Destabilization of U(1) Dirac spin liquids on two-dimensional nonbipartite lattices by quenched disorder

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The stability of the Dirac spin liquid on two-dimensional lattices has long been debated. It was recently demonstrated [Nat. Commun. **10**, 4254 (2019) and Phys. Rev. B **93**, 144411 (2016)] that the staggered π-flux Dirac spin-liquid phase on the nonbipartite triangular lattice may be stable in the clean limit. However, quenched disorder plays a crucial role in determining whether such a phase is experimentally viable. For SU(2) spin systems, the effective zero-temperature low-energy description of Dirac spin liquids in $(2+1)$ dimensions is given by the compact quantum electrodynamics (cQED_{2+1}) which admits monopoles. It is already known that generic quenched random perturbations to the noncompact version of QED_{2+1} (where monopoles are absent) lead to strong-coupling instabilities. In this paper we study cQED_{2+1} in the presence of a class of time-reversal invariant quenched disorder perturbations. We show that in this model, random non-Abelian vector potentials make the symmetry-allowed monopole operators more relevant. The disorder-induced underscreening of monopoles, thus, generically makes the gapless spin-liquid phase fragile.

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I. INTRODUCTION

Quantum spin liquids with their topologically ordered ground states, fractionalized excitations, and long-range entanglements offer a fascinating insight into many-body quantum correlations [1–3]. Experimentally, the observation of a spin-liquid phase has been fraught with the complications arising from spatial inhomogeneities in real materials, which often leads to symmetry breaking towards a spin-glass ground state [4–6]. Although the role of quenched disorder in a frustrated spin system may vary considerably [7–10], in a number of examples [11–13] it has been shown that the topological properties of frustrated systems are considerably affected by quenched disorder. In this paper we consider the gapless Dirac spin-liquid state with $2N$ flavors of matter fermions and compact U(1) gauge symmetry and investigate its stability in the presence of random gauge fluctuations.

As a prototypical spin-liquid state with linearly dispersing gapless fractionalized spinons and minimally coupled compact U(1) gauge fields, the Dirac spin-liquid state has been discussed as a parent state for different competing orders [14,15], deconfined quantum critical points between topological phases [16], and as the prospective ground states of the kagome lattice Heisenberg model [17,18] and the triangular lattice Heisenberg model with next-nearest-neighbor exchange interaction [19]. The variational Dirac spin-liquid state can be derived from the mean-field decomposition of the SU(2) Heisenberg Hamiltonian,

$$H=\sum_{i,j} J_{ij}\vec{S}_i\cdot\vec{S}_j, \tag{1}$$

in terms of fermionic spinons. Here, J_{ij} are exchange couplings between nearest- and next-nearest-neighbor spins. In this picture, a spin-1/2 operator at site i is rewritten as $\vec{S}_i=(1/2)f^\dagger_{i,\alpha}\vec{\sigma}_{\alpha\beta}f_{i,\beta}$ with the physical constraint $\sum_\alpha f^\dagger_{i,\alpha}f_{i,\alpha}=1$. Here $f_{i,\alpha}$ are fractionalized fermionic spinons with $\alpha=\uparrow,\downarrow$ being the spin indices. The mean-field decomposition with bond variables $t_{ij}=-\langle f^\dagger_{i,\alpha}f_{j,\alpha}\rangle$ reduces the Hamiltonian to the quadratic form

$$H_{\mathrm{MF}}=\sum_{i,j}\frac{J_{ij}}{2}[|t_{ij}|^2+(t_{ij}f^\dagger_{i,\alpha}f_{j,\alpha}+\mathrm{H.c.})], \tag{2}$$

with the mean-field ansatz of bond variables t_{ij} chosen suitably to minimize the variational energy. In the spinon decomposition, the compact U(1) gauge symmetry is manifest with the transformation $f_{i,\alpha}\to e^{iA_i}f_{i,\alpha}$, which ultimately leads to the emergence of dynamical U(1) gauge field fluctuations. The emergent gauge group is necessarily compact as it is a subgroup of the larger compact SU(2) gauge group related to the fractionalization of the physical SU(2) spins [1,20].

On honeycomb, kagome, and triangular lattices, a Dirac dispersion band of the spinons are realized with a suitable choice of nearest-neighbor mean-field parameter t_{ij}. However, the fluctuations around the mean-field state may destablize it and in that case the mean-field spin-liquid state does not correspond to a physical state of the original spin Hamiltonian. In the triangular lattice the nearest-neighbor π-flux mean-field ansatz with no fluxes through the lower triangular plaquettes, $\prod_{(ij)\in\nabla}t_{ij}=1$ and π fluxes through the upper triangular plaquettes $\prod_{(ij)\in\triangle}t_{ij}=-1$ yield a Dirac spin-liquid state with four gapless Dirac cones, i.e., two Dirac nodes (valleys) for each spin flavor. Variational Monte Carlo studies [19,21] have indicated that on the J_1-J_2 next-nearest-neighbor triangular lattice Heisenberg antiferromagnet, the Dirac spin-liquid state is energetically favorable compared to other magnetically

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ordered and spin-liquid type states across a certain region of the parameter space. As a clear demonstration of the stability (against fluctuations) and energetic favorability of the U(1) Dirac spin liquid awaits for other lattices, we presently focus on the microscopic realization of the Dirac spin-liquid ground state of the triangular lattice and consider quenched random perturbation to its clean limit. However, as our disorder study is performed in the continuum, the findings are applicable to all Dirac spin-liquid ground states with compact U(1) gauge field fluctuations.

At zero temperature and in the long-wavelength low-energy limit, the Dirac spin-liquid state and its gauge fluctuations are described by the action of the $(2+1)$-dimensional compact quantum electrodynamics [15,22],

$$S_{\mathrm{cQED}} = \int d\tau\, d^2r \left[\bar{\psi}_i \gamma^\mu (\partial_\mu + iA_\mu)\psi_i + \frac{1}{4e^2} {F_{\mu\nu}}^2 \right], \quad (3)$$

where ψ are $2N$ copies of two-component fermionic fields which are descendants of the fermionic spinors $f_{i,\alpha}$ with $i \in 1, \ldots, 2N$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the usual field strength tensor for a Maxwell gauge theory. The number N is determined by the number of Dirac nodes of the microscopic dispersion per spin, i.e., $N = 2$ for the triangular lattice Dirac spin liquid. In the following, we will suppress all the fermionic flavor indices. Here the three Dirac γ matrices γ^μ are taken to be two-component [23,24], and they obey the usual Clifford algebra, $\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu} I_2$. The gauge charge has scaling dimension $[e^2] = +1$ and it flows to infinity in the deep infrared. Consequently, the infrared fixed point of the action as written has conformal symmetry in the large-N limit [25]. The action also has an emergent SU($2N$) symmetry under which the fermions ψ transform as vectors.

From the lattice regularization, the gauge fields A_μ are 2π-periodic compact variables and, therefore, it can be shown that the cQED_{2+1} action must admit monopole operators of charge q which insert $4\pi q$ units of magnetic flux locally [20,26,27]. It was shown originally by Ref. [20] that the proliferation of these monopoles strongly confines the electric charges of the pure compact Maxwell gauge theory. The argument follows from considering a dilute gas of elementary monopoles $q = \pm 1/2$ which can be described by the three-dimensional sine-Gordon model in the continuum limit [25,26],

$$S_{\mathrm{sG}} = \int d^3r \left[\frac{1}{2}\left(\frac{e}{2\pi}\right)^2 (\partial_\mu \chi)^2 - 2y \cos\, \chi \right]. \quad (4)$$

In the Hamiltonian picture, the operator $e^{i\chi}$ adds 2π magnetic flux and creates a monopole operator in three space-time dimensions. Thus, following the Coulomb gas picture y is interpreted as the the monopole fugacity. In the absence of any matter coupling the monopole fugacity with the scaling dimension, $[y] = 3$ flows to strong coupling in the infrared and confines the pure gauge theory. However, in the presence of matter fields coupling the Coulomb interaction between the monopole charges are screened with a modified renormalization of the monopole fugacity [25],

$$\frac{dy}{dl} = (3 - \Delta_{\mathcal{M}})y. \quad (5)$$

Here, $\Delta_{\mathcal{M}}$ is the effective scaling dimension of the monopole creation operator $e^{i\chi}$ in the presence of the matter coupling. The relevance of the monopole fugacity operator now depends on whether $\Delta_{\mathcal{M}}$ is less than space-time dimension 3.

The fate of the compact U(1) gauge theories minimally coupled to gapless fermion spinons with Dirac dispersion has been controversial [28,29]. However, it has been shown the (anomalous) scaling dimension of the monopole operator in the presence of a large (even) number of $2N$ fermionic spinons grows as $\Delta_{\mathcal{M}} \propto 2N$, indicating a stable deconfined phase sufficiently large fermionic flavors [30,25,26]. In particular, within a large-$2N$ approximation the scaling dimension of the monopole operators is found to be of the form [31]

$$\Delta_{\mathcal{M}^{(q)}} = (2N)\lambda_0^{(q)} + \lambda_1^{(q)} + O(1/2N). \quad (6)$$

For the lowest charge $q = 1/2$ monopole operators, computation using state-operator correspondence [27,32] have yielded, $\lambda_0^{(1/2)} = 0.265$ and $\lambda_1^{(1/2)} = -0.0383$. Therefore, if the monopole operators of the lowest charge are allowed, the minimum number of fermionic flavors needed to avoid confinement is $2N_C \geqslant 12$. This number is more than the number of fermionic flavors obtained in the known mean-field Dirac spin-liquid states ($2N = 4$ for the kagome [18] and triangular lattice [19]; and $2N = 4$ and $2N = 8$, respectively, for the staggered and π-flux Dirac spin-liquid states in square lattices [15,33]). However, for the Dirac spin-liquid states in nonbipartite triangular and kagome lattice geometries, it has been recently shown that the monopole operators of the lowest charges are prohibited by lattice symmetries [24,34]. Reference [24] demonstrated that for the triangular lattice Dirac spin liquid only monopole operators with charges $q \geqslant 3/2$ are allowed by microscopic symmetries and these higher charge monopole operators are all irrelevant if the large-$2N$ approximated monopole scaling dimension $\Delta_{\mathcal{M}^{(q)}}$ [32] is extrapolated to $2N = 4$. This indicates the possibility of a stable deconfined Dirac spin-liquid phase in the triangular lattice. The same analysis found that for the kagome lattice the smallest allowed monopole operators are very close to being marginal but relevant within the large-$2N$ approximation. Indeed, on the triangular lattice next-nearest-neighbor J_1-J_2 Heisenberg model the Dirac spin-liquid phase is found to be stable and energetically favorable in variational Monte Carlo simulations [19,21] and density matrix renormalization (DMRG) calculations [35]. A spin-liquid phase was found to be stable for $0.07 < J_2/J_1 < 0.15$ by a separate DMRG study [36]. In other reports [37,38], a chiral spin liquid is found in the same parameter range in the presence of a time-reversal symmetry-breaking perturbation, which is consistent with a viable Dirac spin-liquid phase in the time-reversal symmetric limit.

In our treatment, we examine the fate of the cQED_{2+1} action [Eq. (3)] as an effective theory of the Dirac spin liquid in the presence of time-reversal symmetric microscopic perturbations. Microscopically, the triangular lattice Dirac spin liquid will be our focus as a promising candidate in the clean limit. Theoretical efforts [39–41] to study QED_{2+1} in the presence of various quenched random perturbation has so far been focused on the noncompact limit which neglects the monopole operators. It has been established that random perturbations which break the time-reversal symmetry and/or break completely the emergent SU($2N$) symmetry of the

$cQED_{2+1}$ action drive a renormalization group (RG) flow to a strong disorder coupling fixed point, which in the microscopic sense indicates the destruction of the spin liquid [39]. In this paper, we, therefore, focus on time-reversal symmetric disorder which breaks the SU(2N) symmetry only partially. It has been shown that weak random perturbations which break the SU(2N) symmetry down to U(1) × SU(N) flow to a finite disorder conformal fixed line [39] and, consequently, the Dirac spin-liquid phase may be expected to survive [39,42]. In this context, we consider the compact nature of the effective theory and perturbatively calculate the disorder-induced modification to the scaling dimension of the monopole operator to further clarify the fate of the algebraic Dirac spin-liquid phase.

In Sec. II we introduce the RG marginal random couplings that we consider as a perturbation to the $e^2 \to \infty$, $N \to \infty$ conformal fixed point of the theory and discuss their microscopic origin. Adapting the state-operator correspondence method described in Sec. III, in Sec. IV we calculate the scaling dimension of the monopole operators in the dirtied $cQED_{2+1}$ within a controlled expansion in large-N and perturbative disorder strength. We find that disorder significantly reduces the scaling dimension of the monopole operators and enhances the possibility of confinement of the spinons which carry electric gauge charges. In Sec. V we consider the combined flow of the monopole fugacity and the perturbative disorder couplings and show that even when disorder in itself remains marginal, the monopole fugacity may flow to strong coupling and confine the theory. In the concluding Sec. VI, we comment on the instabilities introduced by disorder-driven spinon confinement within the context of the Dirac spin-liquid phase and argue why among other possibilities a glassy random-singlet like ground state is a likely outcome for even small to moderate disorder in this scenario.

II. QUENCHED DISORDER IN $cQED_{2+1}$

The QED_{2+1} action is an effective low-energy description and the spatial inhomogeneities in the lattice translate to random coupling perturbations to the theory. Reference [39] showed that there are no relevant random perturbations to QED_{2+1} in the large-(2N) limit and the only marginal random couplings are the various conserved currents and mass operators associated with the SU(2N) symmetry of the fermions [39]. In our discussion, we choose σ^α and τ^b as the $(2N)^2-1$ generators of SU(2N) where σ^α are 2 × 2 Pauli matrices with $\alpha = x,\ y,\ z$, and τ^b are N^2-1 traceless $N\times N$ Hermitian matrices with the normalization $\mathrm{tr}[\tau^\alpha\tau^\beta] = \delta^{\alpha\beta}/2$. The generators satisfy the usual commutation relations $[\sigma^\alpha,\sigma^\beta] = i\epsilon_{\alpha\beta\gamma}\sigma^\gamma$ and $[\tau^a,\tau^b] = if_c^{ab}\tau^c$, where f_{abc} are structure constants of the corresponding SU(N) Lie algebra. In this notation σ^α operate on the spin space and τ^β operate on the fermion-doubled valley space originating from the Dirac node structure of the parent mean-field state. Associated with these symmetry generators are SU(2N) current,

$$J_\mu^{\alpha b} = i\bar\psi\sigma^\alpha\tau^b\gamma_\mu\psi,\quad J_\mu^{\alpha 0} = i\bar\psi\sigma^\alpha\gamma_\mu\psi,\quad J_\mu^{0b} = i\bar\psi\tau^b\gamma_\mu\psi, \tag{7}$$

and mass terms,

$$M^{\alpha b} = \bar\psi\sigma^\alpha\tau^b\psi,\quad M^{\alpha 0} = \bar\psi\sigma^\alpha\psi,\quad M^{0b} = \bar\psi\tau^b\psi. \tag{8}$$

It is to be noted that terms not containing σ^α are related to the spin-singlet local (bilinear) operators of the microscopic model whereas the rest maps to the spin-triplet operators. In the clean limit the conserved SU(2N) currents, e.g., $i\bar\psi\sigma^\alpha\gamma^\mu\psi$ have the scaling dimension $\Delta = 2$ to all orders in $1/(2N)$ but the SU(2N) mass terms, e.g., $i\bar\psi\sigma^\alpha\psi$ acquire anomalous scaling dimensions $\propto 1/(2N)$ [15]. Let us consider quenched random coupling to an arbitrary operator $O(\vec r,\tau)$ such that the perturbing action is $S_{\rm dis} = \int d\tau\, d^2r\, h(\vec r)O(\vec r,\tau)$ with uncorrelated random conjugate fields $\overline{h(\vec r)h(\vec r')} = \rho_O\delta^{(2)}(\vec r-\vec r')$. Following standard replica technique, $\overline{F} = \overline{\ln Z} = \lim_{n\to 0}(Z^n-1)/n \sim \lim_{n,\to 0}\prod_{r=1}^n Z_r$, a replicated partition sum emerges

$$Z_{\rm replica} = \int \mathcal{D}[\psi_r, A_r]\exp\Bigg(-\sum_r\int d\tau\, d^2r\,\psi_r[\partial\!\!\!/ + iA\!\!\!/_r]\psi_r + \rho_O\sum_{rs}\int d\tau\, d\tau' d^2r\, O_r(\vec r,\tau)O_s(\vec r,\tau)\Bigg). \tag{9}$$

From power counting it clearly follows that $\Delta_{\rho_O} = 2+2z-2\Delta_O$ where $z = -[\tau]$ is the dynamical critical exponent. Therefore, in the large-N limit when $z=1$, random couplings to the various SU(2N) current and mass terms are marginal at the tree level. Similarly, the random couplings to simple mass terms $\sim\bar\psi\psi$ are also RG marginal but such mass terms break time-reversal symmetry in the (2 + 1) dimension. Quenched disorder breaks Lorentz invariance and, consequently, the scaling dimension of both the random SU(2N) mass and current disorder couplings are modified beyond the tree level. In the absence of monopoles, previous works [39–41,43] have established that if such random couplings break the fermionic SU(2N) symmetry or the time-reversal symmetry the combined RG flow generically moves to a strong-coupling fixed point. However, Ref. [39] has shown that for time-reversal symmetric random perturbations, if the symmetry is only partially broken to U(1) × SU(N), a finite disorder conformal fixed line is obtained, parametrized by the corresponding coupling strengths. Technically, this fixed line is demarcated by the breakdown of the microscopic SU(2) symmetry down to U(1).

In keeping with the goal of calculating the scaling dimension of the monopole operators by invoking the state-operator correspondence of radial quantization [44], we will presently only consider the SU(N) symmetric random current (RC) perturbations,

$$S_{\rm dis} = \int d\tau\, d^2r\, V_{\alpha j}(\vec r) i\bar\psi\sigma^\alpha\gamma^j\psi(\vec r,\tau),\qquad P[V] = \exp\left[-\frac{1}{2\rho_\alpha}\int d^2r\, V_{\alpha j}^2(\vec r)\right], \tag{10}$$

where a Gaussian distribution for the conjugate random field $V_{\alpha j}$ has been considered for convenience with ρ_α being the corresponding disorder strength. The index j of Dirac matrices here runs strictly over the spatial components as the disorder is static in space.

Microscopically, the time-reversal invariant local random perturbations are usually either random bond type $P_{ij} = \vec{S}_i \cdot \vec{S}_j$ or vector-chirality type $\vec{C}_{ij} = \vec{S}_i \times \vec{S}_j$. With time-reversal invariance, the former behaves as scalars in spin space and, therefore, is associated with the spin-singlet mass and current terms J_j^{0b}, M^{0b}, and random Abelian vector potentials, whereas the latter is associated with the spin-triplet mass and current terms $J_j^{\alpha 0}$ and $M^{\alpha 0}$. Although the random Abelian vector potential is known to be an irrelevant perturbation for noncompact QED_{2+1} [39,41], Ref. [39] showed that the random spin-singlet SU($2N$) current and mass terms are, however, relevant perturbations, and, therefore, it can be surmised that random bondlike perturbations are destructive to the Dirac spin-liquid phase. On the other hand, the same treatment revealed the presence of a fixed line for U(1) × SU(N) symmetric random couplings to the spin-triplet terms $J_j^{\alpha 0}$ and $M^{\alpha 0}$, the former of which we presently consider. In (2 + 1) dimensions random the random current terms, $J_j^{\alpha b} = i\bar{\psi}\sigma^\alpha\tau^b\gamma_j\psi$ preserve the time-reversal symmetry [39]. It is to be noted that we are only considering time-reversal symmetric static disorder in this treatment, the origin of which lies in nonmagnetic structural impurities.

III. MONOPOLE SCALING DIMENSION OF CLEAN cQED_{2+1}

In the absence of monopole operators, the cQED_{2+1} action [Eq. (3)] has an additional topological symmetry $\text{U}(1)_{\text{topo}}$ attributed to the conserved current $J^\mu = (1/2\pi)\epsilon^{\mu\nu\lambda}\partial_\nu A_\lambda$. However, there exists stable static and singular gauge field configurations which carry q units of the $\text{U}(1)_{\text{topo}}$ charge. These are the monopole operators that spontaneously breaks the topological symmetry to create $4\pi q$ magnetic flux locally while satisfying the Dirac quantization constraint $2q \in \mathbb{Z}$ [20]. Although these are local operators, they cannot be constructed as polynomials of the fundamental fields of the theory, which makes it difficult to calculate their scaling dimension using direct methods of Feynman diagrams. However, the $e^2 \to \infty$, $N \to \infty$ fixed point of cQED_3 is conformal, and for conformal field theories (CFT) the scaling dimension of local operators can be determined using state-operator correspondence of the radial quantization picture.

For a D-dimensional quantum field theory, the usual quantization has the Hilbert space of states defined on a $(D-1)$-dimensional subspace with the remaining direction involving the time evolution generated by the Hamiltonian. In radial quantization the states are defined on concentric $(D-1)$-dimensional spheres of varying radii with the radial evolution generated by the dilatation operator $D = -ix^\mu\partial_\mu$. It further follows that a local operator O of a CFT inserted at the origin of flat $\mathbb{R}^3$ space-time has a one-to-one correspondence to normalizable states of the CFT on $S^2 \times \mathbb{R}$. For a CFT with a trivial vacuum $|0\rangle$, it can be shown that a state $|\Delta_O\rangle$, created by a conformal primary operator $O(0)|0\rangle = |\Delta_O\rangle$ at the origin is an eigenstate of the dilatation generator [44],

$$D|\Delta_O\rangle = i\Delta_O|\Delta_O\rangle. \tag{11}$$

Following a cylindrical transformation $\tau = \ln r$, it is easy to see that the dilatation generator plays the role of the Hamiltonian for such radial states. Therefore, the scaling dimension Δ_O of the operator on $\mathbb{R}^3$ is equal to the energy of the corresponding state on $S^2 \times \mathbb{R}$. In this scenario the energy eigenvalue of the state corresponding to the monopole operator $\mathcal{M}^{(q)}$ of charge q at the origin amounts diagonalizing the cQED_{2+1} action on $S^2 \times \mathbb{R}$ in the presence of the $4\pi q$ unit of magnetic flux [27]. However, the cQED_{2+1} action can only be diagonalized in the large-$2N$ limit with all fluctuations suppressed. For perturbations around the large-$2N$ limit, the ground-state (free) energy $F^{(q)} = -\ln Z^{(q)}_{S^2\times\mathbb{R}}$ of the flux inserted action has to be computed order by order such that the scaling dimension is obtained as [32]

$$\Delta_{\mathcal{M}^{(q)}} = F^{(q)} - F^{(0)},$$

with

$$F^{(q)} = -\ln Z^{(q)}_{S^2\times\mathbb{R}} = -\lim_{\beta\to\infty}\frac{1}{\beta}\ln Z^{(q)}_{S^2\times S^1_\beta}. \tag{12}$$

In the last line we have interpreted the ground-state energy as the zero-temperature limit ($\beta \to \infty$) of the free energy where all the spatial and temporal directions are compact [45]. The above expression for the scaling dimension subtracts a potentially divergent background free energy in the absence of any monopoles which does not affect physical quantities.

We have to consider the cQED_{2+1} action in the curved $S^2 \times \mathbb{R}$ space-time. From the Euclidean signature the vierbein e^a_μ can be introduced to get a curved space metric $g_{\mu\nu} = e^a_\mu e^a_\nu$. Eliminating any spin connection by performing appropriate unitary rotation, the cQED_{2+1} action in the curved space-time can be written as

$$S_{\text{cQED}} = \int d^3r\sqrt{g}\bar{\psi}e^\mu_a\gamma^a[\partial_\mu + iA_\mu]\psi, \tag{13}$$

where γ^a are the three spinor matrices defined on the flat space-time and $\sqrt{g}$ is shorthand for the square root of the metric determinant $\sqrt{\det g_{\mu\nu}}$. It is introduced to define the coordinate invariant volume measure $d^3r\sqrt{g}$. Insertion of a monopole of charge q amounts to embedding a q unit of magnetic flux at the origin by introducing a singular gauge field configuration. The static gauge field contribution due to the monopole at the center is $\vec{\mathcal{A}}_q = \frac{q}{2}\frac{1-\cos\theta}{r\sin\theta}\hat{e}_\phi$. Mapping to the cylindrical space-time $\mathbb{S}^2 \times \mathbb{R}$ with the metric $ds^2 = g_{\mu\nu}dr^\mu dr^\nu = d\tau^2 + (d\theta^2 + \sin\theta^2 d\phi^2)$ from the usual spherical coordinates in $\mathbb{R}^3$ with the metric $ds^2 = dr^2 + r^2(d\theta^2 + \sin\theta^2 d\phi^2)$ is obtained by putting $r = e^\tau$ and performing a Weyl rescaling,

$$\begin{aligned} g_{\mu\nu} &\to e^{-2\tau}g_{\mu\nu}, \qquad \psi, \bar{\psi} \to e^{-\tau}\psi, e^{-\tau}\bar{\psi}, \\ e^\mu_a &\to e^{-\tau}e^\mu_a, \qquad A_\mu \to A_\mu. \end{aligned} \tag{14}$$

The transformed Dirac operator in the presence of a magnetic monopole of charge q in the cylindrical space-time is given by [27]

$$\not{D} = \gamma_r\left[\frac{\partial}{\partial\tau} - \left(J^2 - L^2 + \frac{1}{4}\right) + q\gamma_r\right], \tag{15}$$

where $\gamma_r = \hat{r}\cdot\vec{\gamma}$. Here, $\vec{J}$ and $\vec{L}$ are the generalized total and orbital angular momenta, respectively, in the presence of the monopole magnetic flux. At this level dynamical contributions towards the gauge fields are ignored. This is strictly valid in the large-N limit and their subleading effect can be incorporated back within a controlled $1/(2N)$ expansion [32].

Following earlier work by Ref. [46], it was shown by Ref. [27] that the Dirac operator in the presence of a monopole generated background gauge field can be diagonalized by a special monopole harmonics basis. In the presence of a monopole of charge q, the monopole harmonics are defined as $L^2 Y_{q,lm} = l(l+1)Y_{q,lm},\ L_z Y_{q,lm} = mY_{q,lm}$ with $l = |q|, |q|+1, |q|+2, \dots$ and $m = -l, \dots, l$. The Dirac equation is not diagonal in the monopole harmonics basis. Instead a basis involving two separate modes of the total angular momentum $j = l \pm \frac{1}{2}$ needs to be considered

$$\begin{aligned} T_{q,lm}(\theta,\varphi) &= \begin{pmatrix} \sqrt{\frac{l+m+1}{2l+1}} Y_{q,lm}(\theta,\varphi) \\ \sqrt{\frac{l-m}{2l+1}} Y_{q,l(m+1)}(\theta,\varphi) \end{pmatrix} : j = l + \frac{1}{2}, \\ S_{q,lm}(\theta,\varphi) &= \begin{pmatrix} -\sqrt{\frac{l-m}{2l+1}} Y_{q,lm}(\theta,\varphi) \\ \sqrt{\frac{l+m+1}{2l+1}} Y_{q,l(m+1)}(\theta,\varphi) \end{pmatrix} : j = l - \frac{1}{2}, \end{aligned} \tag{16}$$

which brings the monopole Dirac equation to an almost diagonal form. Following the notation of Ref. [32] we can write down the 2×2 eigenvalue equation of the Dirac operator in the basis $(T_{q,(l-1)m}, S_{q,lm})^T$,

$$\not{D}\begin{pmatrix} T_{q,(l-1)m}e^{-i\omega\tau} \\ S_{q,lm}e^{-i\omega\tau} \end{pmatrix} = d_{q,l}(\omega)\begin{pmatrix} T_{q,(l-1)m}e^{-i\omega\tau} \\ S_{q,lm}e^{-i\omega\tau} \end{pmatrix}, \tag{17}$$

where $d_{q,l}(\omega) = A_{q,l}(-i\omega + B_{ql})$ is the eigenvalue matrix given by

$$\begin{aligned} A_{q,l} &= \begin{pmatrix} -\frac{q}{l} & -\sqrt{1-\frac{q^2}{l^2}} \\ -\sqrt{1-\frac{q^2}{l^2}} & \frac{q}{l} \end{pmatrix}, \\ B_{q,l} &= \begin{pmatrix} l\left(1-\frac{q^2}{l^2}\right) & -q\sqrt{1-\frac{q^2}{l^2}} \\ -q\sqrt{1-\frac{q^2}{l^2}} & -l\left(1-\frac{q^2}{l^2}\right) \end{pmatrix}. \end{aligned} \tag{18}$$

The monopole harmonics are defined for $l \geqslant |q|$, and for the case involving $l = q$ the matrices $A_{q,l}$ and $B_{q,l}$ are one dimensional with the only term given by their bottom-right entry. In this semidiagonal basis, the zeroth-order ground-state energy is easily obtained by integrating out the fermions from the path integral, and we get

$$\begin{aligned} F_0^{(q)} &= -\frac{1}{\beta}\mathrm{Tr}_{S^2\times S^1_\beta} \ln[\not{D}] \\ &= -(2N)\int \frac{d\omega}{2\pi}\sum_{l=q}^{\infty}\sum_{m=-l}^{l-1} \mathrm{ł},\ \det[d_{q,l}(\omega)] \\ &= -(2N)\int \frac{d\omega}{2\pi}\sum_{l=q}^{\infty} 2l\ \ln(\omega^2 + l^2 - q^2) \\ &= (2N)\lambda_0^{(q)}, \end{aligned} \tag{19}$$

where $\lambda_0^{(q)}$ is a regulated sum which can be computed and has been tabulated for various q's in Ref. [32]. After appropriate regularization the expression obtained in Ref. [32] yields $\lambda_0^{(0)} = 0$ so that the regulated free energy for $q > 0$ is equal to the finite free (Casimir) energy difference of adding a q charge monopole in $S^2 \times \mathbb{R}$. The dynamical gauge fields which have been ignored so far can be introduced as loop corrections to the above free energy. Roughly, this involves expanding the full trace-logarithm $F^{(q)} = -\mathrm{Tr}\ \ln[\not{D} + i\not{A}]$ in the gauge field strength and integrating them out. Its contribution is suppressed by a factor of $1/(2N)$, and the scaling dimension of the monopole operators beyond the large-$2N$ limit is, therefore, given by

$$\Delta_{\mathcal{M}^{(q)}} = (2N)\lambda_0^{(q)} + \lambda_1^{(q)} + O[1/(2N)]. \tag{20}$$

The subleading correction $\lambda_1^{(q)}$ for the first few charges q are provided in Ref. [31]. In this same spirit, we now consider random perturbations to the cQED_{2+1} action and calculate the perturbative correction to the monopole scaling dimension obtained in the clean limit.

IV. MONOPOLE SCALING DIMENSION OF DIRTY cQED_{2+1}

We wish to include the effect of RC perturbation on the monopole free energy obtained above within the large-N perturbation theory. The application of state-operator correspondence requires two ingredients: (1) radial quantization where the Hilbert space is defined on concentric spheres, and (2) conformal symmetry which ensures that the states corresponding to primary operators with well-defined scaling dimensions at the origin are eigenstates of the dilatation generators. Furthermore, we need to map the radially quantized theory on a cylinder which requires Weyl invariance (14) of the action. In principle, there is no problem with using the radial quantization picture with Hamiltonians involving quenched disorder. Conformal symmetry is definitely absent for a particular realization of disorder. However, we will consider the state-operator correspondence only after performing the disorder averaging which restores the homogeneity of space-time. The disorder averaged theory may lack conformal symmetry in the infrared limit, and this will be signaled by the unitarity violation of the primary operators of the theory [47]. Quenched randomness is static in time, and if we seek to establish a connection with the radial quantization picture, such random couplings must be parametrized by the coordinates on the two-sphere.

In the standard treatments, quenched random couplings are parametrized on the planar spatial $\mathbb{R}^2$ submanifold of the $(2+1)$-dimensional space-time manifold. To obtain the correspondence between the disorder strengths of random couplings defined on a spacelike plane and a spacelike sphere we consider a one-point compactification of the two-dimensional plane,

$$(x, y) = \left(\tan\frac{\theta}{2}\cos\phi, \tan\frac{\theta}{2}\sin\phi\right), \tag{21}$$

which transforms the planar spatial metric $ds_\parallel^2 = dx^2 + dy^2$ into $ds_\parallel^2 = \frac{1}{4}\sec^4\frac{\theta}{2}d\theta^2 + \tan\frac{\theta}{2}^2 d\phi^2$. Naturally, the induced metric on the sphere $ds_\parallel^2$ differs from the usual spherical metric $ds_{S^2}^2 = d\theta^2 + \sin^2\theta\, d\phi^2$. In the compactified space the

Gaussian weight for the random couplings in (10), therefore, becomes

$$P[V] = \exp\left(-\frac{1}{2\rho_\alpha}\int d^2r_\parallel\sqrt{g_\parallel}V_{\alpha j}^2\right). \quad (22)$$

Here, following our notation, $d^2r_\parallel\sqrt{g_\parallel}$ is the invariant volume measure of the compactified sphere, but ρ_α's are the same disorder strengths as defined on the flat space. Through this transformation, we have mapped the content of the physical information about the quenched disorder, i.e., the autocorrelation of the random couplings into a radially quantized theory. Once we are able to calculate the scaling dimension of the monopole operator of this theory we can transform back to usual $\mathbb{R}^3$ space-time and retrieve the standard on-site autocorrelation of random couplings defined on the flat space.

In the $S^2\times\mathbb{R}$ space, the disordered perturbation to the clean cQED$_{2+1}$ action is given by

$$S_{\text{dis}} = \int d^3r\sqrt{g}\bar{\psi}(iV_{\alpha j}\sigma^\alpha e_a^j\gamma^a)\psi. \quad (23)$$

The Weyl rescaling (14) leaves the RC fields $V_{\alpha j}$ unchanged. This distinguishes them from random mass perturbations (which do not stay invariant under the Weyl rescaling) and allows us to move forward with the radial quantization technique. For random mass-type disorder the Weyl rescaling leads to a scaling of the random coupling field $M\to e^{-\tau}M$ and introduces infrared divergences to the theory. With the chosen disorder distribution (22), the on-site correlation between the random couplings is given by $\overline{V_{\alpha j}(\theta,\phi)V_{\beta k}(0,0)} = \rho_\alpha\delta_{\alpha\beta}\delta_{jk}\delta(\theta)\delta(\phi)/\sqrt{g_\parallel}$. In the integrated form it yields

$$\int d^3r\sqrt{g}f(\tau)\overline{V_{\alpha j}(\theta,\phi)V_{\beta k}(0,0)} = 4\int d\tau\, f(\tau)\rho_\alpha\delta_{\alpha\beta}\delta_{jk}, \quad (24)$$

here formally the τ is integrated from 0 to β in the temporal direction of the $S^2\times\mathbb{R}$ cylindrical space. This additional factor of 4 establishes a correspondence between the strengths of the random couplings on the sphere and the plane. After we extract the scaling dimension of a single monopole operator inserted at the center in the disordered background we will consider the situation of a dilute concentration of monopole gas (4) in the presence of quenched disorder in the usual space-time. In the limit of vanishing dilution of the monopole operators, we will use the known expression [39,41] for the RG flow of the random couplings in the noncompact disordered QED$_{2+1}$ and disorder-modified RG flow of the monopole fugacity to fully characterize the disorder-driven instabilities of the compact theory. The Coulomb gas of the monopoles does modify the renormalization-group flow of the minimally coupled gauge charge in the matter sector [25,26], but to the leading order considered here, the gauge charge renormalization (and, therefore, the renormalization of the monopole fugacity) does not contribute to the renormalization of random couplings [39]. It is to be noted that the gauge charge flows to strong coupling in this context [26,39,41] and can be left out of the RG flow equations. Following disorder averaging the scaling dimension of the monopole operator can be simply extracted from the difference of the free energies $\Delta_{\mathcal{M}^{(q)}} = \overline{F^{(q)}(V_{\alpha j})} - \overline{F^{(0)}(V_{\alpha j})}$.

As the RCs couple quadratically to fermionic fields, we can formally integrate out the fermions from the generic disordered cQED$_{2+1}$ action, and similar to what is performed with the dynamical gauge fields, perturbatively expand the resulting expression in random coupling strength and perform direct disorder averaging. It is convenient to exploit the homogeneity of the $S^2\times S^1_\beta$ space-time postdisorder averaging and compute the functional trace in the space-time basis as $\text{Tr}_{S^2\times S^1_\beta}\overline{A} = \mathcal{V}(S^2)\mathcal{V}(S^1_\beta)\text{tr}\langle r_0|\overline{A}|r_0\rangle$, where $\mathcal{V}$ denotes the volume of the space, r_0 is any given point in the space, and tr is a trace within the Dirac spinor and SU$(2N)$ flavor space. In the following we choose the north-pole coordinates $r_0 = (\tau=0,\ \theta=0,\ \phi=0)$ and obtain

$$\begin{aligned} F^{(q)}(\rho_\alpha) &= -\overline{\text{Tr}_{S^2\times S^1_\beta}\ln\left[\not{D} + i\not{A} + iV_{\alpha j}\sigma^\alpha e_a^j\gamma^a\right]} + O[1/(2N)] \\ &= (2N)\lambda_0^{(q)} + \lambda_1^{(q)} + \lim_{\beta\to\infty}\frac{\mathcal{V}(S^2)\mathcal{V}(S^1_\beta)}{2\beta}\int d^3r\sqrt{g}\overline{\text{tr}\left[G^{(q)}(r_0,r)iV_{\alpha j}(\theta,\phi)\sigma^\alpha e_a^j\gamma^a G^{(q)}(r,r_0)iV_{\beta k}(0,0)\sigma^\beta e_b^k\gamma^b\right]} \\ &\quad + O[1/(2N), V^4] \\ &= (2N)\lambda_0^{(q)} + \lambda_1^{(q)} + 2\pi(4\rho_\alpha)\int d\tau\,\text{tr}\left[G^{(q)}(\tau)(i\sigma^\alpha e_a^j\gamma^a)G^{(q)}(-\tau)\left(i\sigma^\alpha e_b^j\gamma^b\right)\right] + O[1/(2N),\rho_\alpha^2], \end{aligned} \quad (25)$$

where $G(\tau)$ is the monopole Green's function between coincident angles

$G(\tau) = \langle r_0|\not{D}^{-1}|r\rangle$ with $r = (\tau,0,0)$. For a controlled perturbative double expansion we must have $\rho_\alpha\sim 1/(2N)$. In that way, following the trace over the vertex matrices, the first-order disorder contribution is an $O(1)$ perturbation to the zeroth-order free energy (19). The spectral decomposition of the Green's function matrix is expressed in the 2×2 monopole spherical harmonic basis (16) as

$$G(\tau) = \int\frac{d\omega}{2\pi}e^{-i\omega\tau}\sum_{l=q}^{\infty}\sum_{m=-l}^{l-1}\left(T_{q,l-1,m}\quad S_{q,l,m}\right)d_{q,l}(\omega)^{-1}\begin{pmatrix}T^\dagger_{q,l-1,m}\\ S^\dagger_{q,l,m}\end{pmatrix}\Bigg|_{r_0}. \quad (26)$$

The full expression of the Green's function is given in Ref. [32] and involves complicated special functions. The present scenario, however, is simpler, and at the north pole using the property $Y_{q,lm} = \delta_{q,-m}\sqrt{(2l+1)/(4\pi)}$ [31], we have a much simpler expression for the coincident angle Green's function,

$$G(\tau) = -\frac{\operatorname{sgn}(\tau)}{2}\left[\frac{q}{4\pi}(I+\gamma^0) + \sum_{l=q+1}^{\infty}\frac{l}{2\pi}e^{-\sqrt{l^2-q^2}|\tau|}\gamma^0\right], \tag{27}$$

where I is a 2×2 identity matrix. In the contribution to the free energy, the Green's function convolution includes a sum over the Dirac matrices. Evaluated at the north pole the sum product of the Dirac matrices yields $(i\gamma^1)\gamma^0(i\gamma^1) + (i\gamma^2)\gamma^0(i\gamma^2) = 2\gamma^0$ and $(i\gamma^1)I(i\gamma^1) + (i\gamma^2)I(i\gamma^2) = -2I$. Additionally, the sum and trace over the SU($2N$) matrices contribute a numerical prefactor $= 2\times \mathrm{tr}[\sigma^\alpha\sigma^\alpha 1_{NN}] = 4N$. In the following, we will also abbreviate $\sum_\alpha \rho_\alpha = \rho$.

The singular piece of the free energy is determined from the short-distance ultraviolet (UV) behavior of the Green's function [Eq. (27)]. In its written form a small $|\tau|$ expansion of Eq. (27) is not very useful. Instead, we expand the expression in small q and perform the resulting convergent sums over l directly (see Ref. [48]) and then perform an asymptotic expansion to find

$$G(\tau) = -\frac{\operatorname{sgn}(\tau)}{2}\left[\frac{q}{4\pi}1 + \left(\frac{1}{2\pi\tau^2} - \frac{1}{24\pi} + \frac{q\{8+q[12+q(-8+\pi^2 q)]\}|\tau|}{96\pi} + \frac{(1-20q^2)\tau^2}{480\pi} + O(q^5,\tau^3)\right)\gamma^0\right]. \tag{28}$$

It is simpler to track the UV contribution of the free energy by going to the frequency space. With the Fourier transformations $FT[\operatorname{sgn}(\tau)] = -\frac{2}{i\omega}$, $FT[\frac{1}{\tau^2}] = -\pi|\omega|$, $FT[|\tau|] = -\frac{2}{\omega^2}$ and so forth, it follows that in the given order in the perturbation theory the only terms which depend on a UV frequency cutoff Λ in the free energy [Eq. (25)] are independent of q. The q-independent singular pieces are unimportant as they drop out from the contribution to the scaling dimension $\Delta_{\mathcal{M}^{(q)}} = \overline{F^{(q)}(\rho)} - \overline{F^{(0)}(\rho)}$. As the nonanalytic portion of the Green's function is independent of q, the inference about the UV properties obtained from the current asymptotic expansion holds irrespective of at which order of q we truncate the above series.

Having established that the quenched RC perturbation does not introduce any physical UV singularities we now set out to extract the finite part of the free energy. After performing the time integral and taking the trace over the Dirac and SU($2N$) matrices we obtain

$$\overline{F^{(q)}(\rho)} = (2N)\lambda_0^{(q)} + \lambda_1^{(q)} - \frac{2\rho(2N)}{\pi}\left[q\sum_{l=q+1}^{\infty}\frac{l}{\sqrt{l^2-q^2}} + \sum_{l,l'=q+1}^{\infty}\frac{ll'}{\sqrt{l^2-q^2}+\sqrt{l'^2-q^2}}\right] + O[1/(2N),\rho_\alpha^2]. \tag{29}$$

It is to be noted that the constant τ-independent piece of the Green's function [Eq. (27)] does not contribute a constant contribution to the integrand, and, therefore, the convolution does not lead to any infrared singularities. This is particular to the case of the random non-Abelian vector potential perturbation that we have considered. Indeed, e.g., for a random scalar potential which has the Dirac matrix $e_a^r\gamma^a$ on $S^2\times\mathbb{R}$ (γ^0 in the flat space-time) in its vertex, the contribution from the constant piece does not drop out.

For the present case we have a disordered correction for the free energy [Eq. (29)] with formally divergent summations over the quantum numbers of the monopole harmonic basis (16). The zeroth-order contribution to the free energy (19) has been formally obtained by using the ζ-function regularization technique [32]. We employ the same method for the disorder correction term. With ζ-function regularization, the finite part of an apparently divergent infinite series summation can be obtained using analytic continuation of the Hurwitz ζ function [49],

$$\sum_{l=0}^{\infty}(l+z)^s = \zeta(-s,z) = -\frac{B_{s+1}(z)}{s+1} \quad \forall\, s\neq 1. \tag{30}$$

For the first sum, $I_1(q) = q\sum_{l=q+1}^{\infty} l/\sqrt{l^2-q^2}$ we note that the summand remains $\propto 1$ for large l. The regularized summation can be obtained by considering a different quantity $l/(l^2-q^2)^{s/2}$ for an s where the summation is perfectly convergent, and then the result can be analytically continued to $s=1$. This is a well-known technique to regularize the free energies of radially quantized CFTs [31,32,45,48,50] and one of the central computational elements for treating spherically symmetric problems in similar contexts. For this purpose we need to add and subtract the asymptotic form of the summand and obtain

$$I_1(q)/q = \lim_{s\to 1}\left[\sum_{l=q+1}^{\infty}\left(\frac{l}{(l^2-q^2)^{s/2}} - l^{1-s}\right) + \sum_{l=q+1}^{\infty} l^{1-s}\right] = R_1(q) + \zeta(0,q+1), \tag{31}$$

where in the second step we can take the limit $s=1$ by using the Hurwitz ζ function identity for the formally divergent term. Here $R_1(q)$ is a perfectly convergent summation which can be evaluated up to arbitrary numerical accuracy. The same technique can be extended to obtain the finite contribution from the second sum $I_2(q) = \sum_{l,l'=q+1}^{\infty}\frac{ll'}{\sqrt{l^2-q^3}+\sqrt{l'^2-q^2}}$, but owing to the presence of a double summation, the resulting expression is cumbersome. The complete expression for the regularized double summation contribution with an unimportant q-independent piece subtracted $\tilde{I}_2(q) = I_2(q) - I_2(0)$ has been provided in the Appendix.

From the disorder-averaged free energy the bare scaling dimension of the monopole operator of charge q is, therefore, given by

$$\Delta_{\mathcal{M}^{(q)}} = (2N)\lambda_0^{(q)} + \lambda_1^{(q)} - \frac{2\rho(2N)}{\pi}[I_1(q) + \tilde{I}_2(q)] + O\big[1/(2N),\rho_\alpha^2\big], \tag{32}$$

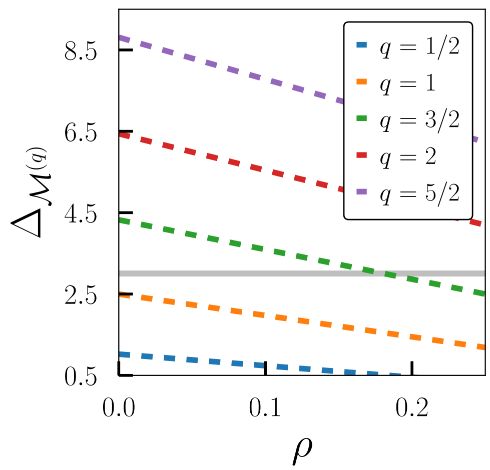

FIG. 1. The bare scaling dimension of monopole operators [Eq. (32)] as a function of bare random current strength ρ with $(2N) = 4$ flavors of fermions. The gray line denotes the threshold minimum $\Delta_{\mathcal{M}^{(q)}} = 3$, below which the monopoles proliferate. In the triangular lattice staggered π-flux spin-liquid state which only allows monopole operators of charges $q \geqslant 3/2$, disorder reduces the scaling dimension of monopoles of all charges.

where the disorder contribution $\rho = \sum_\alpha \rho_\alpha$ is to be summed over the SU(2) indices α, depending on the residual symmetries of the RC disorder. The unitarity bound dictates that the scaling dimension of a conformal scalar operator has to be $\geqslant 0.5$ in $D = 2 + 1$. It clearly follows that at the critical disorder strength $\rho^* = \pi[(2N)\lambda_0^{(q)} + \lambda_0^{(q)} - 0.5]/\{2(2N)[I_1(q) + \tilde{I}_2(q)]\}$, the bound is saturated for the monopole operator of charge q. This signals the breakdown of the conformal symmetry of the disordered infrared fixed point.

However, more importantly, it is the RG relevance of the monopole fugacity operator $y^{(q)}$, which has the scaling dimension $3 - \Delta_{\mathcal{M}^{(q)}}$ (5) in three space-time dimensions that dictates the suppression or proliferation of the monopoles [25,51]. From Eq. (32) we see that the presence of disorder underscreens the monopole operator such that the monopole fugacity is made more relevant. This is an important finding of our paper. In the triangular lattice staggered π-flux Dirac spin-liquid phase the allowed monopole charges are $q \geqslant 3/2$. Considering generic RC perturbation with disorder strength $\rho = \sum_\alpha \rho_\alpha$, it clearly emerges from Fig. 1 that with increasing disorder strength, higher charged monopole fugacities become relevant but the instability is still instigated by the proliferation of the monopoles of the lowest allowed charge $q = 3/2$. In the large-$2N$ limit the $q = 1$ monopole in the kagome lattice is allowed as a composite operator along with the spinons [24]. Although this is already a relevant operator in the clean limit, from Fig. 1 it clearly emerges that finite RC perturbation makes such a composite operator even more relevant. The proliferation of the monopoles leads to the confinement of fractionalized spinons of the destruction of the spin-liquid phase. For generic couplings to random SU($2N$) currents, the disorder strengths ρ_α also renormalize [39–41]. In that case the fate of the conformal fixed point is governed by the combined renormalization of all the couplings of the problem.

V. DISORDERED RG FLOW WITH MONOPOLES

To study the fate of the deconfined fixed point of the cQED_{2+1} action, the renormalization of all the associated couplings and the related instabilities need to be considered together [25]. In our case, this boils down to the renormalization-group flow of the monopole fugacity $y^{(q)}$ (which has bare scaling dimension $3 - \Delta_{\mathcal{M}^{(q)}}$) and the random disorder couplings. The infrared flow of the gauge electric charge $e^2 \to \infty$ is to the leading order unaffected by the disorder couplings [26,39] and can be ignored for the present discussion.

The RG flow equation of RC couplings ρ_α was obtained in Ref. [39] and in the absence of any other form of disorder we can adapt their expression to write $\frac{d\rho_\alpha}{dl} = (4/\pi)|\epsilon_{\alpha\beta\gamma}|\rho_\beta\rho_\gamma$ where $\epsilon_{\alpha\beta\gamma}$ is the usual Levi-Civita tensor. Compared to Ref. [39] we have the opposite sign convention of the RG flow, and our random current strengths are defined as twice of theirs (see Eqs. (24), (34), and (44) from Ref. [39] for comparison). This equation describes a flow of the genetic disorder strength to its strong-coupling fixed point, and, therefore, such random couplings introduce instability to the conformal fixed point of cQED_{2+1}. Including monopoles in the picture we have, to the leading order, a combined RG flow equation,

$$\frac{dy^{(q)}}{dl} = \left(3 - \sum_\alpha \Delta_{\mathcal{M}^{(q)}}(\rho_\alpha)\right) y^{(q)},$$
$$\frac{d\rho_\alpha}{dl} = \frac{4|\epsilon_{\alpha\beta\gamma}|}{\pi}\rho_\beta\rho_\gamma, \tag{33}$$

where the contributions to the scaling dimension of the monopole operator from all the disorder couplings have been added together. The presence of all three ρ_α couplings would indicate that the emergent SU($2N$) flavor symmetry has been broken down to a reduced SU(N). From the coupled flow equation it is clear that the SU(N) symmetric disorder moves to a strong-coupling fixed point. Consequently, due to its linear regressive dependence on the disorder strength, the monopole fugacity also flows to strong coupling (see Fig. 2). This observation leads to the clear indication that the Dirac spin-liquid phase is destroyed by SU(N) symmetric RC disorder as magnetic monopoles proliferate, and confinement ensues. From Fig. 1 we understand that the fugacity of the monopoles with the lowest microscopically allowed charge turns relevant first as the disorder strength flows to its strong-coupling limit.

Reference [39] further showed that if the random perturbations to the action obey more symmetries, the effect of disorder may be less drastic. Particularly, for a random coupling ρ_z to only one of the three components of the SU(2) subgroup of SU($2N$) vector currents ($\rho_x = \rho_y = 0$), the RC is U(1) × SU(N) symmetric, and in this case, following the same RG equation from above, it turns out that the disorder coupling ρ_z is marginal under RG. However, as demonstrated in Fig. 3, for the case of the triangular lattice where monopole operators of charge $q < 3/2$ are prohibited, a confinement transition ensues at a finite disorder strength, and the spin-liquid phase is destabilized at a finite critical value of $\rho_z^c \sim 0.175$. As per our definition of the RC perturbation [Eq. (10)], this critical disorder strength is a dimensionless

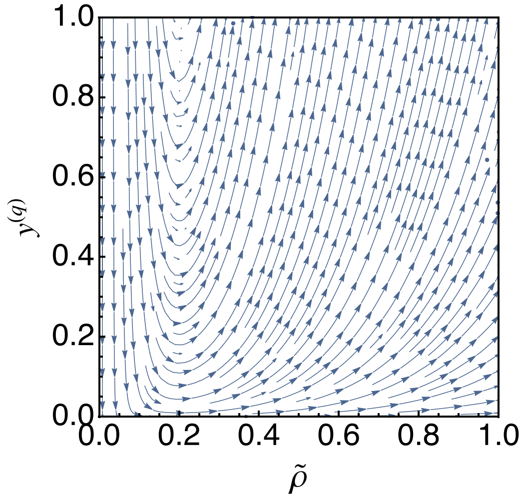

FIG. 2. The combined RG flow of SU(N) symmetric random current coupling $\rho_x = \rho_y = \rho_z = \tilde{\rho}$ and monopole fugacity $y^{(q)}$ of the $q = 3/2$ monopole operator which is allowed for the triangular lattice staggered π-flux U(1) Dirac spin-liquid state.

phenomenological number, and the precise form of its magnitude as a function of the inhomogeneities present in the lattice depends on the microscopic details. From Fig. 3 it also follows that effect of the disorder is offset by spinon flavor numbers, and, therefore, stronger disorder is needed to drive the confinement transition. However, more generic nonsymmetric random perturbation in the disorder coupling has a runaway flow to strong coupling (such as in Fig. 2) and, consequently, the Dirac spin-liquid state suffers a smearing transition to a symmetry-broken phase for any magnitude of the disorder strength.

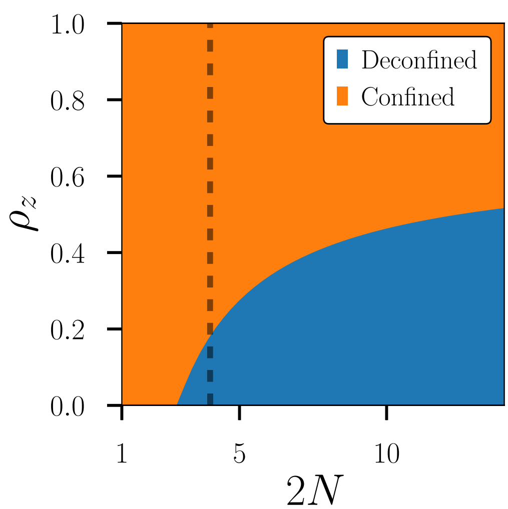

FIG. 3. The phase diagram of the U(1) Dirac spin liquid with $2N$ flavors of fermions coupled to RG marginal U(1) $\times$ SU(N) symmetric random current perturbations when only the monopole operators with charge $q \geqslant 3/2$ are symmetry allowed. In the triangular lattice the microscopic lattice symmetries disallow monopoles of charge $q < 3/2$ for the staggered π-flux ansatz. The gray line indicates the case of the triangular lattice staggered π-flux Dirac spin-liquid state which has $(2N) = 4$.

VI. CONCLUSION

From the available studies [39–41] it is known that the most generic local random perturbations to two-dimensional Dirac spin-liquid Hamiltonians with noncompact U(1) gauge fields lead to strong disorder instabilities and, consequently, emergence of a confined phase. In a realistic material, this outcome can, however, be affected by two key aspects, the symmetric nature of the random perturbations [39] and the effects of the microscopic monopole excitations of the compact gauge field fluctuations that arise in a lattice [20]. Our paper extends the findings of the earlier works on this problem to consider the interaction between quenched disorder and the monopole operators.

We have adapted the radial quantization techniques to calculate the scaling dimensions of local primary operators in a $(2+1)$-dimensional CFT with quenched random couplings. By computing the renormalized scaling dimension of the monopole operators of the effective cQED_{2+1} description, we establish that the spin liquid is further destablized due to the disorder-induced underscreening of the monopole operators. In the absence of monopoles, Ref. [39] found that the random vector-chiralitylike microscopic perturbations, which break the microscopic SU(2) spin symmetry down to U(1) and introduces random current perturbations to the effective theory, flows to a finite disorder fixed point where the spin liquid may yet survive despite some quantitative modifications [42]. We find that this finite disorder fixed point is, in fact, also fragile once the monopole operators are considered. On the other hand, more generic forms of random perturbations are seen to drive the RG flow towards a strong-coupling fixed point where both the monopole fugacity and disorder strength turn relevant. Our paper carries an important input towards the search of U(1) Dirac spin-liquid phases in frustrated two-dimensional spin systems especially on the triangular lattice where it is anticipated that a monopole-driven confinement of the gapless spinon excitations is avoided due to the lattice's nonbipartite nature [24,34].

On the experimental side, observations of spin-liquid like signatures in certain triangular lattice organic salts, such as $\kappa-(\text{ET})_2\text{Cu}_2(\text{CN})_3$ [52,53] and $\text{EtMe}_4\text{Sb}[\text{Pd}(\text{dmit})_2]_2$ [54,55] have triggered discussions surrounding the viability of a stable U(1) Dirac spin-liquid phase at zero temperature. Similarly, experimental studies on compounds such as Herbertsmithite have inspired the possibility of observing a quantum spin-liquid ground state on the kagome lattice [56]. Although, the nature of the nonmagnetic ground state of the kagome lattice Heisenberg antiferromagnet is still a matter of active debate [56], some very recent studies indicate that the U(1) Dirac spin-liquid state is preferred over other candidate Z_2 spin-liquid states [57]. However, many of the prospective triangular lattice and kagome lattice spin-liquid compounds are also noted to include significant quenched randomness effects which may mimic spin-liquid behaviors [58] or show prominent spin-glass type signatures [6].

For time-reversal invariant random exchangelike perturbations, the effective CFT describing U(1) Dirac spin liquids has a RG flow to strong disorder coupling where the spin liquid is purportedly destabilized [39,41]. Our paper indicates that time-reversal invariant random vector chirality perturbations generically turn the symmetry allowed monopole operators relevant on the triangular and kagome lattices, thereby introducing strong instabilities to the spin liquid and confining the Dirac-dispersing spinons. This result unambiguously establishes that in the presence of a large class of generic random perturbations, a Dirac spin liquid is not stable in two-dimensional nonbipartite lattices as the monopoles which are irrelevant in the clean limit turn relevant due to disorder effects and confine the perturbed theory.

VII. DISCUSSION AND OUTLOOK

Now, we turn our attention to the nature of the disordered phase that emerges when the monopoles become relevant due to quenched random perturbations. The microscopic monopole operators of the compact U(1) Dirac spin-liquid states on two-dimensional nonbipartite lattices are either spin-singlet or spin-triplet excitations [34]. In a clean frustrated spin system, the proliferation of the singlet-type monopoles is associated with VBS ordering (which breaks the lattice symmetries) whereas the triplet monopole proliferation leads to spiral Neél ordering [24,34]. Although the complicated interactions among the quenched random perturbation, the monopole operators and the order parameters of these competing long-range ordered phases are difficult to track, some recently established no-go results on disordered frustrated spin systems can help us understand the character of the disorder-driven confined phases proximate to the two-dimensional Dirac spin-liquid phases on such lattices.

For frustrated SU(2) [or U(1)] symmetric spin systems in two spatial dimensions (e.g., the triangular lattice Heisenberg model), it has been shown that both spiral Néel and VBS orders are unstable against small random exchange perturbations and ultimately give rise to short or quasi-long-ranged ordered glassy phases [5,10,13]. Following the work in Ref. [59], the same inference can be extended to random vector-chiralitylike perturbation effects on spiral ordering. SU(2) symmetric random exchange couplings, which are associated with spin-singlet random perturbations to the effective theory of the Dirac spin liquid (see Sec. II), would naturally lead to the proliferation of spin-singlet monopole operators and, consequently, an instability to VBS-type ordering in the disordered background. However, following the recent arguments on frustrated two-dimensional spin systems even weak disorder leads to the destruction of long-range VBS ordering in favor of domain formation and nucleation of spinons [5]. For intermediate to strong disorder a glassy random singlet ground state has been observed in numerical simulations of the random-bond triangular lattice Heisenberg antiferromagnet [60]. On the other hand, random vector-chiralitylike couplings, which generate disordered spin-triplet perturbations (Sec. II), lead to proliferation of spin-triplet monopole operators and, consequently, introduce a putative magnetic spiral ordering instability. However, based on the recent results, such spiral ordering on frustrated spin systems are actually destabilized in favor of a spin-glass phase for weak exchange disorder [10].

Altogether, it is, therefore, plausible that in the presence of generic random perturbations, the Dirac spin-liquid ground states of the triangular and kagome lattice Heisenberg antiferromagnets are destabilized, most likely in favor of short-range ordered ground states where monopole operators are confining and disorder flows to strong coupling. This observation is consistent with the quantum critical behavior seen in the compound $\kappa-(\mathrm{ET})_2\mathrm{Cu}_2(\mathrm{CN})_3$ where it is anticipated that a gapless spin-liquid state enters a glassy phase in the presence of random Dzyaloshinskii-Morya and multispin chiral interaction at low temperatures [61]. However, a more accurate characterization of the disordered phases proximate to the Dirac spin-liquid phases on these nonbipartite lattices requires further microscopic studies which have been left as future tasks.

Finally, it is to be noted that the methodology discussed in this treatment have wider applicabilities. The radial quantization scheme with quenched random coupling can be adapted to a number of $(2+1)$-dimensional U(1) conformal gauge theories [50] perturbed by quenched disorder. Among them, the $\mathbb{CP}^{N_b-1}$ theory of unit-norm N_b-component complex bosonic spinons constitute a viable example. This theory captures the transition between collinear Néel ordering and VBS ordering on bipartite quantum antiferromagnets [48]. The monopole operators of this model are interpreted as the order parameter of VBS order. Therefore, it will be worthwhile to apply and extend the methodology of this paper within the context of the collinear Néel to VBS transition in the presence of quenched disorder and investigate how the role of the monopole operators are affected at the critical point.

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APPENDIX: REGULARIZATION OF THE DOUBLE SUMMATION IN THE DISORDERED QED_{2+1} free energy

The second contribution appearing in the expression for the free-energy Eq. (25) involves a formally divergent double summation over angular momentum indices. In this Appendix, the divergent summation is regularized using the ζ-function method. It is convenient to split the double summation in two parts,

$$\begin{aligned} I_2(q) &= \sum_{l,l'=q+1}^{\infty} \frac{ll'}{\sqrt{l^2-q^2}+\sqrt{l'^2-q^2}} \\ &= \sum_{l=q+1}^{\infty} \frac{l^2}{2\sqrt{l^2-q^2}} + \sum_{\substack{l,\,l'\,=\,q+1 \\ l\,\neq\, l'}}^{\infty} \frac{ll'}{\sqrt{l^2-q^2}+\sqrt{l'^2-q^2}}. \end{aligned} \tag{A1}$$

The first single sum grows as $\propto l$ asymptotically. Following the same regularization technique as in the main text a perfectly converged sum may instead be considered $(1/2)\sum_{l=q+1}^{\infty} l^2/(l^2-q^2)^{1/2+s}$. It is possible to subtract and then add back from this expression its asymptotic dependence and then analytically continue the result to $s\to 0$ using the identities of the Hurwitz ζ function.

However, in certain cases where the resulting expression contains essential singularities in the limit $s\to 0$, a modification to this regularization scheme is better suited [62]. Let us consider $A(s)$ as a quantity which we want to analytically continue to $\lim_{s\to 0} A(s)=a_0$ but $A(s)=a_{-m}s^{-m}+a_{-(m-1)}s^{-(m-1)}+\cdots a_0+a_1 s+\cdots$ is singular. We can instead take the operator,

$$\mathcal{D}\left[\frac{d^n}{ds^n}[s^n A(s)]\right]=a_0, \tag{A2}$$

and consider any $n>m$ such that the regularized finite part $a_0=\lim_{s\to 0}A(s)$ is obtained without encountering any singularities. The original scheme corresponds to $n=0$.

Now following this strategy the manipulated summand yields the regularized finite contribution,

$$\begin{aligned}\frac{1}{2}\sum_{l=q+1}^{\infty}&\left\{\frac{l^2}{(l^2-q^2)^{1/2+s}}-\left[l^{1-2s}+\left(s+\frac{1}{2}\right)\frac{q^2}{2}l^{-1-2s}\right]\right\}_{s\to 0}\\ &+\frac{1}{2}\sum_{l=q+1}^{\infty}\left[l^{1-2s}+\left(s+\frac{1}{2}\right)\frac{q^2}{2}l^{-1-2s}\right]_{s\to 0},\\ &=R_2(q)+\frac{1}{2}\left[\zeta(-1,q+1)-\frac{q^2}{2}\psi(q+1)\right].\end{aligned} \tag{A3}$$

Here $R_2(q)=\frac{1}{2}\sum_{l=q+1}^{\infty}(\frac{l^2}{\sqrt{l^2-q^2}}-[l+\frac{q^2}{2l}])$ is now a convergent summation even after taking the limit $s\to 1$.

The remaining double summation offers more difficulty. To make progress the summation may be cast in a different form

$$\sum_{l\neq l'}\frac{ll'(\sqrt{l^2-q^2}-\sqrt{l'^2-q^2})}{l^2-l'^2}=\sum_{l\neq l'}\frac{2ll'}{l^2-l'^2}\sqrt{l^2-q^2}, \tag{A4}$$

which is helpful to obtain the sum over one of the indices in a purely analytical form. Thus, the sum over l' of the quantity $\frac{2l'}{l^2-l'^2}$ is first considered. The summand grows as $\propto -\frac{2}{l'}$ and a finite value to it can be assigned by considering the modified regularization $\mathcal{D}[\cdots]$. For $l\geqslant q+2$ it follows that,

$$\begin{aligned}\sum_{\substack{l'=q+1\\ l'\neq l}}^{\infty}\frac{2l'}{l^2-l'^2}&=\sum_{\substack{l'=q+1\\ l'\neq l}}^{\infty}-\left(\frac{1}{l'-l}+\frac{1}{l'+l}\right),\\ &=\frac{1}{2l}+\psi(l-q)+\psi(l+q+1)\quad\forall\, l\geqslant q+2,\end{aligned} \tag{A5}$$

which then reduces the double summation to a single summation over l,

$$\begin{aligned}\sum_{\substack{l,l'=q+1\\ l\neq l'}}^{\infty}&\frac{ll'}{\sqrt{l^2-q^2}+\sqrt{l'^2-q^2}}\\ &=\sum_{l'=q+2}^{\infty}\frac{2(q+1)l'\sqrt{2q+1}}{(q+1)^2-l'^2}\\ &\quad+\sum_{\substack{l=q+2,\,l'=q+1\\ l'\neq l}}^{\infty}\frac{2ll'}{l^2-l'^2}\sqrt{l^2-q^2}\\ &=\sum_{l=q+2}^{\infty}\frac{2l(q+1)\sqrt{2q+1}}{(q+1)^2-l^2}\\ &\quad+\sum_{l=q+2}^{\infty}l\sqrt{l^2-q^2}\left(\frac{1}{2l}+\psi(l-q)+\psi(l+q+1)\right).\end{aligned} \tag{A6}$$

The first term in the above expression can be computed similarly as a principle value to yield the finite contribution,

$$\begin{aligned}-(q+1)\sqrt{2q+1}&\sum_{l=q+2}^{\infty}\left(\frac{1}{l-(q+1)}+\frac{1}{l+(q+1)}\right)\\ &=(q+1)\sqrt{2q+1}[\gamma+\psi(2q+3)],\end{aligned} \tag{A7}$$

where $\gamma=0.57721\cdots$ for the Euler-Mascheroni constant.

The second term is also formally divergent due to its asymptotic growth $\propto 2l^2\ln l+l/2-q^2\ln l-(1/6+q+q^2)-q^2/(4l)$. The logarithmically growing portion can be regularized by using the identity $\ln l=-\frac{d}{ds}l^{-s}|_{s=0}$ such that one has

$$\sum_{l=q+2}^{\infty}\ln l=-\frac{d}{ds}\zeta(s,q+2)\bigg|_{s=0}=-\zeta'(0,q+2). \tag{A8}$$

In a similar manner the other logarithmically growing term gets the finite expression $\sum_{l=q+2}^{\infty}l^2\ln l=-\zeta'(-2,q+2)$. The terms which grow as l and $1/l$ can be regularized using the various identities already invoked above. Subtracting the diverging part from the summand and adding its regularized value back to the summation as above, the regularized double summation is, thus, obtained to be

$$\begin{aligned}\sum_{\substack{l,l'=q+1\\ l\neq l'}}^{\infty}&\frac{ll'}{\sqrt{l^2-q^2}+\sqrt{l'^2-q^2}}\\ &=(q+1)\sqrt{2q+1}[\gamma+\psi(2q+3)]+R_3(q)\\ &\quad+\left[-2\zeta'(-2,q+2)+\frac{\zeta(-1,q+2)}{2}+q^2\zeta'(0,q+2)\right.\\ &\quad\left.-\left(\frac{1}{6}+q+q^2\right)\zeta(0,q+2)+\frac{q^2}{4}\psi(q+2)\right],\end{aligned} \tag{A9}$$

where

$$R_3(q) = \sum_{l=q+2}^{\infty} \left(l\sqrt{l^2 - q^2}\left[\frac{1}{2l} + \psi(l-q) + \psi(l+q+1)\right] - \left[2l^2 \ln l + \frac{l}{2} - q^2 \ln l - (1/6 + q + q^2) - \frac{q^2}{4l}\right]\right) \tag{A10}$$

is once again a convergent sum. Putting together all of these pieces the complete and finite regularized expression for the second contribution to the disorder-averaged scaling dimension [Eq. (32)] is found to be

$$I_2(q) - I_2(0) = R_2(q) + [R_3(q) - R_3(0)] + [f(q) - f(0)], \tag{A11}$$

where $f(q)$ combines the contributions with an analytical expression,

$$f(q) = (q+1)\sqrt{2q+1}H_{2q+2} + \frac{q^2}{2}\ln\frac{\Gamma(2+q)^2}{2\pi} - \frac{4 - q[4 + 35q + 12(3+q)q^2]}{12(1+q)} - 2\zeta'(-2, q+2). \tag{A12}$$

Here H_z is the harmonic number, $\Gamma(z)$ is the Γ function, and both of these special functions can be evaluated up to arbitrary precision.

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