. Freitas and J. P. Paz, Phys. Rev. E **95**, 012146 (2017).
[14] M. Perarnau-Llobet, H. Wilming, A. Riera, R. Gallego, and J. Eisert, Phys. Rev. Lett. **120**, 120602 (2018).
[15] A. Bruch, C. Lewenkopf, and F. von Oppen, Phys. Rev. Lett. **120**, 107701 (2018).
[16] W. Dou, M. A. Ochoa, A. Nitzan, and J. E. Subotnik, Phys. Rev. B **98**, 134306 (2018).
[17] P. Haughian, M. Esposito, and T. L. Schmidt, Phys. Rev. B **97**, 085435 (2018).
[18] P. Strasberg, Phys. Rev. E **100**, 022127 (2019).
[19] E. A. Martinez and J. P. Paz, Phys. Rev. Lett. **110**, 130406 (2013).
[20] N. Freitas and J. P. Paz, Phys. Rev. E **90**, 042128 (2014).
[21] U. Vool and M. H. Devoret, Int. J. Circuit Theory Appl. **45**, 897 (2017).
[22] S. M. Girvin, *Circuit QED: Superconducting Qubits Coupled to Microwave Photons* (Oxford University Press, Oxford, 2014).
[23] C. Gaul and H. Büttner, Phys. Rev. E **76**, 011111 (2007).
[24] A. Asadian, D. Manzano, M. Tiersch, and H. J. Briegel, Phys. Rev. E **87**, 012109 (2013).
[25] F. Nicacio, A. Ferraro, A. Imparato, M. Paternostro, and F. L. Semião, Phys. Rev. E **91**, 042116 (2015).
[26] N. Freitas, E. A. Martinez, and J. P. Paz, Phys. Scr. **91**, 013007 (2015).
[27] A. Polkovnikov, Ann. Phys. (NY) **325**, 1790 (2010).
[28] D. R. Schmidt, R. J. Schoelkopf, and A. N. Cleland, Phys. Rev. Lett. **93**, 045901 (2004).
[29] A. V. Timofeev, M. Helle, M. Meschke, M. Möttönen, and J. P. Pekola, Phys. Rev. Lett. **102**, 200801 (2009).
[30] P. S. Riseborough, P. Hanggi, and U. Weiss, Phys. Rev. A **31**, 471 (1985).
[31] J. Pendry, J. Phys. A **16**, 2161 (1983).
[32] A. Greiner, L. Reggiani, T. Kuhn, and L. Varani, Phys. Rev. Lett. **78**, 1114 (1997).
[33] F. Mascherpa, A. Smirne, A. D. Somoza, P. Fernández-Acebal, S. Donadi, D. Tamascelli, S. F. Huelga, and M. B. Plenio, Phys. Rev. A **101**, 052108 (2020).
[34] L. M. A. Pascal, H. Courtois, and F. W. J. Hekking, Phys. Rev. B **83**, 125113 (2011).
[35] J. P. Pekola, Nat. Phys. **11**, 118 (2015).
[36] A. Ronzani, B. Karimi, J. Senior, Y.-C. Chang, J. T. Peltonen, C. Chen, and J. P. Pekola, Nat. Phys. **14**, 991 (2018).
[37] H. Li, L. J. Fernández-Alcázar, F. Ellis, B. Shapiro, and T. Kottos, Phys. Rev. Lett. **123**, 165901 (2019).
[38] O. Maillet, D. Subero, J. T. Peltonen, D. S. Golubev, and J. P. Pekola, Nat. Commun. **11**, 4326 (2020).