Edge modes in one-dimensional topological charge conserving spin-triplet superconductors: Exact results from Bethe ansatz

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Charge conserving spin singlet and spin triplet superconductors in one dimension are described by the $U(1)$ symmetric Thirring Hamiltonian. We solve the model with open boundary conditions on a finite line segment by means of the Bethe ansatz. We show that the ground state displays a fourfold degeneracy when the bulk is in the spin triplet superconducting phase. This degeneracy corresponds to the existence of zero energy boundary bound states localized at the edges which may be interpreted, in the light of the previous semiclassical analysis due to Kesselman and Berg, as resulting from the existence of fractional spin $\pm 1/4$ localized at the two edges of the system.

I. INTRODUCTION

One of the hallmarks of topological phases of matter is the existence of protected gapless modes localized at their ends [1]. This is typically the case of one-dimensional, proximity induced, topological superconductors which host Majorana bound states at their edges [2–4]. Since the topological protection of the edge modes relies on the presence of a finite energy gap in the bulk, the question has been raised [5] whether purely one-dimensional superconductors, with the superconductivity induced by intrinsic attractive charge conserving interactions, could support protected localized gapless end modes. Indeed, in these systems charge conservation induces strong quantum fluctuations leaving the system gapless with only quasi-long-range superconducting correlations. Due to the massless charge degrees of freedom, one would expect the edge modes to leak into the bulk and to be only power-law localized instead of being exponentially localized.

However, it has been argued that, provided there exist enough symmetries, exponentially localized zero energy end modes may also be present in gapless systems and several such phases have been proposed in the literature [1,6–15]. Among these, maybe the most representative example is provided by 1D spin-triplet topological superconductors (e.g., with dominant triplet superconducting correlations) which are predicted to host exponentially localized *fractional* spin-$\frac{1}{4}$ at both ends of an open chain [1] leading to a fourfold ground state degeneracy in the thermodynamic limit.

These predictions are supported by extensive DMRG calculations [1] in related lattice models, but most of the arguments leading to the existence of such localized fractional zero energy modes are based on semiclassical or mean-field arguments. Therefore, we find it important to provide an exactly solvable model which displays, over a wide range of coupling constants, spin triplet superconducting correlations

in the bulk and localized zero energy modes at the two edges of an open geometry. To this end we shall diagonalize in this work the Hamiltonian of the $U(1)$-symmetric Thirring with open boundary conditions (OBC) imposed on the fermions. The Hamiltonian is given by $H = \int_{-L/2}^{L/2} dx \, \mathcal{H}$ where

$$
\begin{aligned}
\mathcal{H} = &-iv(\psi_{Ra}^{\dagger}\partial_x\psi_{Ra} - \psi_{La}^{\dagger}\partial_x\psi_{La}) \\
&+ \psi_{Ra}^{\dagger}\psi_{Rb}\big[\, g_{\parallel}\,\sigma_{ab}^z\sigma_{cd}^z + g_{\perp}\,(\sigma_{ab}^x\sigma_{cd}^x + \sigma_{ab}^y\sigma_{cd}^y)\big]\psi_{Lc}^{\dagger}\psi_{Ld}.
\end{aligned}
\tag{1}
$$

In the above equation, $\sigma^{x,y,z}$ are the Pauli matrices and the two-components spinor fields $\psi_{L(R)}(x)$, which describe left and right moving fermions carrying spin $1/2$ with components $a = (\uparrow, \downarrow)$.

The $U(1)$ Thirring model, which is an anisotropic XXZ-type deformation of the $SU(2)$ invariant Thirring model (or the Chiral invariant two-flavors Gross-Neveu model), describes both singlet and spin triplet 1D charge conserving superconductors as well as the quantum phase transition between them, as a function of the couplings $(g_{\parallel}, g_{\perp})$.

The model has has been shown to be integrable with periodic boundary conditions (PBC) [16–18]. However we are not aware of a solution of the model on a finite line segment with OBC,

$$
\psi_{La}(\pm L/2) + \psi_{Ra}(\pm L/2) = 0, \quad a = (\uparrow, \downarrow). \tag{2}
$$

It is only, to our knowledge, in the $SU(2)$ invariant case (i.e., $g_{\parallel} = g_{\perp}$) that an exact solution has been obtained recently on the system with one open edge with a Kondo impurity coupled to it [19]. As we shall demonstrate in Sec. IV, the model is integrable for arbitrary couplings when the OBC (2) are imposed on the fermions and it remains integrable also in the presence of more general boundary conditions

which are *asymmetric* with respect to the left and the right edges.

Solving the model we find it possesses both topologically trivial and nontrivial phases corresponding to spin singlet and spin triplet superconducting correlations, respectively. While the topologically trivial SSS phase phase is unique and nondegenerate a more interesting situation arises in the topological STS phase. The exact solution shows that for an infinitesimal asymmetric OBC a four degenerate ground state structure emerges resulting from the existence of two zero energy boundary bound states localized at the two ends of the system. Two of the ground states have a total z-component spin $S^z = 0$ and fermion parity $\mathcal{P} = +1$ while the other two have $S^z = \pm 1/2$ and fermion parity $\mathcal{P} = -1$. This fourfold degeneracy can be consistently interpreted in the light of the semiclassical analysis [1] as *fractional* spin-$\frac{1}{4}$ boundary states. We remark however that when the asymmetry between the left and right edges is removed we explicitly find only a *threefold* degenerate ground state in the thermodynamic limit. We shall argue that the fourth state can be obtained by acting with a symmetry operator, analogous to the spin lowering operator that needs to be applied to the highest weight spin state provided by the solution to the Bethe ansatz equations of SU(2) symmetric models in order to complete the multiplets. In the topological phase such an operator would be given by either one of the two zero energy Majorana modes, localized at the two edges of the system, characterizing the topological degeneracy in a given fermionic parity sector. Overall, our exact results indicate that quantum fluctuations do not spoil the topological nature of the spin triplet phase found in the semiclassical limit.

The paper is organized as follows. We begin in Sec. II by reviewing the bulk properties of (1) using both fermionic and bosonic languages. In Sec. III we elaborate on the semiclassical arguments given by Keselman and Berg [1] leading to existence of localized *fractional* spin-$\frac{1}{4}$ boundary states in the spin triplet superconducting phase. In Sec. IV we solve the model using Bethe ansatz in the scaling limit where universal answers can be obtained for the ground state as well as for boundary excitations. We shall also consider the effect of integrable asymmetric boundary conditions between left and right edges. We finally discuss our results and open questions in Sec. V.

II. BULK PROPERTIES OF THE $U(1)$-SYMMETRIC THIRRING MODEL

In this section we shall present, to be self-consistent, some of the known results regarding the model. They follow from the exact solution given in Ref. [18] for PBC and bosonization. Most of these results hold when OBC are considered as far as bulk properties are concerned.

A. Symmetry properties

We start by briefly discussing the symmetry properties of the model. The Hamiltonian (1) displays, for generic couplings $(g_\parallel, g_\perp)$, a $U(1)_c \otimes U(1)_s$ symmetry corresponding to the changes, $\psi_{L(R)} \to e^{i\alpha_c} \psi_{L(R)}$ and $\psi_{L(R)} \to e^{i\alpha_s \sigma^z/2} \psi_{L(R)}$.

As a consequence the total spin S^z and the total number of fermions N,

$$S^z = \frac{1}{2} \int dx \, (\psi_L^\dagger \sigma^z \psi_L + \psi_R^\dagger \sigma^z \psi_R), \qquad (3)$$

$$N = \int dx \, (\psi_L^\dagger \psi_L + \psi_R^\dagger \psi_R), \qquad (4)$$

are conserved quantum numbers. The model displays also a number of discrete symmetries. On top of the chiral symmetry, i.e., $\psi_{L(R)} \to \psi_{R(L)}$, (1) is time-reversal (TR) symmetric, $\mathcal{T}\psi_{L(R)} = i\sigma^y \psi_{R(L)}$ ($\mathcal{T}^2 = -1$), and is invariant under space parity $x \to -x$. The $U(1)$-Thirring model is also invariant upon reversing the spins of all the fermions, i.e., $\Psi_{L(R),\uparrow} \leftrightarrow \Psi_{L(R),\downarrow}$. The latter symmetry has a $\mathbb{Z}_2 = \{1, \tau\}$ group structure with

$$\tau \psi_{L(R)} = \sigma^x \psi_{L(R)}, \quad \tau^2 = 1, \qquad (5)$$

and, similarly to time reversal $\mathcal{T}$, it reverses the sign of the total spin S^z, $S^z \to -S^z$. Finally, on the line $g_\parallel = g_\perp$, the $U(1)_s$ symmetry in the spin sector is enlarged to $SU(2)$. On this line, the $U(1)$-Thirring model (1) is invariant under the shift, $\psi_{L(R)} \to e^{i\vec{\alpha}\cdot\vec{\sigma}/2} \psi_{L(R)}$, and is nothing but the Gross-Neveu (GN) model.

Duality symmetry

On top of the above symmetries, the model displays also a *duality* symmetry Ω [20] which acts asymmetrically on the left and right fermions

$$\begin{aligned} \widehat{\psi} &= \Omega \psi \\ \widehat{\psi}_L &= \psi_L, \quad \widehat{\psi}_R = i\sigma^z \psi_R. \end{aligned} \qquad (6)$$

The duality Ω relates exactly, and at all scales, different models with opposite couplings $g_\perp$ and $-g_\perp$:

$$\mathcal{H}(\psi, g_\parallel, g_\perp) = \mathcal{H}(\widehat{\psi}, g_\parallel, -g_\perp). \qquad (7)$$

In particular it relates the correlation functions between any set of operators $\mathcal{O}_j(\psi)$ and their duals $\widehat{\mathcal{O}}_j \equiv \mathcal{O}_j(\widehat{\psi})$ in the two ground states of $\mathcal{H}(\psi, g_\parallel, \pm g_\perp)$, i.e., $\langle \mathcal{O}_1...\mathcal{O}_n \rangle_{g_\perp} = \langle \widehat{\mathcal{O}}_1...\widehat{\mathcal{O}}_n \rangle_{-g_\perp}$. This property allows us in principle to deduce the properties of dual models with couplings with say $(g_\parallel, -g_\perp < 0)$ from those of models with couplings $(g_\parallel, g_\perp > 0)$. It is a symmetry of the phase diagram. Of particular interest is the model described by (1) on the line $g_\parallel = -g_\perp$ which is dual to the $SU(2)$ Gross-Neveu model $\mathcal{H}_{GN}(\psi_a, g, g)$ and displays a dual $\widehat{SU(2)}$ nonlocal symmetry

$$\psi_L \to e^{i\vec{\alpha}\cdot\vec{\sigma}/2} \psi_L, \quad \psi_R \to (\sigma^z e^{i\vec{\alpha}\cdot\vec{\sigma}/2} \sigma^z)\psi_R. \qquad (8)$$

We shall refer to the model described by Hamiltonian $\mathcal{H}_{\widehat{GN}}(\psi, g, -g)$ as the dual $\widehat{SU(2)}$ Gross-Neveu model that we shall denote $\widehat{GN}$ in the following.

B. Bosonization

The Hamiltonian (1) may also be expressed in terms of bosonic fields using the equivalence between the $U(1)$-Thirring model and the sine-Gordon (SG) model. The latter correspondence is valid in the long distance and low energy limit [18,21] and is achieved using the bosonization of the

Fermi field

$$\psi_{L(R)a}(x) = \frac{\kappa_a}{\sqrt{2\pi a_0}} \exp\left[-i\sqrt{\pi}(\theta_a(x) \pm \phi_a(x))\right], \quad (9)$$

where $a = (\uparrow, \downarrow)$, a_0 is a short distance cutoff and $[\phi_a(x), \theta_b(y)] = -i\delta_{ab}\theta(x - y)$, $\theta(u)$ being the Heaviside step function [not to be confused with the dual bosonic field $\theta_b(u)$]. The operators $\kappa_{a=(\uparrow,\downarrow)}$ are anticommuting Klein factor satisfying $\{\kappa_a, \kappa_b\} = 2\delta_{ab}$ which ensure that fermions of different spins anticommute. The long distance and low energy limit of (1) is described by two scalar bosons fields, a massless charge field Φ_c and a SG spin field Φ_s as well as their duals Θ_c and Θ_s,

$$\Phi_c = (\phi_\uparrow + \phi_\downarrow)/\sqrt{2}, \quad \Theta_c = (\theta_\uparrow + \theta_\downarrow)/\sqrt{2} \quad (10)$$

$$\Phi_s = \frac{2\sqrt{\pi}}{\beta}(\phi_\uparrow - \phi_\downarrow), \quad \Theta_s = \frac{\beta}{4\sqrt{\pi}}(\theta_\uparrow + \theta_\downarrow), \quad (11)$$

in terms of which the total number of fermions N and the total spin S^z can be expressed as

$$N = \sqrt{\frac{2}{\pi}} \int dx\, \partial_x \Phi_c, \quad S^z = \frac{\beta}{4\pi} \int dx\, \partial_x \Phi_s. \quad (12)$$

The Hamiltonian (1) written in terms of these low energy fields decomposes as $\mathcal{H} = \mathcal{H}_{LL} + \mathcal{H}_{SG}$ where

$$\mathcal{H}_{LL} = \frac{u_c}{2}\left[\frac{1}{K_c}(\partial_x \Phi_c)^2 + K_c(\partial_x \Theta_c)^2\right], \quad (13)$$

a Luttinger liquid Hamiltonian which describes the gapless charge degrees of freedom and

$$\mathcal{H}_{SG} = \frac{u_s}{2}[(\partial_x \Phi_s)^2 + (\partial_x \Theta_s)^2] - \chi \frac{m_0^2}{\beta^2} \cos(\beta \Phi_s), \quad (14)$$

is the sine-Gordon (SG) Hamiltonian which describes the spin degrees of freedom. Although in the integrable $U(1)$ Thirring model the charge Luttinger parameters $u_c = v$ and $K_c = 1$, for sake of generality, we shall keep them as generic parameters in the following. In the spin sector $\chi = \text{sgn}(g_\perp)$ and the couplings u_s, m_0^2, and β of the SG model (14) are related to the couplings $(g_\parallel, g_\perp)$ of the $U(1)$ Thirring model (1) in a nonuniversal way except in the weak coupling limit ($|g_\parallel|, |g_\perp| \ll 1$) where $m_0^2/\beta^2 \approx g_\perp u_s/(\pi a_0)^2$ and $\beta^2/8\pi \approx 1 - g_\parallel/(\pi v)$.

Finally, to be complete, let us quote how the discrete symmetries of the model act on the boson fields. The $\mathbb{Z}_2$ symmetry generator τ in (5) acts on the spin boson fields as

$$\tau \Phi_s = -\Phi_s, \quad \tau \Theta_s = -\Theta_s, \quad (15)$$

while the duality symmetry Ω (6) on the fermion fields translate as

$$\Omega \Phi_s = \Phi_s + \frac{\pi}{\beta}, \quad \Omega \Theta_s = \Theta_s - \frac{\pi}{\beta}. \quad (16)$$

C. Phase diagram

The phase diagram of the $U(1)$ Thirring model depicted in Fig. 1 is well known and was obtained in Ref. [18] in the case of periodic boundary conditions. The plane $(g_\parallel, g_\perp)$ is divided into six regions. For $g_\perp > 0$ one distinguishes between three regions: A for $g_\parallel > g_\perp > 0$, B for $g_\parallel < -g_\perp < 0$, and C for $|g_\parallel| > g_\perp$. For $g_\perp < 0$ the other three regions are: $\widehat{A}$ for $g_\parallel >$
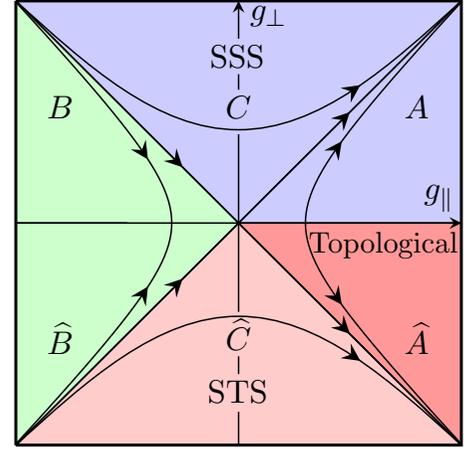

FIG. 1. Weak-coupling phase diagram of the $U(1)$ Thirring model. In green is the Luther-Emery phase which includes the regions B and $\widehat{B}$. In blue is the spin singlet superconducting (SSS) phase. It includes the two regions A and C as well as the $SU(2)$ invariant GN line at $g_\parallel = g_\perp$. In red is the spin triplet superconducting (STS) phase. It includes the two regions $\widehat{A}$ (dark red) and $\widehat{C}$ (light red) and the $\widehat{GN}$ line at $g_\parallel = -g_\perp$. The semiclassical regime corresponding to large $g_\parallel \gg 1$ is not displayed in the figure. In a system with OBC we show using Bethe ansatz that, in the region A and on the GN line of the SSS phase, the ground state is nondegenerate. In contrast, in the region $\widehat{A}$ and on the $\widehat{GN}$ line of the STS phase (dark red) the ground state is fourfold degenerate. This degeneracy results from zero energy boundary bound states localized at the edges. We interpret them in the light of the semiclassical analysis of Kesselman and Berg [1] as the result of localized spin $\pm 1/4$ at the two edges of the system. We hence give support that the topological phase found in the strong anisotropic regime in Ref. [1] survives quantum fluctuations down to the weak coupling regime. In the regions C and $\widehat{C}$ we are unable to conclude from the Bethe ansatz analysis about the ground state degeneracy in the universal regime.

$-g_\perp > 0$, $\widehat{B}$ for $g_\parallel < g_\perp < 0$, and $\widehat{C}$ for $|g_\parallel| < -g_\perp$. They are the dual to the $g_\perp > 0$ regions in the sense of (6). On top of these six regions are the two invariant GN and $\widehat{GN}$ lines obtained for $g_\parallel = g_\perp$ and $g_\parallel = -g_\perp$. On these lines, the model displays enlarged $SU(2)$ and $\widehat{SU(2)}$ symmetries.

In the regions B and $\widehat{B}$ the four fermion term, proportional to $g_\perp$ in (1), is irrelevant and both spin and charge sectors remain massless. The low energy sector of the theory is described by two Luttinger liquids, one in each sector. This is the Luther-Emery phase [22]. The same conclusion holds for the GN and $\widehat{GN}$ lines when $g_\parallel < 0$.

In contrast, in the remaining regions, A ($\widehat{A}$), C ($\widehat{C}$), and on the GN ($\widehat{GN}$) lines for $g_\parallel > 0$, the four fermion term is relevant: There is a dynamical mass generation m, corresponding to the opening of a spin gap, while a massless charge excitation decouples from the spectrum. When $g_\perp > 0$, the (A, C) regions and the GN line define the same phase. When $g_\perp < 0$ the two regions $\widehat{A}$, $\widehat{C}$ and the $\widehat{GN}$ line define a different, dual, spin gapped phase. The reason why we distinguish between two regions in a single phase is that, while the theory is asymptotically free in the ultraviolet in the regions A($\widehat{A}$) as well as on the GN and $\widehat{GN}$ lines, in regions C($\widehat{C}$), it is

nontrivial in both the infrared and the ultraviolet regimes. In all these regions the (massive) spectrum and the Bethe equations are different but the ground state properties do not qualitatively change within each of the two spin gapped phases. In particular we stress that there are no phase transitions on either the GN and $\widehat{GN}$ lines between the regions A and C or $\widehat{A}$ and $\widehat{C}$. The only phase transition between the two spin gapped phases is between regions A and $\widehat{A}$ on the line $g_\perp = 0$ where the $U(1)$ Thirring model is described by two charge and spin Luttinger liquids. We refer the reader interested in more details of these topics to Ref. [18].

Ground state instabilities

As well known in one dimension, due the massless charge degree of freedom, there can be no true long range order of any local, superconducting, and/or charge density wave types. The different phases are instead characterized by both the ground state degeneracy and the dominant instabilities (i.e., the power-law asymptotics of their correlation functions) they support. In all the phases, the ground state is not degenerate for PBC and we distinguish between two spin gapped phases.

(i) *The SSS phase.* This is a spin singlet superconducting phase which is stabilized when $g_\perp > 0$ and corresponds to the regions A, C and the GN line. In this phase a nonzero mass gap m develops and the system displays both spin singlet superconducting and charge density wave (CDW) instabilities with order parameters,

$$\mathcal{O}_{\text{SSS}} = \psi_L^\dagger \sigma^y \psi_R^\dagger, \quad \mathcal{O}_{\text{CDW}} = \psi_L^\dagger \psi_R, \quad (17)$$

which, using bosonization, can be shown to display quasi-long-range order

$$\langle \mathcal{O}_{\text{SSS}}^\dagger(x) \mathcal{O}_{\text{SSS}}(0) \rangle \propto |x|^{-1/K_c}, \quad (18)$$

$$\langle \mathcal{O}_{\text{CDW}}^\dagger(x) \mathcal{O}_{\text{CDW}}(0) \rangle \propto |x|^{-K_c}, \quad (19)$$

where K_c is the charge Luttinger parameter.

(ii) *The STS phase.* This is a spin triplet superconducting phase which is obtained when $g_\perp < 0$ in the regions $\widehat{A}, \widehat{C}$, and $\widehat{GN}$. In this phase both SSS and CDW correlation functions are short range and the instabilities are of the spin triplet superconducting (STS) and the spin density wave (SDW) types with order parameters,

$$\mathcal{O}_{\text{STS}} = \psi_L^\dagger \sigma^x \psi_R^\dagger, \quad \mathcal{O}_{\text{SDW}} = i\psi_L^\dagger \sigma^z \psi_R, \quad (20)$$

and asymptotics

$$\langle \mathcal{O}_{\text{STS}}^\dagger(x) \mathcal{O}_{\text{STS}}(0) \rangle \propto |x|^{-1/K_c}, \quad (21)$$

$$\langle \mathcal{O}_{\text{SDW}}^\dagger(x) \mathcal{O}_{\text{SDW}}(0) \rangle \propto |x|^{-K_c}. \quad (22)$$

The two types of instabilities in (17) and (20) are mutually nonlocal and are actually dual to each other in the sense of (6), i.e., $\mathcal{O}_{\text{STS}} = \widehat{\mathcal{O}_{\text{SSS}}}$ and $\mathcal{O}_{\text{SDW}} = \widehat{\mathcal{O}_{\text{CDW}}}$. They only coexist on the line of fixed points $g_\perp = 0$ where the spin gap closes. The two SSS and STS phases define therefore two different phases separated by the quantum phase transition line ($g_\perp = 0$, $g_\parallel \geqslant 0$). Notice that in a general 1D electron gas [23] the superconducting and density waves instabilities have different power-law asymptotics, i.e., $\propto |x|^{-1/K_c}$ and $\propto |x|^{-K_c}$, respectively. Which instability dominates depends on K_c and

it is custom to label the phases by their dominant instability: Either CDW or SDW when $K_c < 1$ and either SSS or STS phase when $K_c > 1$. In the present integrable model, where $K_c = 1$, both types of instabilities are equally dominant and we choose to label the two phases by the type of superconducting instability they support, i.e., SSS when $g_\perp > 0$ and STS when $g_\perp < 0$.

III. OPEN BOUNDARIES: SEMICLASSICAL APPROXIMATION

We now consider the effect of the OBC (2) on the fermions. Before going into the detailed analysis of our exact solution, we discuss the model in the strongly anisotropic limit: $g_\parallel \gg 1, |g_\perp| \ll 1$ (called quasiclassical regime in Ref. [18]) where, as far as bulk properties are concerned, the $U(1)$ Thirring model can be regarded as a regularized integrable version of the SG model (14) with $\beta^2/4\pi < 1$ and m_0^2 small. In the quantum regime, i.e., when $1 < \beta^2/4\pi < 2$, this ceases to be true and the equivalence between the $U(1)$ Thirring model and the SG model is only valid in the asymptotic low-energy/long-distance limit.

The OBC on the fermions (2) translate to the charge bosonic field (13) and the spin bosonic field of the SG model (14) as

$$\Phi_c(-L/2) = 0, \quad \Phi_c(+L/2) = \sqrt{\frac{\pi}{2}} N, \quad (23)$$

$$\Phi_s(-L/2) = 0, \quad \Phi_s(+L/2) = \frac{4\pi}{\beta} S^z. \quad (24)$$

Therefore, since the total number of particle of each spin component $N_{(\uparrow,\downarrow)}$ enclosed in a finite chain is an integer, $N \pm 2S^z$ has to be an even integer. This implies that the total particle number sector N of the Luttinger liquid Hamiltonian (13) and the total S^z sector of the SG model (14) are not independent but constrained by

$$N \text{ even} \Leftrightarrow S^z \in \mathbb{Z}, \ N \text{ odd} \Leftrightarrow S^z \in \mathbb{Z} + \tfrac{1}{2}. \quad (25)$$

Besides the latter constraint, the physics of the $U(1)$ Thirring model in the presence of OBC in the semiclassical regime boils down to that of the SG model (14) with the OBC (24). The effect of Dirichlet boundary conditions on the spectrum of the SG model has been extensively studied [24–26]. It appears that, in the presence of the OBC (24), boundary bound states are likely to be stabilized in an extended part of the phase diagram which corresponds to a region which is deep in the STS phase. As we shall see, these boundary bound states emerge straightforwardly in the semiclassical analysis and are responsible, as first stressed out in Ref. [1], of the topological nature of the STS phase in the limit of large $g_\parallel \gg 1, |g_\perp| \ll 1$.

Semiclassical analysis

The semiclassical limit of the SG model corresponds to the limit $\beta \to 0$ (keeping m_0^2 small fixed) and to large $g_\parallel \gg 1$. It is well understood as far as PBC are concerned [27]. For OBC, as we shall see, the ground state degeneracy dramatically changes when going from the SSS phase ($g_\perp > 0$) to the STS phase $g_\perp > 0$. In the limit $\beta \to 0$, as argued by Keselman and Berg [1], the system in the STS phase hosts symmetry

protected edge states with fractional spin $S^z = \pm 1/4$, which are exponentially localized at each edge. These edge states become degenerate in the thermodynamical limit leading to a fourfold ground state degeneracy. In the following we shall follow the lines of arguments presented in Ref. [1] and elaborate on the nature of the edge states Hilbert space.

1. Classical edge kinks

In the limit, $\beta \to 0$, the boson spin field is locked to the minima $\Phi_{s,n}$ of the cosine term in (14) which depend on the sign of $g_\perp$

$$\Phi_{s,n} = n\frac{2\pi}{\beta}, \quad \chi > 0, \tag{26}$$

$$\Phi_{s,n} = \left(n + \frac{1}{2}\right)\frac{2\pi}{\beta}, \quad \chi < 0, \tag{27}$$

where we recall that $\chi = \mathrm{sgn}(g_\perp)$ and $n \in \mathbb{Z}$. Clearly in the SSS phase ($\chi > 0$) given the OBC (24), there is a unique minimum at $\Phi_{s,n} = 0$ which corresponds to a total spin $S^z = 0$. In the STS phase ($\chi < 0$) the situation radically changes as none of the minima in (27) match with the OBC (24). The lowest energy states in this case consist of classical kinks configurations of the spin field $\Phi_s(x)$ which interpolate between $\Phi_s(-L/2) = 0$ and $\Phi_s(+L/2) = \frac{4\pi}{\beta}S^z$ and match in the bulk (i.e., when $-L/2 \ll x \ll +L/2$) with one of the classical ground states (27). Due to the spin gap m in the bulk, these kinks are exponentially localized near both left and right edges at $x = -L/2$ and $x = +L/2$, respectively, and, to the exponential accuracy in the system size [i.e., to $\mathcal{O}(e^{-mL})$], each edge can be treated separately. Hence, for large system sizes, i.e., when $mL \gg 1$, the kinks can be seen as the sum of left and right kinks which interpolate between $\Phi_s(-L/2) = 0$ and $\Phi_{s,n} = (n + \frac{1}{2})\frac{2\pi}{\beta}$ in bulk and between $\Phi_{s,n} = (n + \frac{1}{2})\frac{2\pi}{\beta}$ in bulk and $\Phi_s(+L/2) = \frac{4\pi}{\beta}S^z$.

Consider first the left edge at $x = -L/2$. As depicted in Fig. 2, there the lowest energy states consist of classical kinks interpolating between $\Phi_s(-L/2) = 0$ and one of the two classical ground states $\Phi_s = \pm\pi/\beta$ in the bulk. Due to (12) these two left kinks correspond to an accumulation of a *fractional* spin at the left edge,

$$S^z_{\mathcal{L}} = \frac{\beta}{4\pi}\int_{-L/2}^{y} dx\,\partial_x\Phi_s = \pm 1/4, \tag{28}$$

where y is some point deep in the bulk. These two left kinks cost a finite energy but have the same energy owing to the $\mathbb{Z}_2$ symmetry (15) which exchanges the two ground states in the bulk, i.e., $\pi/\beta \leftrightarrow -\pi/\beta$. A similar analysis can be done at the right edge at $x = L/2$. The situation there depends on the total spin S^z enclosed in the system. When $S^z = 0$ the lowest energy states consist of right kinks interpolating between $\Phi_s = \pm\pi/\beta$ in the bulk and $\Phi_s(+L/2) = 0$. They correspond to an accumulation of a fractional spin,

$$S^z_{\mathcal{R}} = \frac{\beta}{4\pi}\int_{y}^{L/2} dx\,\partial_x\Phi_s = \mp 1/4, \tag{29}$$

at the right edge. When $S^z = \pm 1/2$ the right kinks interpolate between $\Phi_s = \pm\pi/\beta$ in the bulk and $\Phi_s(+L/2) = \pm 2\pi/\beta$ at the edge. They also accumulate a fractional spin at the right
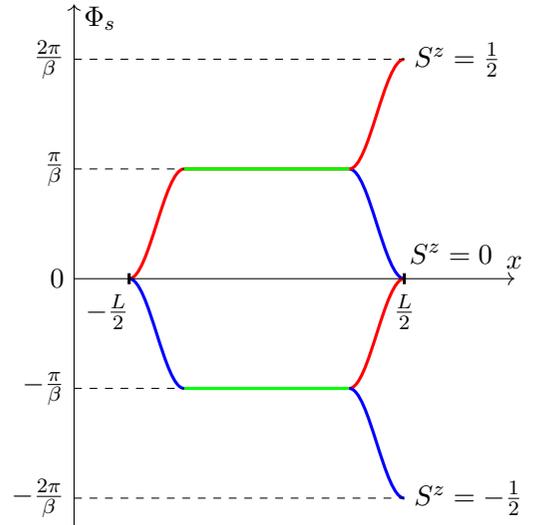

FIG. 2. Classical kink configurations in the STS phase in the presence OBC as taken from Ref. [1]. The red (blue) lines correspond to edge kinks with an accumulation of spin $\Delta S^z = \frac{1}{4}(-\frac{1}{4})$. The two green horizontal lines correspond to the constant values that the spin field Φ_s takes in the two bulk ground states $\Phi_s = \pm\frac{\pi}{\beta}$.

edge $S^z_{\mathcal{R}} = \pm 1/4$. Since kinks depend only on the accumulation of spin they carry, there are only two independent right classical kinks with spin $\pm 1/4$. Due to the $\mathbb{Z}_2$ symmetry (15) these right kinks have the same energy. All together there are four classical kinks states, two at the left and two at the right edges, each carrying fractional spins $\pm 1/4$ and having, to $\mathcal{O}(e^{-mL})$ accuracy, the same classical energy thanks to the $\mathbb{Z}_2$ symmetry (15).

2. Quantum edge kinks

To promote the above left and right classical kinks to quantum states one needs to *assume* the existence of quantum kinks at each edge

$$\left|\pm\tfrac{1}{4}\right\rangle_{\mathcal{L}}, \quad \left|\pm\tfrac{1}{4}\right\rangle_{\mathcal{R}}, \tag{30}$$

labelled by their local fractional spins. Concurrently this implies the existence of local quantum spin operators $S^z_{\mathcal{L}}$ and $S^z_{\mathcal{R}}$, $[S^z_{\mathcal{L}}, S^z_{\mathcal{R}}] = 0$, which, when acting on the left and right quantum kinks states, have fractional eigenvalues

$$S^z_{\mathcal{L}}\left|\pm\tfrac{1}{4}\right\rangle_{\mathcal{L}} = \pm\tfrac{1}{4}\left|\pm\tfrac{1}{4}\right\rangle_{\mathcal{L}},$$
$$S^z_{\mathcal{R}}\left|\pm\tfrac{1}{4}\right\rangle_{\mathcal{R}} = \pm\tfrac{1}{4}\left|\pm\tfrac{1}{4}\right\rangle_{\mathcal{R}}. \tag{31}$$

With these assumptions, the edge state Hilbert space is given by the tensor product of left and right kinks states (30) and consists of the four states

$$\left|\pm\tfrac{1}{4}\right\rangle_{\mathcal{L}} \otimes \left|\pm\tfrac{1}{4}\right\rangle_{\mathcal{R}}, \tag{32}$$

which can be sorted out according to the total spin

$$S^z = S^z_{\mathcal{L}} + S^z_{\mathcal{R}} \tag{33}$$

into two $S^z = 0$ states

$$\left|+\tfrac{1}{4}\right\rangle_{\mathcal{L}} \otimes \left|-\tfrac{1}{4}\right\rangle_{\mathcal{R}}, \quad \left|-\tfrac{1}{4}\right\rangle_{\mathcal{L}} \otimes \left|+\tfrac{1}{4}\right\rangle_{\mathcal{R}} \tag{34}$$

and two $S^z = \pm 1/2$ states

$$\left|+\tfrac{1}{4}\right\rangle_{\mathcal{L}} \otimes \left|+\tfrac{1}{4}\right\rangle_{\mathcal{R}}, \quad \left|-\tfrac{1}{4}\right\rangle_{\mathcal{L}} \otimes \left|-\tfrac{1}{4}\right\rangle_{\mathcal{R}}. \tag{35}$$

Although the left and right edge states carry fractionalized spins $\pm 1/4$ the total spins of the four ground states have integer and half integer values as it should.

3. Majorana modes and symmetry fractionalization

The above analysis matches the mean field approaches [15,28] based on coupled Majorana chains that we shall label "x" and "y" chains. It leads the existence of four local Majorana modes (two at each edge)

$$\left(\sigma_{\mathcal{L}}^x, \sigma_{\mathcal{L}}^y\right) \text{ and } \left(\sigma_{\mathcal{R}}^x, \sigma_{\mathcal{R}}^y\right), \tag{36}$$

which satisfy the Clifford algebra, i.e., $\{\sigma_r^\mu, \sigma_{r'}^\nu\} = 2\delta_{rr'}\delta^{\mu\nu}$ where $(\mu, \nu) = (x, y)$ and $(r, r') = (\mathcal{L}, \mathcal{R})$. At each edge, the low energy Hilbert space is described by the two $SO(2)_{(\mathcal{L},\mathcal{R})}$ *spinors* (30) which span two irreducible projective representations of the $U(1)$ symmetry group generated by

$$s_{(\mathcal{L},\mathcal{R})} = \tfrac{1}{4i}\left[\sigma_{(\mathcal{L},\mathcal{R})}^x, \sigma_{(\mathcal{L},\mathcal{R})}^y\right]. \tag{37}$$

These two representations are eigenvectors of the local fermionic parity operators

$$\begin{aligned}
\mathcal{P}_{\mathcal{L}} &= -i\sigma_{\mathcal{L}}^x\sigma_{\mathcal{L}}^y = \sigma_{\mathcal{L}}^z, \\
\mathcal{P}_{\mathcal{R}} &= -i\sigma_{\mathcal{R}}^x\sigma_{\mathcal{R}}^y = \sigma_{\mathcal{R}}^z,
\end{aligned} \tag{38}$$

with $[\mathcal{P}_{\mathcal{L}}, \mathcal{P}_{\mathcal{R}}] = 0$ and $\mathcal{P}_{(\mathcal{L},\mathcal{R})} = \pm 1$. They are related to the local spin operators by

$$\sigma_{(\mathcal{L},\mathcal{R})}^z = -ie^{i2\pi S^z_{(\mathcal{L},\mathcal{R})}}, \tag{39}$$

or equivalently by $S^z_{(\mathcal{L},\mathcal{R})} = \tfrac{1}{4}\sigma^z_{(\mathcal{L},\mathcal{R})}$, so that the states $|\pm\tfrac{1}{4}\rangle_{(\mathcal{L},\mathcal{R})}$ have local fermionic parities $\mathcal{P}_{(\mathcal{L},\mathcal{R})} = \pm 1$. One may verify, using (25), that the total fermionic parity operator fractionalizes into

$$\mathcal{P} = -\mathcal{P}_{\mathcal{L}}\mathcal{P}_{\mathcal{R}} = (-1)^N. \tag{40}$$

In this scheme, the two $S^z = 0$ states in (34) with an even total fermionic parity $\mathcal{P} = +1$ are the tensor products of states with opposite local parities. The two states (35) with total spins $S^z = \pm 1/2$ have an odd total fermionic parity $\mathcal{P} = -1$ and are the tensor products of states with the same local parities.

When projected onto the low energy subspace of the edges kinks, the $\mathbb{Z}_2$ symmetry group (5) also fractionalizes between the two edges into local $\mathbb{Z}_{2,(\mathcal{L},\mathcal{R})}$ symmetry groups

$$\mathbb{Z}_{2,(\mathcal{L},\mathcal{R})} = \left\{1, \sigma_{(\mathcal{L},\mathcal{R})}^x\right\}, \quad \left(\sigma_{(\mathcal{L},\mathcal{R})}^x\right)^2 = 1. \tag{41}$$

The local spin flip operators $\sigma_{(\mathcal{L},\mathcal{R})}^x$ reverse the spins $S^z_{(\mathcal{L},\mathcal{R})}$ (as well as the fermion parities $\mathcal{P}_{(\mathcal{L},\mathcal{R})}$) at each edge and the total spin flip operator τ of the $\mathbb{Z}_2$ symmetry in (5) is given by

$$\tau = i\sigma_{\mathcal{L}}^x\sigma_{\mathcal{R}}^x. \tag{42}$$

Since $[\tau, \mathcal{P}] = 0$, in each fermionic parity sector $\mathcal{P} = \pm$, one may sort out the states (34) and (35) into symmetric and antisymmetric states $|\mathcal{P} = \pm, \tau = \pm\rangle$ with respect to the total spin flip operator τ, i.e.,

$$\mathcal{P}|\pm, \tau\rangle = \pm|\pm, \tau\rangle, \quad \tau|\mathcal{P}, \pm\rangle = \pm|\mathcal{P}, \pm\rangle, \tag{43}$$

where

$$\begin{aligned}
|+, \pm\rangle &= \sqrt{\tfrac{1}{2}}\left(\left|+\tfrac{1}{4}\right\rangle_{\mathcal{L}} \otimes \left|-\tfrac{1}{4}\right\rangle_{\mathcal{R}} \pm \left|-\tfrac{1}{4}\right\rangle_{\mathcal{L}} \otimes \left|+\tfrac{1}{4}\right\rangle_{\mathcal{R}}\right), \\
|-, \pm\rangle &= \sqrt{\tfrac{1}{2}}\left(\left|+\tfrac{1}{4}\right\rangle_{\mathcal{L}} \otimes \left|+\tfrac{1}{4}\right\rangle_{\mathcal{R}} \pm \left|-\tfrac{1}{4}\right\rangle_{\mathcal{L}} \otimes \left|-\tfrac{1}{4}\right\rangle_{\mathcal{R}}\right).
\end{aligned} \tag{44}$$

While working in this basis it is suitable to introduce new Majorana operators, $\lambda_{(\mathcal{L},\mathcal{R})}$ and $\eta_{(\mathcal{L},\mathcal{R})}$, which are associated with the two commuting $\mathbb{Z}_2$ symmetries of the problem: the $\mathbb{Z}_2 = \{1, \tau\}$ symmetry (5) associated with the spin flip symmetry of the Hamiltonian and the $\mathbb{Z}_2^F = \{1, \mathcal{P}\}$ fermion parity symmetry. With the correspondence

$$\begin{aligned}
\lambda_{\mathcal{L}} &= \sigma_{\mathcal{L}}^z, \quad \lambda_{\mathcal{R}} = -i\sigma_{\mathcal{L}}^y\sigma_{\mathcal{R}}^x, \\
\eta_{\mathcal{L}} &= \sigma_{\mathcal{L}}^z\sigma_{\mathcal{R}}^x, \quad \eta_{\mathcal{R}} = \sigma_{\mathcal{R}}^y,
\end{aligned} \tag{45}$$

we check that $[\lambda_r, \eta_{r'}] = 0$, $\{\lambda_r, \lambda_{r'}\} = \{\eta_r, \eta_{r'}\} = 2\delta_{rr'}$, and

$$\tau = i\lambda_{\mathcal{L}}\lambda_{\mathcal{R}}, \quad \mathcal{P} = i\eta_{\mathcal{L}}\eta_{\mathcal{R}}. \tag{46}$$

The problem then decouples into two commuting λ and η Majorana modes which are associated with the $\mathbb{Z}_2$ and $\mathbb{Z}_2^F$ symmetries, respectively. In a given total fermion parity sector the Majorana $\lambda_{(\mathcal{L},\mathcal{R})}$ exchange symmetric and antisymmetric states $|\mathcal{P}, \tau\rangle \leftrightarrow |\mathcal{P}, -\tau\rangle$ while the Majorana $\eta_{(\mathcal{L},\mathcal{R})}$ reverse the fermion parity of either the symmetric or the antisymmetric states $|\mathcal{P}, \tau\rangle \leftrightarrow |-\mathcal{P}, \tau\rangle$.

4. Particle number conservation

So far we have described the spin sector only. Including the charge degrees of freedom is the equivalent to enforce particle number conservation together with the constraint (25). In a system with overall conservation of the total number of fermions N, the states with different fermionic parities $\mathcal{P}$ must differ by an odd number of fermions. Hence the minimum energy difference between the states with $\mathcal{P} = +1$ and $\mathcal{P} = -1$ in (44) is given by the energy cost of adding or removing a charge in the system. This is the charging energy which, in a Luttinger liquid, goes to zero as $1/L$ in the thermodynamical limit. Thus the four states (34), (35) or equivalently (44) are degenerate in the $L \to \infty$ limit. At finite size though, the effect of particle number conservation is to lift the fourfold degeneracy of the edge states. In a finite system the ground state of the system is (to the exponential accuracy in the system size) only doubly degenerate. The states with opposite fermion parities $|\pm\mathcal{P}, \tau\rangle$ are separated by a gap of order $1/L$ whereas, in each parity sector, the symmetric and antisymmetric states $|\mathcal{P}, \pm\tau\rangle$ are (quasi)degenerate with a much smaller energy splitting $\delta E \sim e^{-mL}$. The resulting twofold (quasi)degeneracy is exhausted by the two zero energy Majorana modes $(\lambda_{\mathcal{L}}, \lambda_{\mathcal{R}})$ [see Eq. (45)] which are localized at each edge of the system and confer the STS phase a topological degeneracy which results from the $\mathbb{Z}_2$ symmetry of the problem.

Strictly speaking the arguments leading to the existence of the symmetry protected zero energy edge modes (30), and concurrently to the topological degeneracy, are valid in the semiclassical regime which corresponds to the strongly

anisotropic regime of the $U(1)$ Thirring model. Hence, the mere existence of a stable topological phase that extends in the whole STS phase rely on the assumption that quantum fluctuation do not spoil the nice picture described above. This is particularly true in the weak coupling regime, i.e., $0 < g_{\parallel} \ll 1, 0 < -g_{\perp} \ll 1$, where quantum fluctuations are strong. We also stress that at the heart of the existence of *fractional* edge states (30) is the assumption that the fractionalization of the spin quantum number at the edges is a genuine phenomenon. This implies that the fractional spin operators $S^z_{(\mathcal{L},\mathcal{R})}$, defined semiclassically in (28) and (29), have to be given a proper sense in the full quantum theory as sharp quantum observable with zero variance in the thermodynamic limit, i.e.,

$$\lim_{L\to\infty} \langle (S^z_{(\mathcal{L}(\mathcal{R})})^2 \rangle - (\langle S^z_{(\mathcal{L},\mathcal{R})} \rangle)^2 = 0. \qquad (47)$$

In a strongly interacting theory this is a highly nontrivial issue. It is only, to our knowledge, for free massive fermions interacting with a classical soliton-anti-soliton field that the fractionalization at the fermion charge at the locations of the soliton and antisoliton fields has been clearly established [29]. In the present context, which corresponds to $\beta^2 = 4\pi$, the soliton and the antisoliton can be seen as the two left and right boundaries and the fractionalized fermion charge $\pm 1/2$ corresponds to the fractional spin $\pm 1/4$ quantum edge states.

This supports the results obtained above in semiclassical analysis. However, it says nothing about quantum regime ($\beta^2 > 4\pi$) where quantum fluctuations are strong. To investigate this regime other methods are required. This will be provided in the next section when solving the model (1) using the Bethe ansatz.

IV. OPEN BOUNDARIES: THE BETHE ANSATZ SOLUTION

In this section we solve exactly, using the Bethe ansatz, the $U(1)$ Thirring model when OBC are imposed on the fermions. We shall present the Bethe equations for all values of the couplings $g_{\parallel}$ and $g_{\perp}$. In the following we shall present our results for the regions A($\widehat{A}$) of the phase diagram (see Fig. 1) and on the GN($\widehat{GN}$) lines at $g_{\parallel} = \pm g_{\perp}$ where universal answers can be obtained in the scaling limit. In the latter limit, the cutoff D is taken to infinity while the mass (the spin gap) m is kept fixed. This corresponds to the region of small couplings ($|g_{\parallel}|, |g_{\perp}|) \ll 1$ where $\beta^2 \sim 8\pi$ in the SG model (14) and this is precisely the quantum regime we want to investigate. In the regions C($\widehat{C}$), as well known, taking the scaling limit is a nontrivial issue since, as seen in Fig. 1, the limit of infinite cutoff corresponds to a strong coupling fixed point (the theory is not asymptotically free) whose nature is, to our knowledge, unknown. In the following we shall thus concentrate on the portions of both the SSS and STS phases that correspond to the regions A($\widehat{A}$) and the GN($\widehat{GN}$) lines in Fig. 1.

A. Overview of Bethe ansatz solution

Our main focus will be on the the effects of the open boundaries on the ground state properties and we shall not discuss their effects on the gapped excitations. Before proceeding to the actual calculation we present here an outline of the results. We shall show that the model when defined on a line segment with open boundary conditions is integrable and

that its properties are given by a set of algebraic equations, the Bethe ansatz equations, which in addition to incorporating the scattering dynamics of the model also incorporate the boundary conditions. We shall derive the equations and discuss in detail the boundary effects that follow from the presence of the boundary terms in the Bethe ansatz equations. The solutions of the equations, the Bethe roots, together with the total spin S^z usually characterize the eigenstates of the Hamiltonian, its ground state in particular (see below).

Analyzing the equations we shall find that in the SSS phase the ground state is unique with a total spin $S^z = 0$ as in the periodic boundary conditions case [18]. In this state all the Bethe roots are real. In the STS phase on the other hand we find *three* ground states which are degenerate in the thermodynamic limit. Two of the states, denoted $|\widehat{\frac{1}{2}}\rangle$ and $|-\widehat{\frac{1}{2}}\rangle$, have spins $S^z = 1/2$ and $S^z = -1/2$, respectively. The state with spin $S^z = 1/2$ is constructed from Bethe reference state with all spin up and the state with spin $S^z = -1/2$ is constructed from Bethe reference state with all spin down. These states have all real Bethe roots and have identical Bethe root distribution. The third state $|\widehat{0}\rangle$ has spin $S^z = 0$. It is constructed by adding a purely imaginary solution to either of the states $|\widehat{\frac{1}{2}}\rangle$, $|-\widehat{\frac{1}{2}}\rangle$. Purely imaginary Bethe roots are referred to as boundary strings and correspond to boundary bound states [26].

So far our Bethe ansatz solution for symmetric OBC appears to disagree with the semiclassical predictions which predict two $S^z = 0$ states. We note however that in the presence of symmetries the solutions to Bethe equations might not give all the states in the Hilbert space. To obtain these states, one needs to apply operators associated with the symmetries to the states obtained directly from the Bethe equations—examples are spin raising operators applied to a highest weight state obtainable as Bethe ansatz state so as to complete a $SU(2)$ multiplet.

In the present case the space parity symmetry $x \to -x$ induces two equal boundary terms in the Bethe equations of the STS phase, due to which the boundary string occurs as a double pole. One expects that this double pole corresponds to two states, namely, there exists another singlet state $|\widehat{0'}\rangle$ in addition to the state $|\widehat{0}\rangle$. As this state $|\widehat{0'}\rangle$ cannot be obtained directly from the Bethe equations and the construction of a corresponding generating symmetry operator in the Bethe ansatz framework is a nontrivial task, we resolve this issue by considering a slightly *asymmetric* boundary conditions allowing a small twist ϵ' between the left and right moving fermions at the right boundary [to be defined in (92)]. This splits the double pole giving rise to the expected additional boundary string solution, the fourth state $|\widehat{0'}\rangle$, which is obtained in the limit where $\epsilon' \to 0$.

This asymmetric BC which also breaks the $\mathbb{Z}_2$ symmetry does not change the number of ground states in the SSS phase which remains unique with spin $S^z = 0$ in the scaling limit. In the STS phase we obtain four states, two with spin $S^z = \pm 1/2$ and two with spin $S^z = 0$ in the scaling limit. As before, the state $|\widehat{\frac{1}{2}}\rangle_{\epsilon'}$ with spin $S^z = 1/2$ is constructed from Bethe reference state with all spin up and the state $|-\widehat{\frac{1}{2}}\rangle_{\epsilon'}$, with spin $S^z = -1/2$, is constructed from Bethe reference state with all spin down. These two states have all real roots

TABLE I. Total spin and energy in the scaling limit of low lying states in the STS phase with asymmetric boundary conditions.

State	Total spin	Energy
$|\widehat{0'}\rangle_{\epsilon'}$	0	0
$|-\widehat{\frac{1}{2}}\rangle_{\epsilon'}$	$-1/2$	0
$|\widehat{0}\rangle_{\epsilon'}$	0	$\delta E_{\epsilon'}$
$|\widehat{\frac{1}{2}}\rangle_{\epsilon'}$	$1/2$	$\delta E_{\epsilon'}$

but have now slightly different Bethe root distributions and they differ in their energy: $E_{S^z=1/2} - E_{S^z=-1/2} = \delta E_{\epsilon'}$. Again, a state $|\widehat{0}\rangle_{\epsilon'}$ with spin $S^z = 0$ can be obtained by adding a purely imaginary solution to the state with spin $S^z = 1/2$ and is degenerate with it. However, due to the presence of the asymmetry another purely imaginary solution exists, which when added to the state with spin $S^z = 1/2$ gives a state $|\widehat{0'}\rangle_{\epsilon'}$ with spin $S^z = 0$ which is degenerate with the state with spin $S^z = -1/2$.

In the limit $\epsilon' \to 0$, when the asymmetry vanishes, the energy difference $\delta E_{\epsilon'} \to 0$ and the states $|\widehat{\frac{1}{2}}\rangle_{\epsilon'}$, $|-\widehat{\frac{1}{2}}\rangle_{\epsilon'}$ transform into states $|\widehat{\frac{1}{2}}\rangle$ and $|-\widehat{\frac{1}{2}}\rangle$, respectively. The two singlet states $|\widehat{0}\rangle_{\epsilon'}$ and $|\widehat{0'}\rangle_{\epsilon'}$ yield two different spin singlet states $|\widehat{0}\rangle$ and $|\widehat{0'}\rangle$ which are quasidegenerate in the limit $\epsilon' \ll 1$. The state $|\widehat{0'}\rangle$ is precisely the state we wished to construct in the symmetric BC case. We thus conclude that the fourfold ground state degeneracy found in the semiclassical analysis by Kesselman and Berg [1] in the strongly anisotropic regime ($g_\parallel \gg 1$) survives strong quantum fluctuations down to weak couplings in the region $\widehat{A}$ and on the dual Gross-Neveu line $\widehat{GN}$ (Table I).

B. Bethe equations

Since the Hamiltonian (1) commutes with total particle number N (4), $\mathcal{H}$ can be diagonalized by constructing the exact eigenstates in each N sector. From here on, for notational convenience, we shall use the notation $(+, -)$ to indicate the chirality index of the fermions replacing (R, L) notation. The N-particle eigenstate takes the standard reflection Bethe ansatz form of a plane wave expansion in $N! \, 2^N$ different regions of coordinate space. The state is labeled by momenta k_j, $j = 1 \cdots N$, the same in all regions, and is given by

$$|\{k_j\}\rangle = \sum_{Q,\vec{a},\vec{\sigma}} \int d^N x \, \theta(x_Q) A^{\{\sigma\}}_{\{a\}}[Q] \prod_j^N e^{i\sigma_j k_j x_j} \psi^\dagger_{a_j \sigma_j}(x_j)|0\rangle \tag{48}$$

with energy eigenvalue $E = \sum_j k_j$. In the above equation, the sum is to be taken over all spin and chirality configurations specified by $\{a\} = \{a_1 \ldots a_N\}$, $\{\sigma\} = \{\sigma_1 \ldots \sigma_N\}$ as well as different orderings of the N particles. These different orderings correspond to elements of the symmetric group $Q \in \mathcal{S}_N$. Here $\theta(x_Q)$ denotes the Heaviside function which is nonzero only for that particular ordering Q. The amplitudes $A^{\vec{\sigma}}_{\vec{a}}[Q]$ are related to each other by the various S matrices. Amplitudes which differ by changing the chirality of the rightmost and leftmost particle are related by the boundary S matrices which

are identities in our system owing to the open boundary conditions (2). Amplitudes which are related by swapping the order of particles with different chiralities are related by the particle-particle S matrix, which is given by [17]

$$S^{ij} = \begin{pmatrix} 1 & & & \\ & \dfrac{\sinh(f)}{\sinh(f+\eta)} & \dfrac{\sinh(\eta)}{\sinh(f+\eta)} & \\ & \dfrac{\sinh(\eta)}{\sinh(f+\eta)} & \dfrac{\sinh(f)}{\sinh(f+\eta)} & \\ & & & 1 \end{pmatrix}, \tag{49}$$

where $\eta = -iu$ and f, u are related to $g_\parallel$ and $g_\perp$ through the relations

$$\cos(u) = \frac{\cos(g_\parallel)}{\cos(g_\perp)}, \quad \frac{\sin(u)}{\tanh(f)} = \frac{\sin(g_\parallel)}{\cos(g_\perp)}. \tag{50}$$

An additional S matrix, denoted by W^{ij}, is also required. It relates amplitudes that differ by exchanging particles of the same chirality. This is given by

$$W^{ij} = P^{ij}. \tag{51}$$

The consistency of the solution is then guaranteed as the S and W matrices satisfy the Yang-Baxter and Reflection equations [30–32].

Imposing the boundary condition at $x = \pm L/2$ quantizes the single particle momenta k_j which are expressed in terms of M parameters λ_β, the Bethe rapidities or Bethe roots, which satisfy a set of coupled nonlinear equations called the Bethe equations. In a state, M denotes the number of down spins and $N - M$ is the number of up spins and vice versa. We use the method of boundary algebraic Bethe ansatz to obtain the logarithmic form of Bethe equations, which take different forms in different regions of Fig. 1.

For definiteness we give the explicit form of the Bethe equations in the regions A and $\widehat{A}$ below.

$$\sum_{\sigma=\pm} N\Theta(\lambda_\alpha + \sigma f/2u, 1/2) - 2\Theta(\lambda_\alpha + i\tau\pi/2u, 1/2)$$

$$= \sum_{\beta=1}^{M} \sum_{\sigma=\pm} \Theta(\lambda_\alpha + \sigma\lambda_\beta, 1) + 2i\pi I_\alpha, \tag{52}$$

$$k_j = \frac{\pi n_j}{L} + \frac{i}{2L} \sum_{\beta=1}^{M} \sum_{\sigma=\pm} \Theta(f/2u + \sigma\lambda_\beta, 1/2), \tag{53}$$

where $\Theta(x, y) = \log\left(\frac{\sinh(u(x+iy))}{\sinh(u(x-iy))}\right)$. The second term in Eq. (52) is a boundary term where $\tau = 1$ in the region A, and $\tau = 0$ in region $\widehat{A}$. The parameters f, u are real in the regions A and $\widehat{A}$. The GN and $\widehat{GN}$ lines correspond to the isotropic limit ($f, u \to 0$, $f/u = 1/g$) of the Bethe equations of regions A and $\widehat{A}$, respectively [18]. We work in the region where $u < \pi/2$ which corresponds to $4\pi < \beta^2 < 8\pi$ in SG.

The boundary term in the topological phase leads to a dramatic change in the degeneracy of the ground state in the region $\widehat{A}$ and on the $\widehat{GN}$ line. The Bethe roots govern the spin degrees of freedom of the system and $M \leqslant N/2$ gives the total z component of spin, $S^z = N/2 - M$. The solutions to equations of type (52) are well studied in the literature [33,34]. The solutions λ_α can be real or take complex values in the form of strings. In order to have a nonvanishing wave function

they must all be distinct, $\lambda_\alpha \neq \lambda_\beta$. In addition, the values $\lambda_\alpha = (0, i\pi/2u)$ should also be discarded as they result in a vanishing wave function [34]. Bethe equations of the type (52) are reflective symmetric, that is they are invariant under $\lambda_\alpha \to -\lambda_\alpha$ transformation. Due to this symmetry, solutions to the Bethe equations occur in pairs $\{-\lambda_\alpha, \lambda_\alpha\}$. The integers n_j and I_α arise from the logarithmic branch and serve as the quantum numbers of the states. The quantum numbers I_α correspond to the spin degrees of freedom while the quantum numbers n_j are associated with the charge degrees of freedom and they must all be different. I_α and n_j can be chosen independently implying the charge spin decoupling. Minimizing the ground state energy results in a cutoff such that the $\pi|n_j|/L < \pi D$ where $D = N/L$ is the density [16].

C. The SSS phase

This corresponds to the regions A and the GN line, as displayed in Fig. 1. We shall consider them separately in the following.

1. Region A

The ground state is given by the particular choice of charge and spin quantum numbers n_j^0, I_α^0, where n_j^0 are consecutively filled from the lower cutoff $-LD$ upwards, and the integers I_α^0 take consecutive values which corresponds to real valued λ_α roots in the region A. In the limit $N \to \infty$ the Bethe roots fill the real line and the ground state can be described by $\rho(\lambda)$ the density of solutions λ, from which the properties of the ground state can be obtained. Reflection symmetry of the Bethe equations (52) allows us to define $\lambda_{-\alpha} = -\lambda_\alpha$, $\lambda_0 = 0$ [35] and introduce the counting function $\nu(\lambda)$ such that $\nu(\lambda_\alpha) = I_\alpha$. Differentiating (52), and noticing that $\rho(\lambda) = \frac{d}{d\lambda}\nu(\lambda)$ [36], we obtain the following integral equation,

$$h_A(\lambda) = \rho_A(\lambda) + \int_{-\infty}^{+\infty} d\mu \, a_2(\lambda - \mu)\rho_A(\mu), \quad (54)$$

where ρ_A stands for the ground state density distribution in the region A and $h_A(\lambda) = Na_1(\lambda + \sigma f/2u) + a_2(\lambda) + a_1(\lambda) - b_1(\lambda)$ where

$$a_n(x) = \frac{u}{\pi} \frac{\sin(nu)}{\cosh(2ux) - \cos(nu)}, \quad (55)$$

$$b_n(x) = -\frac{u}{\pi} \frac{\sin(nu)}{\cosh(2ux) + \cos(nu)}. \quad (56)$$

Note that we have excluded the root $\lambda = 0$ and also applied the restriction $\lambda_\alpha \neq \lambda_\beta$.

Solving (54) by Fourier transformation [37] we obtain the Fourier transformed ground state distribution of Bethe roots in the region A

$$\tilde{\rho}_A(\omega) = \frac{N\cos\left[\frac{f\omega}{2u}\right] + \frac{1}{2}\left(\frac{\sinh((\pi - 2u)(\omega/2u)) + \sinh(\omega/2)}{\sinh((\pi - u)(\omega/2u))} + 1\right)}{\cosh\left[\frac{\omega}{2}\right]}. \quad (57)$$

The term which is proportional to N corresponds to the bulk contribution while the terms of order $N^{(0)}$ can be associated with the boundaries at $x = (-L/2, L/2)$. The number of

Bethe roots M_A in the ground state of region A is given by

$$2M_A + 1 = \int_{-\infty}^{+\infty} d\lambda \, \rho_A(\lambda), \quad (58)$$

from which the z component of spin $(S^z)_A$ of the ground state in this region is obtained using the relation $S_A^z = N/2 - M_A$. Taking into account that $\tilde{\rho}(0) = \int d\lambda \, \rho(\lambda)$ along with (57) we find that in the scaling limit, i.e., when $|g_\parallel| \ll 1$, $|g_\perp| \ll 1$, $u \ll 1$,

$$(S^z)_A = 0. \quad (59)$$

We thus find from (25) that the ground state in the region A has an even number of fermions and hence an *even* fermion parity $\mathcal{P} = +1$. It is nondegenerate and is a $\mathbb{Z}_2$ singlet.

2. GN line

On the GN line, the Bethe equations can be obtained by taking the limit $f, u \to 0$, $f/u = 1/g$, which leads to the limit

$$\log\left(\frac{\sinh(u(x+iy))}{\sinh(u(x-iy))}\right) \to \log\left(\frac{x+iy}{x-iy}\right),$$

$$\log\left(\frac{\cosh(u(x+iy))}{\cosh(u(x-iy))}\right) \to 1 \quad (60)$$

in (52). We obtain the following integral equation

$$h_{GN}(\lambda) = \rho_{GN}(\lambda) + \int_{-\infty}^{\infty} d\mu \, \rho_{GN}(\mu) \, \varphi(\lambda - \mu, 1), \quad (61)$$

where ρ_{GN} stands for the ground state density distribution on the GN line and $h_{GN}(\lambda) = \sum_{\sigma=\pm} 2N\varphi(2\lambda + \sigma/g, 1) + 2\varphi(2\lambda, 1) + \varphi(\lambda, 1)$ where

$$\varphi(x, a) = (1/\pi)(a^2 + x^2)^{-1}. \quad (62)$$

Solving (61) by Fourier transformation [19] we obtain the ground state distribution on the GN line,

$$\tilde{\rho}_{GN}(\omega) = \frac{N\cos\left[\frac{\omega}{2g}\right] + \frac{1}{2} + \frac{1}{2}e^{-\frac{|\omega|}{2}}}{\cosh\left[\frac{\omega}{2}\right]}. \quad (63)$$

The number of Bethe roots is given by an equation similar to (58), using which we obtain

$$(S^z)_{GN} = 0, \quad (64)$$

for the ground state in the region GN. Exactly as in the region A in the SSS phase, the ground state on the GN line is nondegenerate and has an even fermion parity $\mathcal{P} = +1$. It is actually, on top of being a $\mathbb{Z}_2$ singlet, an $SU(2)$ singlet.

In summary, we have seen that, although the descriptions of the ground state in terms of the Bethe root distribution is different in the region A and on the GN line, the ground state belongs to the even fermion parity sector $\mathcal{P} = +1$. It is nondegenerate, has a zero total spin $S^z = 0$, and is at least a $\mathbb{Z}_2$ singlet. Labeling the SSS ground state as $|0\rangle$ we have

$$S^z|0\rangle = 0, \ \mathcal{P}|0\rangle = |0\rangle, \ \tau|0\rangle = |0\rangle, \quad (65)$$

where $\mathcal{P} = (-1)^N$ is the fermionic parity operator and τ is the total spin flip operator generating the $\mathbb{Z}_2$ symmetry group (5). As in the case where periodic boundary conditions are imposed, the ground state properties are the same in these regions and there is no phase transition between them.

D. The STS phase

This corresponds to the regions $\widehat{A}$ and the $\widehat{GN}$ line. As already emphasized, in the STS phase the boundary term in the Bethe equations (52) leads to a change in the ground state degeneracy as we shall now see.

1. Region $\widehat{A}$

By following the same procedure as before, we obtain the following integral equation

$$h_{\widehat{A}}(\lambda) = \rho_{\widehat{A}}(\lambda) + \int_{-\infty}^{\infty} d\mu \, a_2(\lambda - \mu) \rho_{\widehat{A}}(\mu), \quad (66)$$

where $\rho_{\widehat{A}}$ stands for the ground state density distribution in the region $\widehat{A}$ and $h_{\widehat{A}}(\lambda) = N a_1(\lambda + \sigma f/2u) + a_2(\lambda) - a_1(\lambda) + b_1(\lambda)$. Solving (66) by Fourier transform we obtain the ground state distribution in region $\widehat{A}$

$$\tilde{\rho}_{\widehat{A}}(\omega) = \frac{N \cos\left[\frac{f\omega}{2u}\right] + \frac{1}{2}\left(\frac{\sinh((\pi - 2u)(\omega/2u)) - \sinh(\omega/2)}{\sinh((\pi - u)(\omega/2u))} - 1\right)}{\cosh\left[\frac{\omega}{2}\right]}. \quad (67)$$

Notice that the second and third term of the boundary contribution in the above expression have opposite sign compared to those in region A (57). As a consequence, using (58), we find that the ground state in region $\widehat{A}$ has a nonzero spin

$$(S^z)_{\widehat{A}} = \frac{1}{2}, \quad (68)$$

which, from (25), corresponds to an odd number of particles N and hence has an *odd* fermion parity $\mathcal{P} = -1$. Due to the $\mathbb{Z}_2$ symmetry (5) we immediately deduce that there is another ground state in the same fermion parity sector, degenerate with the above, which has the opposite spin

$$(S^z)_{\widehat{A}} = -\frac{1}{2}. \quad (69)$$

Actually, this state can be obtained by choosing the Bethe reference state with all spins down instead of up [38]. The two states $S^z = \pm 1/2$ have the same Bethe root distribution and transform into each other under the action of the $\mathbb{Z}_2$ generator τ. This is to be contrasted with the situation in the region A where the ground state, having $S^z = 0$, has an even fermion parity $\mathcal{P} = +1$ and is a $\mathbb{Z}_2$ singlet.

2. $\widehat{GN}$ line

The Bethe equations on the $\widehat{GN}$ line are rational just as in the case of GN. They can be obtained by taking the isotropic limit (60) of the Bethe equations in region $\widehat{A}$. We obtain the following integral equation

$$h_{\widehat{GN}}(\lambda) = \rho_{\widehat{GN}}(\lambda) + \int_{-\infty}^{\infty} d\mu \, \rho_{\widehat{GN}}(\mu) \, \varphi(\lambda - \mu, 1), \quad (70)$$

where $\rho_{\widehat{GN}}$ stands for the ground state density distribution on the $\widehat{GN}$ line and $h_{\widehat{GN}}(\lambda) = \sum_{\sigma = \pm} 2N\varphi(2\lambda + \sigma/g, 1) - 2\varphi(2\lambda, 1) + \varphi(\lambda, 1)$.

Solving (70) by Fourier transformation we obtain the following distribution of Bethe roots in the ground state on the $\widehat{GN}$ line

$$\tilde{\rho}_{\widehat{GN}}(\omega) = \frac{N \cos\left[\frac{\omega}{2g}\right] - \frac{1}{2} + \frac{1}{2} e^{-\frac{|\omega|}{2}}}{\cosh\left[\frac{\omega}{2}\right]}. \quad (71)$$

Using an equation similar to (58) we obtain two degenerate ground states with spins

$$(S^z)_{\widehat{GN}} = \pm \frac{1}{2}. \quad (72)$$

As in the region $\widehat{A}$ these two states have an odd fermion parity and transform into each other under τ. Notice that, contrarily to the ground state on the GN line, they are obviously not $SU(2)$ singlet states. This is consistent with the fact that on the $\widehat{GN}$ line the model is only $U(1)_s \otimes \mathbb{Z}_2$ symmetric and that the enlarged $\widehat{SU(2)}$ symmetry (8) is nonlocal.

Just as in the SSS phase, despite having a different description in terms of the Bethe root distributions, there is no phase transition between these regions in the STS phase. Here the ground state belongs to the *odd* fermion parity sector $\mathcal{P} = -1$ and is doubly degenerate, each ground state having spins $S^z = \pm 1/2$. The degeneracy here is to be understood as the consequence of the nonvanishing of the spin in the ground state and of the $\mathbb{Z}_2$ symmetry which reverses the total spin. This is to be contrasted with what happens in the SSS phase where the ground state, having a spin zero, is not degenerate.

As we shall see, besides the two degenerate states with $S^z = \pm 1/2$, there is one more state with $S^z = 0$ which is degenerate with the ground states in the large system size limit $L \to \infty$. This state is a solution of the Bethe equations that involve a boundary string which corresponds to a boundary mode.

E. Bulk excitations

Excitations correspond to states whose quantum numbers n_j or I_α have been modified from their ground state configurations. Note that we can choose n_j and I_α independently, meaning that the spin and charge degrees of freedom are decoupled [16,39,40]. In the charge sector the excitations are constructed by removing a number $n^h < 0$ from the sequence n_j^0 and adding an extra $n^p > 0$. The energy of this excitation is $\delta E = 2\pi(n^p - n^h)/L > 0$. Gapless excitations such as this are known as holons. The structure of excitations in the spin sector is more complicated as they arise from solutions to the Bethe ansatz equations (52) for non-ground-state configurations of the I_α quantum numbers. The lowest energy spin bulk excitation is of two spinons which is constructed by removing two arbitrary Bethe roots λ_1^h, λ_2^h from the ground state distribution [16]. Each hole corresponds to a single spinon with spin $+1/2$. The energy of this excitation in all the regions except on the GN and $\widehat{GN}$ lines is

$$\delta E = \sum_{l=1}^{2} D \arctan\left[\frac{\cosh(\pi \lambda_l^h/u)}{\sinh(f\pi/2u)}\right]. \quad (73)$$

From this we find that the system has dynamically generated a superconducting mass gap in the spin sector

$$m = D \arctan\left[\sinh(f\pi/2u)\right]^{-1}. \quad (74)$$

Universality

Having obtained a dynamically generated mass gap we may remove the cutoff D and obtain universal answers, in other words taking the scaling limit $D \to \infty$ while holding the physical mass m fixed. This corresponds to $g_\parallel \ll 1$, $g_\perp \ll 1$, or

$u \ll 1$. In this limit we have that $m = 2De^{-f\pi/2u}$ and the excitation energy of a single spinon becomes $\varepsilon(\lambda) = m \cosh(\lambda)$, where π/u is absorbed into λ. On the GN and $\widehat{GN}$ lines, the excitation energy and the mass gap is obtained by the replacement $f/u \to 1/g$.

F. Boundary excitations

The boundary modes arise as purely imaginary solutions of the Bethe equations. These purely imaginary Bethe roots, which correspond to the bound states, appear as poles in the dressed or physical boundary S matrix [24–26,41]. By observation we see that, in the limit $N \to \infty$, the Bethe equations (52) have a *unique* solution

$$\lambda = \pm i/2, \tag{75}$$

as the two $\pm$ strings lead to the same state by reflection symmetry. This is to be true both in the region $\widehat{A}$ and on the $\widehat{GN}$ line. Adding the boundary string to either the $S^z = 1/2$ or the $S^z = -1/2$ ground state a unique state is found. The reason is that two Bethe states are equivalent if they are described by the same root distribution and in addition have the same total spin S^z values. As already mentioned, both the states with $S^z = \pm 1/2$ have the same root distribution. The states obtained by adding the boundary string solution to these states with $S^z = \pm 1/2$ will again have the same root distribution. It turns out that these resulting states both have $S^z = 0$ and hence they both are equivalent.

1. Region $\widehat{A}$

Adding the boundary string (75) to the Bethe equations in region $\widehat{A}$ (52) results in the following equation

$$-2i\pi I_\alpha + \sum_{\sigma=\pm} N\Theta(\lambda_\alpha + \sigma f/2u, 1/2) - 2\Theta(\lambda_\alpha, 1/2)$$
$$= \Theta(\lambda_\alpha, 1/2) + \Theta(\lambda_\alpha, 3/2) + \sum_{\beta=1}^{M-1}\sum_{\sigma=\pm} \Theta(\lambda_\alpha + \sigma\lambda_\beta, 1). \tag{76}$$

The above equation can be solved by following the same procedure as in the ground state. We obtain the following distribution of Bethe roots

$$\tilde{\rho}_{\widehat{A}}^b = \tilde{\rho}_{\widehat{A}} + \Delta\tilde{\rho}_{\widehat{A}}^b, \tag{77}$$

where $\tilde{\rho}_{\widehat{A}}$ is the ground state distribution given by (67) and the shift

$$\Delta\tilde{\rho}_{\widehat{A}}^b = -\frac{\sinh((\pi - 2u)(\omega/2u))}{\sinh((\pi - u)(\omega/2u))} \tag{78}$$

is due to the presence of the boundary string. In the presence of the boundary string, the relation between the number of Bethe roots and the density distribution also takes a different form as compared to (58). Namely

$$2M_{\widehat{A}}^b - 1 = \int_{-\infty}^{+\infty} d\lambda \, \rho_{\widehat{A}}^b(\lambda), \tag{79}$$

from which, using $(S^z)_{\widehat{A}}^b = N/2 - M_{\widehat{A}}^b$, we find, in the scaling limit $u \ll 0$, the spin of this state

$$(S^z)_{\widehat{A}}^b = 0. \tag{80}$$

Thus the resulting state corresponding to the boundary string (75) is a spin singlet which has fermion parity $\mathcal{P} = +1$. From the analysis of the Bethe equations of XXZ spin chain [26] with equal boundary terms, we expect that the wave function associated with this unique fundamental boundary string is exponentially localized near both the left and right boundaries. Furthermore it is symmetric upon the exchange of both boundaries or under space parity $x \to -x$. However since it has a total spin $S^z = 0$, we cannot infer how this state transforms under the $\mathbb{Z}_2$ symmetry (5). As we shall now see the situation is similar on the $\widehat{GN}$ line.

2. $\widehat{GN}$ line

On the $\widehat{GN}$ line we find that the addition of the boundary string leads to the following change in the distribution of the Bethe roots

$$\tilde{\rho}_{\widehat{GN}}^b = \tilde{\rho}_{\widehat{GN}} + \Delta\tilde{\rho}_{\widehat{GN}}^b, \quad \Delta\tilde{\rho}_{\widehat{GN}}^b = -e^{-|\omega|/2}. \tag{81}$$

Using an equation similar to (79) we find in the scaling limit

$$(S^z)_{\widehat{GN}}^b = 0. \tag{82}$$

Again, as expected, we obtain a unique state with $S^z = 0$ by adding the boundary string to either of the ground states with spins $\pm 1/2$ on the $\widehat{GN}$ line.

3. Boundary string energy

As seen, the addition of the boundary string to the ground state with either spins $S^z = \pm 1/2$ in the STS phase leads, in each of the regions $\widehat{A}$ and $\widehat{GN}$, to a *single* new state with spin $S^z = 0$ that includes a boundary excitation. To get the energy of this state, or of the boundary string, we notice that it is given by the energy difference, up to chemical potential, between the ground states with $S^z = 0$ and $S^z = \pm 1/2$

$$E_B = E_N - \tfrac{1}{2}(E_{N-1} + E_{N+1}). \tag{83}$$

Here E_N refers to the energy of the state with an odd number of particles which, in our system, corresponds to the ground states in the topological phase with spin $S^z = \pm 1/2$. Similarly E_{N+1} and E_{N-1} refer to the energies of the states with an *even* number of particles and spin $S^z = 0$. The latter states include the added boundary string. The expression (83) is defined in Ref. [15] as the binding energy, which precisely measures the energy cost of adding an electron to the system, where it is shown that this is equal to only the charging energy in the topological phase and is equal to the mass gap in the topologically trivial phase.

As it can be shown, the value of the boundary string energy is the same in both the region $\widehat{A}$ and on the $\widehat{GN}$ line. We shall consequently evaluate E_B in the region $\widehat{A}$. To this end we use (53), from which we obtain the following expression for total energy of a state with N fermions

$$E = \sum_{j=1}^{N} \frac{\pi}{L} n_j + \frac{iD}{2} \int_{-\infty}^{\infty} d\lambda \, \Theta\left(\frac{f}{2u} - \lambda, \frac{1}{2}\right) \rho_{\widehat{A}}(\lambda). \tag{84}$$

From (83) we find that E_B has two contributions, one from the charge degrees of freedom and one from the spin degrees of freedom: $E_B = E_{\text{charge}} + E_{\text{spin}}$. The charge contribution is given by the charging energy

$$E_{\text{charge}} = \sum_{j=1}^{N} \frac{\pi}{L} n_j - \frac{1}{2}\left(\sum_{j=1}^{N+1} \frac{\pi}{L} n_j + \sum_{j=1}^{N-1} \frac{\pi}{L} n_j\right). \quad (85)$$

Note that the charge quantum numbers take all the values from the cutoff $-DL$ upwards. In the ground state with $S^z = \pm 1/2$ they fill all the slots from $n_j = -N$ to $n_j = -1$. In the state with one extra particle they fill all the slots from $n_j = -N$ to $n_j = 0$. In the state with one less particle there is an unfilled slot at $n_j = -1$ which corresponds to a holon excitation. Hence we obtain

$$E_{\text{charge}} = -\frac{\pi}{2L}. \quad (86)$$

The spin contribution is given by the expression

$$E_{\text{spin}} = E_0 + \frac{iD}{2} \int_{-\infty}^{+\infty} d\lambda\, \Theta\left(\frac{f}{2u} - \lambda, \frac{1}{2}\right) \Delta\rho_A^b(\lambda), \quad (87)$$

where $E_0 = \frac{iD}{2} \Theta(f/2u, 1)$ and $\Delta\rho_A^b(\lambda)$ is the shift of the Bethe roots distribution due to the boundary string which is given in (77). Evaluating (87) we find that the spin part of the energy of the boundary string is exactly zero in the thermodynamic limit. Hence this corresponds to a zero energy boundary bound state localized at the two ends of the system in a finite system, the boundary string, which is a solution to the Bethe equations in the limit $N \to \infty$, have corrections of the order $1/N$. Since finite size corrections to the Bethe equations are generally expected to be exponentially small in the system size we expect that $E_{\text{spin}} \sim e^{-mL}$ also. As a result, we thus find that the energy of the boundary string is, to the exponential accuracy in the system size, given by the charging energy (85)

$$E_B = -\frac{\pi}{2L} \quad (88)$$

and hence vanishes in the thermodynamical limit.

We thus find that in the regions $\widehat{A}$ and $\widehat{GN}$ of the STS phase the ground state is only *threefold* degenerate in the limit of infinite size in contrast with the *fourfold* degeneracy predicted by the semiclassical analysis of the preceding section (Sec. III). The three ground states in the STS phase are given by the two $S^z = \pm 1/2$ ground states found in the odd fermion parity sector (IV D) plus a single $S^z = 0$ state in the even fermion parity sector which is obtained from them by adding the boundary string $\lambda = \pm i/2$. Labelling the ground states in the STS phase by their spins

$$\left|-\widehat{\tfrac{1}{2}}\right\rangle, \ \left|+\widehat{\tfrac{1}{2}}\right\rangle, \ |\widehat{0}\rangle, \quad (89)$$

with $S^z|\pm\widehat{\tfrac{1}{2}}\rangle = \pm\tfrac{1}{2}|\pm\widehat{\tfrac{1}{2}}\rangle$ and $S^z|\widehat{0}\rangle = 0$, we have $\mathcal{P}|\pm\widehat{\tfrac{1}{2}}\rangle = -|\pm\widehat{\tfrac{1}{2}}\rangle$ and $\mathcal{P}|\widehat{0}\rangle = |\widehat{0}\rangle$. However, although we clearly have $\tau|\pm\widehat{\tfrac{1}{2}}\rangle = |\mp\widehat{\tfrac{1}{2}}\rangle$, we cannot infer from our analysis whether the state $|\widehat{0}\rangle$ is symmetric or antisymmetric under the $\mathbb{Z}_2$ symmetry group generator τ (5). In either case, when comparing the semiclassical prediction given in Eq. (43) to Eq. (89),

one notes that one extra spin singlet state is expected but not obtained from the solution to the Bethe equations. One may then wonder whether the fourfold degeneracy predicted in the semiclassical approximation survives into the full quantum regime. We shall argue in the following that this is the case.

G. Asymmetric boundary conditions

Actually, in the presence of symmetries not all states are given as solutions of the Bethe ansatz equations. A well known example is the bulk spin one triplet excitation of the $SU(2)$ invariant Gross-Neveu model where the $S^z = 0$ component is not given by a solution of the Bethe equations unlike the $S^z = \pm 1$ components. This state is obtained by applying a spin lowering operator to the $S^z = 1$ triplet excitation [42]. In the present case due to the space parity symmetry we obtain a unique boundary string solution which occurs as a double pole in the Bethe equations of the STS phase, see Appendix Eq. (A41). Hence one may expect that it should count as two states, namely, that there exists another state $|\widehat{0}'\rangle$ with $S^z = 0$ in addition to the state $|\widehat{0}\rangle$. Such a state cannot be obtained simply by a lowering operator as was the case for the $SU(2)$ multiplets, discussed earlier, since $\mathbb{Z}_2$ representations are all one dimensional.

To circumvent this problem we break the space parity symmetry by considering asymmetric boundary conditions, which splits the double pole and yields another boundary string solution. As a result the second spin singlet state is obtained as a solution to the Bethe equations leading to two quasidegenerate ground states in the limit of infinitesimal asymmetry with wave functions localized at either the left or the right edge. These two states, in properly renormalized symmetric limit, account for a twofold degeneracy of the ground state missed by the Bethe ansatz analysis of the symmetric case.

We consider now the following asymmetric OBC

$$\Psi_{Ra}(L/2) = -B_{ab}\Psi_{Lb}(L/2), \quad (90)$$

$$\Psi_{Ra}(-L/2) = -\Psi_{Lb}(-L/2), \quad (91)$$

where

$$B_{ab} = \frac{1}{\cosh(f/2)}\begin{pmatrix} \cosh(f/2 + i\epsilon) & 0 \\ 0 & \cosh(f/2 - i\epsilon) \end{pmatrix}, \quad (92)$$

and $\epsilon > 0$ is an asymmetry parameter. The latter boundary conditions, which break both space parity and the $\mathbb{Z}_2$ symmetry (5), give back the symmetric OBC in the limit $\epsilon \to 0$. Remarkably enough the problem is still integrable when $\epsilon \neq 0$ and the resulting Bethe equations (see Appendix A 2) are given by

$$-2\pi I_\alpha + \sum_{\sigma=\pm} N\Theta(\lambda_\alpha + \sigma f/2u, 1/2)$$
$$- \Theta(\lambda_\alpha + i\tau\pi/2u, 1/2)$$
$$- \Theta(\lambda_\alpha + i\tau\pi/2u, (1-\epsilon')/2)/2)$$
$$= \sum_{\beta=1}^{M}\sum_{\sigma=\pm} \Theta(\lambda_\alpha + \sigma\lambda_\beta, 1), \quad (93)$$

where $\tau = (1, 0)$ in the regions A and $\widehat{A}$, respectively, and

$$\epsilon' = 2\epsilon/u. \tag{94}$$

As we can readily see, unlike (52), the latter equations display two different boundary terms. These equations can be solved by following the same procedure as for symmetric boundary conditions. In order to obtain a nontrivial solution in the scaling limit, where $u \to 0$, one needs to take also simultaneously the limit $\epsilon \to 0$ with ϵ' maintained fixed and small. In this limit it is ϵ' that plays the role of the physical asymmetry parameter.

In the SSS phase, we find that the ground state in the region A and on the GN line has total spin $S^z = 0$ in the scaling limit and that there is no boundary string solutions just as in the symmetric case. In this phase the asymmetry in the boundary conditions plays a marginal role. This is not the case in the STS phase. This is due to the fact that, in the symmetric case, the two degenerate ground states carry a nonzero spin $S^z = \pm 1/2$. The asymmetry at the right boundary, which distinguishes between up and down spins, will then lift the degeneracy. Indeed for a positive $\epsilon > 0$ we find that the ground state has a spin $S^z = -1/2$ while the state with $S^z = +1/2$ has a higher energy. On top of that, due to the presence of the two different boundary terms in (93), there exists now *two* different boundary strings at

$$\lambda = \pm i/2, \ \lambda = \pm i(1 - \epsilon')/2. \tag{95}$$

For a positive $\epsilon > 0$ one can only add these two boundary strings to the state with the higher spin $S^z = +1/2$. Doing that we end up with *two different* spin singlet states with $S^z = 0$

$$|\widehat{0}\rangle_{\epsilon'} \to \lambda = \pm i/2,$$
$$|\widehat{0'}\rangle_{\epsilon'} \to \lambda = \pm i(1 - \epsilon')/2. \tag{96}$$

The calculation of the energy of the two boundary strings (95), and hence of the two states (96), proceed as in the symmetric case. For both strings the above energy splits into a charge and a spin part: $E_B = E_{\text{charge}} + E_{\text{spin}}$. While the charge contribution is still $E_{\text{charge}} = -\pi/2L$ for both strings (95), the spin contributions are different. In the limit of large system size it is zero [to order $\mathcal{O}(e^{-mL})$] for the first string while the second string has a finite energy which is precisely the energy splitting between the two $S^z = \pm 1/2$ states. Therefore, as far as the spin degrees of freedom are concerned, in the presence of a nonzero $\epsilon > 0$, the ground state is twofold degenerate and consists of two states $(|\widehat{0'}\rangle_{\epsilon'}, |-\frac{1}{2}\rangle_{\epsilon'})$. The two other states $(|\widehat{0}\rangle_{\epsilon'}, |\frac{1}{2}\rangle_{\epsilon'})$ have a higher energy $\delta E_{\epsilon'} = m \sin(\epsilon'\pi/2)$. The calculation of this energy and discussion about the structure of the ground states and associated symmetries in the presence of asymmetric boundary conditions goes beyond the scope of this work, hence it will be discussed in further works.

When $\epsilon' \to 0$, the energy splitting between these states $\delta E_{\epsilon'}$ goes to zero, and the two spin singlet states (96) are quasidegenerate in the limit of infinitesimally small asymmetry $\epsilon' \ll 1$. They correspond to two zero energy boundary bound state modes which are localized at the two ends of the system. Although the above analysis does not tell us about the status of the two states with respect to the $\mathbb{Z}_2$ (5), i.e., whether they are symmetric or antisymmetric under the action of τ, it

does tell us that there are *two* states in the spin singlet sector when $0 < \epsilon' \ll 1$. On physical grounds, we do not expect anything special to happen to *the number of states* in the symmetric limit which should be two when $\epsilon' \to 0$. Of course, when $\epsilon' = 0$, the two boundary strings (95) become identical and the two states in (96) overlap. However, as in the XXZ spin chain [26], we expect that, in a suitable renormalized limit $\epsilon' \to 0$, the two states $|\widehat{0}\rangle_{\epsilon'}$ and $|\widehat{0'}\rangle_{\epsilon'}$ yield different spin singlet states $|\widehat{0}\rangle$ and $|\widehat{0'}\rangle$ in the symmetric limit.

These two singlet states together with the two odd fermion parity $S^z = \pm 1/2$ spin states account for the fourfold degeneracy found in the semiclassical analysis. However, as in the symmetric case, the present analysis cannot explain the status of the two singlet states with respect to the $\mathbb{Z}_2$ symmetry and hence we are unable to relate these states with the Majorana construction given in the preceding section. We hope to come back to this nontrivial issue in a further publication.

V. DISCUSSION

We have provided the exact solution of the $U(1)$ Thirring model on a finite line segment with both symmetric and asymmetric open boundary conditions (OBC). We showed that the fourfold ground state degeneracy found by semiclassical analysis [1] can be understood as being due to the presence of two zero energy boundary bound states localized at the edges of the system. These bound states correspond to two boundary strings solutions of the Bethe equations in the presence of slightly asymmetric OBC. Our results are consistent with the semiclassical analysis based on the presence of spin $\pm 1/4$ localized at the two edges of the system and support the fact that the massless spin-triplet superconducting topological state, predicted in the anisotropic regime $g_\parallel \gg 1$, survives strong quantum fluctuations at least in the region $\widehat{A}$ and on the dual $\widehat{GN}$ line.

However, our Bethe ansatz approach cannot track down the two zero energy Majorana modes ($\gamma_\mathcal{L}, \gamma_\mathcal{R}$) [see Eqs. (45)], associated with the $\mathbb{Z}_2$ symmetry (5), which are responsible for the topological order in a given fermion parity sector. Probing these Majorana modes would require a detailed calculation of the wave functions in real space associated with the boundary bound states. This is a formidable task in the present fermionic field theory. However, related work on the XXZ spin chain [26], where the boundary bound states wave functions can be obtained with asymmetric boundary fields, suggests that one could possibly probe these Majorana modes in a suitable symmetric limit.

Although the second spin singlet state $|\widehat{0'}\rangle$ was obtained by considering slightly asymmetric OBC, it is not a solution of the Bethe equation in the symmetric case. An alternative way would be to construct an analog of a lowering operator acting on a highest weight spin-1 state to obtain the S^z member of the multiplet, though in itself it cannot be obtained as a solution of the Bethe ansatz equation [42]. Similarly, in our present case one would need to construct a "raising" or "lowering" operator Γ which, when acting on the singlet solution $|\widehat{0}\rangle$, gives the desired state, i.e.,

$$\Gamma|\widehat{0}\rangle = |\widehat{0'}\rangle. \tag{97}$$

In the topological phase such an operator would be provided by one of two zero energy Majorana modes $\Gamma = (\gamma_{\mathcal{L}}, \gamma_{\mathcal{R}})$ of Eq. (45) associated with the $\mathbb{Z}_2$ symmetry (5) with $[\Gamma, H] = [\Gamma, \mathcal{P}] = 0$, $\{\Gamma, \tau\} = 0$, and $\Gamma^2 = 1$.

Last but not least, it would be interesting to understand what happens in the regions $C(\widehat{C})$. Although we have obtained the Bethe equations in these regions we find that there are issues when one wants to obtain universal answers. This leaves open the question of the topological nature of the STS phase in the region $\widehat{C}$. We hope to come upon all these topics in further works.

ACKNOWLEDGMENTS

APPENDIX: BETHE ANSATZ

In this section we derive the Bethe equations of the model subject to the following asymmetric boundary conditions

$$\Psi_{Ra}(L/2) = -B_{ab}\Psi_{Lb}(L/2), \quad \Psi_{Ra}(-L/2) = -\Psi_{Lb}(-L/2) \tag{A1}$$

where

$$B_{ab} = \frac{1}{\cosh(f/2)}\begin{pmatrix} \cosh(f/2 + i\epsilon) & 0 \\ 0 & \cosh(f/2 - i\epsilon) \end{pmatrix}, \tag{A2}$$

and $\epsilon > 0$ is an asymmetry parameter. The left boundary has the usual open boundary condition whereas the right boundary has a 'slightly twisted' boundary condition. This breaks the space parity and $\mathbb{Z}_2$ symmetry which gives rise to two fundamental boundary string solutions. Symmetric boundary condition can be obtained by taking the limit $\epsilon \to 0$ which restores the broken space parity and the $\mathbb{Z}_2$ symmetry.

1. *N*-particle solution

The Hamiltonian commutes with total particle number, $N = \int \psi_+^\dagger(x)\psi_+(x) + \psi_-^\dagger(x)\psi_-(x)$, and H can be diagonalized by constructing the exact eigenstates in each N sector. Since N is a good quantum number we may construct the eigenstates by examining the different N particle sectors separately. We start with $N = 1$ wherein we can write the wave function as an expansion in plane waves,

$$|k\rangle = \sum_{a_j=\uparrow\downarrow,\sigma=\pm} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx\, e^{i\sigma kx} A_{a_1}^\sigma \psi_{\sigma,a_1}^\dagger(x)|0\rangle.$$

$|0\rangle$ is the drained Fermi sea and $A_{a_1}^\sigma$ are the amplitudes for an electron with chirality σ and spin a_1. The two boundary S matrices $S_{a_1b_1}^{1R}$, $S_{a_1b_1}^{1L}$ exchange the chirality of a particle.

$$A_{a_1}^- = S_{a_1b_1}^{1R} A_{b_1}^+ \tag{A3}$$

$$A_{a_1}^+ = S_{a_1b_1}^{1L} A_{b_1}^-. \tag{A4}$$

The asymmetric boundary conditions (A2) lead to the following boundary S matrices

$$S_{ab}^{1R} = B_{ab}^\dagger, \quad S_{ab}^{1L} = I_{ab}. \tag{A5}$$

Applying the boundary condition at the left boundary also quantizes the bare particle momentum k.

We now consider the two particle sector, $N = 2$, were the bulk interaction plays a role. Since the two particle interaction is pointlike we may divide configuration space into regions such that the interactions only occur at the boundary between two regions. Therefore away from these boundaries we write the wave function as a sum over plane waves so that the most general two particle state can be written as

$$|k_1, k_2\rangle = \sum_{\sigma,a} \int_{-\frac{L}{2}}^{\frac{L}{2}} d^2x\, F_{a_1a_2}^{\sigma_1\sigma_2}(x_1, x_2) e^{\sum_{j=1}^2 i\sigma_j k_j x_j}$$
$$\times \psi_{\sigma_1a_1}^\dagger(x_1)\psi_{\sigma_2a_2}^\dagger(x_2)|0\rangle, \tag{A6}$$

where we sum over all possible spin and chirality configurations and the two particle wave function, $F_{a_1a_2}^{\sigma_1\sigma_2}(x_1, x_2)$, is split up according to the ordering of the particles,

$$F_{a_1a_2}^{\sigma_1\sigma_2} = A_{a_1a_2}^{\sigma_1\sigma_2}[12]\theta(x_2 - x_1) + A_{a_1a_2}^{\sigma_1\sigma_2}[21]\theta(x_1 - x_2). \tag{A7}$$

The amplitudes $A_{a_1a_2}^{\sigma_1\sigma_2}[Q]$ refer to a certain chirality and spin configuration, specified by σ_j, a_j as well as an ordering of the particles in configuration space denoted by Q. For $Q = 12$ particle 1 is to the left of particle 2 while for $Q = 21$ the order of the particles are exchanged. Applying the Hamiltonian to (A6) we find that it is an eigenstate with energy $E = k_1 + k_2$ provided that these amplitudes are related to each other via application of S matrices. The amplitudes which differ by exchanging the chirality of the leftmost or the rightmost particle are related by the boundary S matrices.

$$A^{\sigma_1-}[12] = S^{2R} A^{\sigma_1+}[12], \quad A^{+\sigma_2}[12] = S^{1L} A^{-\sigma_2}[12], \tag{A8}$$

$$A^{-\sigma_2}[21] = S^{1R} A^{+\sigma_2}[21], \quad A^{\sigma_1+}[21] = S^{2L} A^{\sigma_1-}[21]. \tag{A9}$$

As discussed above in the one particle case, the boundary S matrices are $S^{1R} = B^\dagger$, $S^{1L} = I$, $S^{2R} = B^\dagger$, $S^{2L} = I$. For ease of notation we have suppressed spin indices. It is understood that S^{1R}, S^{1L} act in the spin space of particle 1 whereas S^{2R}, S^{2L} act in the spin space of particle 2.

There are two types of two particle bulk S matrices denoted by S^{12} and W^{12} which arise due to the bulk interactions and relate amplitudes which have different orderings. The first relates amplitudes which differ by exchanging the order of particles with opposite chirality

$$A^{+-}[21] = S^{12}A^{+-}[12], \tag{A10}$$

$$A^{-+}[12] = S^{12}A^{-+}[21], \tag{A11}$$

where S^{12} acts on the spin spaces of particles 1 and 2. Explicitly it is given by [17]

$$S^{ij} = \begin{pmatrix} 1 & & & \\ & \frac{\sinh(f)}{\sinh(f+\eta)} & \frac{\sinh(\eta)}{\sinh(f+\eta)} & \\ & \frac{\sinh(\eta)}{\sinh(f+\eta)} & \frac{\sinh(f)}{\sinh(f+\eta)} & \\ & & & 1 \end{pmatrix}, \tag{A12}$$

where $\eta = -iu$ and f, u are related to $g_\parallel$ and $g_\perp$ through the relations $\cos(u) = \frac{\cos(g_\parallel)}{\cos(g_\perp)}$ and $\frac{\sin(u)}{\tanh(f)} = \frac{\sin(g_\parallel)}{\cos(g_\perp)}$. In obtaining the above form of the S matrix we have ignored an unimportant overall factor. While the second type of S matrix relates amplitudes where particles of the same chirality are exchanged,

$$A^{--}[21] = W^{12}A^{--}[12], \tag{A13}$$

$$A^{++}[12] = W^{12}A^{++}[21]. \tag{A14}$$

Unlike (A12), W^{12} is not fixed by the Hamiltonian but rather by the consistency of the construction. This is expressed through the Yang-Baxter equations

$$S^{23} S^{13} W^{12} = W^{12} S^{13} S^{23}, \tag{A15}$$

$$W^{23} W^{13} W^{12} = W^{12} W^{13} W^{23}, \tag{A16}$$

$$S^{2R} S^{12} S^{1R} W^{12} = W^{12} S^{1R} S^{12} S^{2R}, \tag{A17}$$

$$S^{2L} S^{12} S^{1L} W^{12} = W^{12} S^{1L} S^{12} S^{2L}, \tag{A18}$$

which need to be satisfied for the eigenstate to be consistent. We take $W^{12} = P^{12}$ which can be explicitly checked to satisfy (A15)–(A18). The relations (A6)–(A14) provide a complete set of solutions of the two particle problem.

We can now generalize this to the N-particle sector and find that the eigenstates of energy $E = \sum_{j=1}^{N} k_j$ are of the form

$$|\{k_j\}\rangle = \sum_{Q,\vec{a},\vec{\sigma}} \int d^N x \, \theta(x_Q) A_{\{a\}}^{\{\sigma\}}[Q] \prod_j^N e^{i\sigma_j k_j x_j} \psi_{a_j \sigma_j}^\dagger(x_j)|0\rangle. \tag{A19}$$

Here we sum over all spin and chirality configurations specified by $\{a\} = \{a_1 \ldots a_N\}$, $\{\sigma\} = \{\sigma_1 \ldots \sigma_N\}$ as well as different orderings of the N particles. These different orderings correspond to elements of the symmetric group $Q \in \mathcal{S}_N$. In addition $\theta(x_Q)$ is the Heaviside function which is nonzero only for that particular ordering. As in the $N = 2$ sector the amplitudes $A_{\vec{a}}^{\vec{\sigma}}[Q]$ are related to each other by the various S matrices in the same manner as before, i.e., amplitudes which differ by changing the chirality of the leftmost particle are equal as $S^{jL} = I$, the amplitudes which differ by changing the chirality of the rightmost particle are related by S^{jR} and the amplitudes which differ by exchanging the order of opposite or same chirality particles are related by S^{ij} and W^{ij} respectively. The consistency of this construction is then guaranteed by virtue of these S matrices satisfying the following Yang-Baxter equations [30–32]

$$W^{jk} W^{ik} W^{ij} = W^{ij} W^{ik} W^{jk}, \tag{A20}$$

$$S^{jk} S^{ik} W^{ij} = W^{ij} S^{ik} S^{jk}, \tag{A21}$$

$$S^{jR} S^{ij} S^{iR} W^{ij} = W^{ij} S^{iR} S^{ij} S^{jR}, \tag{A22}$$

$$S^{jL} S^{ij} S^{iL} W^{ij} = W^{ij} S^{iL} S^{ij} S^{jL}, \tag{A23}$$

where $W^{ij} = P^{ij}$ and as before the superscripts denote which particles the operators act upon.

2. Bethe equations

In this section we derive the Bethe equations (3). Enforcing the boundary condition at $x = -L/2$ on the eigenstate (A19) we obtain the following eigenvalue problem which constrains the k_j,

$$e^{-2ik_j L} A_{\{a\}}^{\{\sigma\}}[\mathbb{1}] = (Z_j)_{\{a\},\{a\}'}^{\{\sigma\},\{\sigma\}'} A_{\{\bar{a}'\}}^{\{\bar{\sigma}'\}}[\mathbb{1}]. \tag{A24}$$

Here $\mathbb{1}$ denotes the identity element of $\mathcal{S}_N$, i.e., $\mathbb{1} = 12 \ldots N$ and the operator Z_j is the transfer matrix for the jth particle given by

$$Z^j = W^{jj-1} \ldots W^{j1} S^{j1} \ldots S^{jj-1} S^{jj+1} \ldots S^{jN} S^{jR} W^{jN} \ldots W^{jj+1}, \tag{A25}$$

where the spin indices have been suppressed. This operator takes the jth particle from one side of the system to the other and back again, picking up S-matrix factors along the way as it moves past the other $N - 1$ particles, first as a right mover and then as a left mover. Using the relations (A20)–(A23) one can prove that all the transfer matrices commute, $[Z^j, Z^k] = 0$ and therefore are simultaneously diagonalizable. In order to determine the spectrum of H we must therefore diagonalize Z^j, $\forall j$. Here we choose to diagonalize Z^1. To do this we use the method of boundary algebraic Bethe ansatz [30,31,34]. In order to use this method we need to embed the bare S matrices in a continuum [36], that is, we need to find the matrices $R(\lambda)$, $K(\lambda)$ such that for certain values of the spectral parameter λ, we obtain the bare S matrices of our model. Note that the S matrix S^{12} is of the form of XXZ R matrix

$$R^{ij}(\lambda) = \begin{pmatrix} 1 & & & \\ & \frac{\sinh(\lambda)}{\sinh(\lambda+\eta)} & \frac{\sinh(\eta)}{\sinh(\lambda+\eta)} & \\ & \frac{\sinh(\eta)}{\sinh(\lambda+\eta)} & \frac{\sinh(\lambda)}{\sinh(\lambda+\eta)} & \\ & & & 1 \end{pmatrix}. \tag{A26}$$

We can see that $R^{ij}(0) = W^{ij}$, $R^{ij}(f) = S^{ij}$. The K matrix is given by [30]

$$K^j(\lambda) = \frac{1}{\cosh(\lambda)} \begin{pmatrix} \cosh(\lambda - i\epsilon) & 0 \\ 0 & \cosh(\lambda + i\epsilon) \end{pmatrix} \tag{A27}$$

and it is related to the right boundary S matrix as $S^{jR} = K^j(f/2)$. The transfer matrix Z_1 is related to the Monodromy matrix $\Xi_\tau(\lambda)$ as $Z^1 = t(\frac{f}{2}) = \text{Tr}_\tau \Xi^\tau(\frac{f}{2})$, where

$$\Xi^\tau(\lambda) = R^{1\tau}\left(\lambda + \frac{f}{2}\right) \ldots R^{N\tau}\left(\lambda + \frac{f}{2}\right) K^\tau(\lambda) R^{N\tau}\left(\lambda - \frac{f}{2}\right) \ldots R^{1\tau}\left(\lambda - \frac{f}{2}\right). \tag{A28}$$

Here τ represents an auxiliary space and Tr_τ represents the trace in the auxiliary space. Using the properties of the R matrices one can prove that $[t(\lambda), t(\mu)] = 0$ [34] and by expanding $t(\mu)$ in powers of μ, obtain infinite set of conserved charges which

guarantees integrability. By following the boundary Algebraic Bethe ansatz approach we obtain the following Bethe equations in the region A, corresponding to the reference state with all up spins

$$e^{2ik_jL} = \beta^{-1}(f/2) \, \Pi_{\alpha=1}^M \Pi_{\sigma=\pm} \, \gamma(f/2, \sigma\lambda_\alpha, u/2), \tag{A29}$$

$$\gamma(x, y, z) = \frac{\sinh(x + y - iz)}{\sinh(x + y + iz)}, \beta(x) = \frac{\cosh(x - i\epsilon)}{\cosh(x)}, \tag{A30}$$

where $\lambda_\alpha, \alpha = 1, \ldots, M$ are the Bethe roots which satisfy the following equations

$$\Pi_{\sigma=\pm}\gamma^N(\lambda_\alpha, \sigma f/2, u/2)\gamma(\lambda_\alpha, i\pi/2, -u/2)\gamma(\lambda_\alpha, i\pi/2, -(u - 2\epsilon)/2) = \Pi_{\beta=1,\sigma=\pm}^M \gamma(\lambda_\alpha, \sigma\lambda_\beta, u). \tag{A31}$$

By rescaling $\lambda_\alpha \to u\lambda_\alpha$ and applying logarithm we obtain the following Bethe equations in the region A with asymmetric boundary conditions.

$$\sum_{\sigma=\pm} N\Theta(\lambda_\alpha + \sigma f/2u, 1/2) - \Theta(\lambda_\alpha + i\pi/2u, 1/2) - \Theta(\lambda_\alpha + i\pi/2u, (1 - \epsilon')/2) = \sum_{\beta=1}^M \sum_{\sigma=\pm} \Theta(\lambda_\alpha + \sigma\lambda_\beta, 1) + 2i\pi I_\alpha \tag{A32}$$

$$k_j = \frac{\pi n_j}{L} + \frac{i}{2L}\left(\log[\beta(f/2)] + \sum_{\beta=1}^M \sum_{\sigma=\pm} \Theta(f/2u + \sigma\lambda_\beta, 1/2)\right), \tag{A33}$$

where $\Theta(x, y) = \log\left(\frac{\sinh(u(x+iy))}{\sinh(u(x-iy))}\right)$ and $\epsilon' = 2\epsilon/u$. To obtain the Bethe equations in the topological region $\widehat{A}$, we can work with $g_\parallel, g_\perp < 0$ in the Hamiltonian and then take the limit $f \to -i\pi - f$ [18] in the obtained Bethe equations. We obtain a different set of Bethe equations

$$e^{2ik_jL} = \widehat{\beta}^{-1}(f/2) \, \Pi_{\alpha=1}^M \Pi_{\sigma=\pm} \, \gamma(f/2, \sigma\lambda_\alpha, u/2), \quad \widehat{\beta}(x) = \frac{\sinh(x - i\epsilon)}{\sinh(x)} \tag{A34}$$

$$\Pi_{\sigma=\pm}\gamma^N(\lambda_\alpha, \sigma f/2, u/2)\gamma(\lambda_\alpha, 0, -u/2)\gamma(\lambda_\alpha, 0, -(u - 2\epsilon)/2) = \Pi_{\beta=1,\sigma=\pm}^M \gamma(\lambda_\alpha, \sigma\lambda_\beta, u). \tag{A35}$$

Applying logarithm to the above equation and rescaling the Bethe roots we obtain the Bethe equations in the region $\widehat{A}$ with asymmetric boundary conditions,

$$\sum_{\sigma=\pm} N\Theta(\lambda_\alpha + \sigma f/2u, 1/2) - \Theta(\lambda_\alpha, 1/2) - \Theta(\lambda_\alpha, (1 - \epsilon')/2) = \sum_{\beta=1}^M \sum_{\sigma=\pm} \Theta(\lambda_\alpha + \sigma\lambda_\beta, 1) + 2i\pi I_\alpha, \tag{A36}$$

$$k_j = \frac{\pi n_j}{L} + \frac{i}{2L}\left(\log[\widehat{\beta}(f/2)] + \sum_{\beta=1}^M \sum_{\sigma=\pm} \Theta(f/2u + \sigma\lambda_\beta, 1/2)\right). \tag{A37}$$

The Bethe equations corresponding to the reference state with all down spins can be obtained by taking the limit $\epsilon \to -\epsilon'$ [30] in the above Bethe equations. Note that when symmetric boundary conditions are applied the Bethe equations corresponding to the reference state with all up spins are the same as those corresponding to the reference state with all down spins.

As already mentioned in the main text the asymmetric boundary conditions (A2) break the $\mathbb{Z}_2$ symmetry; this shifts the spin S^z of all the states in the regions A and $\widehat{A}$ by a term which is proportional to ϵ. In the scaling limit $u \ll 1$, one also needs to take the limit $\epsilon \ll 1$ while holding $\epsilon' = 2\epsilon/u$ fixed. In this limit the shift in the values of S^z of all the states goes to zero, and therefore the ground state in any certain region of the phase diagram with asymmetric boundary condition has the same total spin S^z as that in the corresponding region with symmetric boundary condition. The profound effect of applying the asymmetric boundary conditions is that we now have two fundamental boundary string solutions $\lambda = \pm i/2, \ \lambda = \pm i/2(1 - \epsilon')$.

To obtain the Bethe equations with symmetric boundary conditions in region A we can take the limit $\epsilon' \to 0$ in the equations (A31). We get

$$e^{2ik_jL} = \Pi_{\alpha=1}^M \Pi_{\sigma=\pm} \, \gamma(f/2, \sigma\lambda_\alpha, u/2), \tag{A38}$$

$$\Pi_{\sigma=\pm}\gamma^N(\lambda_\alpha, \sigma f/2, u/2)\gamma^2(\lambda_\alpha, i\pi/2, -u/2) = \Pi_{\beta=1,\sigma=\pm}^M \gamma(\lambda_\alpha, \sigma\lambda_\beta, u). \tag{A39}$$

To obtain the Bethe equations with symmetric boundary conditions in region $\widehat{A}$ we can take the limit $\epsilon' \to 0$ in the equations (A35). We get

$$e^{2ik_jL} = \Pi_{\alpha=1}^M \Pi_{\sigma=\pm} \, \gamma(f/2, \sigma\lambda_\alpha, u/2), \tag{A40}$$

$$\Pi_{\sigma=\pm}\gamma^N(\lambda_\alpha, \sigma f/2, u/2)\gamma^2(\lambda_\alpha, 0, -u/2) = \Pi_{\beta=1,\sigma=\pm}^M \gamma(\lambda_\alpha, \sigma\lambda_\beta, u). \tag{A41}$$

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