Comment on "Chern-Simons theory and atypical Hall conductivity in the Varma phase"

Angel E. Obispo,[1,*] Francisco A. Cruz Neto,[1] Andrés G. Jirón Vicente,[2,1] and L. B. Castro[1,3,†]

[1] *UFMA Universidade Federal do Maranhão-Campus do Bacanga-CCET, 65080-805 São Luís, Maranhão, Brazil*

[2] *UNAC Universidad Nacional del Callao-Campus Central-FCNM. 07001 Bellavista-Callao, Perú*

[3] *UNESP Universidade Estadual Paulista-Campus de Guaratinguetá-DFQ, 12516-410 Guaratinguetá, São Paulo, Brazil*

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In a recent paper [Phys. Rev. B **97**, 075135 (2018)], Menezes *et al.* analyze the topological behavior of an effective bosonic model defined on the Lieb lattice in the presence of an electromagnetic field. In this context, the authors claim to have found an atypical quantum Hall effect for the quasiparticles. However, some inconsistencies related to the treatment of the propagator jeopardize the main result in this system.

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In an interesting paper, Menezes *et al.* [1] analyzed the topological response of an effective bosonic theory defined on the Lieb lattice which is minimally coupled to an external $U(1)$ gauge field in $(2+1)$ dimensions. To this purpose, the authors consider a tight-binding Hamiltonian with three different species of (pseudo-) gapped fermions (see Eqs. (1)–(3) in Ref. [1]), similar to the one proposed in Ref. [2]. Such pseudogap behavior arises from the so-called Varma phase [3] which breaks time-reversal symmetry spontaneously (preserving the translational symmetry of the lattice) and whose realization would be possible in the copper oxygen planes of high-temperature cuprate superconductors [3,4].

As a first result, in Sec. II in Ref. [1], the authors showed that the dynamics of the charge carriers on that Lieb lattice in the low-energy regime present a relativisticlike behavior, correctly described by Duffin-Kemmer-Petiau-like Hamiltonian (DKP). Disregarding irrelevant constants, this effective Hamiltonian is expressed in a simplified version as (see Eq. (4) in Ref. [1]),

$$H_{\mathrm{DKP}}\Psi = E\Psi, \tag{1}$$

$$H_{\mathrm{DKP}} = [\beta^0, \beta^1]k^1 + [\beta^0, \beta^2]k^2 + m\beta^0, \tag{2}$$

where Ψ is the three-component spinor (see Fig. 1 in Ref. [1]),

$$\Psi(\mathbf{k}) = \begin{pmatrix} b(\mathbf{k}) \\ a(\mathbf{k}) \\ c(\mathbf{k}) \end{pmatrix}, \tag{3}$$

$\mathbf{k} = (k^1, k^2)$ is the momentum or wave vector and $\beta^i$are two 3×3 anti-Hermitian matrices which, together with another 3×3 Hermitian matrix β^0, satisfy the so-called DKP algebra,

$$\beta^\mu\beta^\nu\beta^\sigma + \beta^\sigma\beta^\nu\beta^\mu = \beta^\mu\eta^{\nu\sigma} + \beta^\sigma\eta^{\nu\mu}, \tag{4}$$

with the metric tensor $\eta^{\mu\nu} = \mathrm{diag}(1, -1, -1)$. This is the basis on which [1] develops.

In simple terms, the DKP equation is a first-order wave equation that defines spin 0 (scalar sector) and spin 1 (vectorial sector) fields and particles with a rich algebraic structure not capable of being expressed in the traditional Klein-Gordon and Proca theories [5]. Related to this, the authors in Ref. [1] state that the charge carriers on the Lieb lattice are described by *relativistic pseudospin-0 quasiparticles in two spatial dimensions*, i.e., these would exist in the scalar sector of the theory. We rebut this statement via the following argument. In $(3+1)$ spacetime dimensions, the algebra (4) generates a set of 126-independent matrices whose irreducible representations are a trivial representation, a five-dimensional representation for the scalar sector, and a ten-dimensional representation for the vectorial sector [5,6]. Whatever the sector, it is clear that the DKP spinor will have an excess of components. In this case, the theory needs to be complemented by a constraint equation that allows to eliminate the redundant components, which is given by

$$\beta^i\beta^0\beta^0 k_i\Psi = m(1-\beta^0\beta^0)\Psi, \quad i = 1\text{–}3. \tag{5}$$

With this constraint equation, we can express the three (four) components of the spinor by the other two (six) components and their space derivatives in the scalar (vector) sector—for more details, see Ref. [5]. Thereby we can exclude the redundant components and reexpress our system of equations to another that depends only on physical components (one for the scalar sector and four for the vectorial sector) of the DKP theory. If we performed this same analysis in $(2+1)$ dimensions, we will see that the algebra (4) now generates a set of 35-independent matrices whose irreducible representations are a trivial representation, a four-dimensional representation, and two different three-dimensional representations. Note that only the three-dimensional representations adapt to the structure of the Hamiltonian (2), reducing Ψ to a three-component spinor, similar to (3). As the Lieb lattice has three bands at low energy, and we have three components to the spinor in (3), these components come from the three-component pseudospin 1 quasiparticles. The constraint equation (5) (modified to two space dimensions, $i = 1, 2$) eliminates the redundant pseudospin 0 quasiparticles.

*angel.obispo@ufma.br, ae.obispo@gmail.com

†luis.castro@ufma.br

Therefore, (3) describes pseudospin-1 quasiparticles. It is important to highlight that constraint equation also allows one to demonstrate the equivalence between the Hamiltonian form (2) and the DKP equation of motion, which is not a trivial matter [5,7]. In fact, a DKP quantum field theory is possible only if this equivalence is established as can be verified in Refs. [8–11].

Section III in Ref. [1] represents the crucial point of this Comment. In that section, the authors analyzed the topological response generated by one-loop radiative corrections to the two-point function of the gauge and DKP fields in (2 + 1) dimensions. As it is known from usual QED_{2+1}, such a topological term—called the Chern-Simons term—comes from the first order in the external momentum contribution of the vacuum polarization diagram [12]. In Physics of Condensed Matter [13] (perhaps its most notable application), this emergent Chern-Simons theory naturally leads to the transverse conductivity observed from the Hall effect. In this context, the authors claim to have found *an atypical quantum Hall effect for the DKP quasiparticles* (Eq. (15) in Ref. [1]), given by

$$\sigma^{xy} = \mathrm{sgn}(m)\frac{q^2}{4h}. \tag{6}$$

The above expression is their main result and represents a truly atypical result (one should expect to obtain an integer quantum Hall effect), which is relevant in itself because it is obtained from an Abelian Chern-Simons theory whose origin is a non-Dirac system (namely, DKP system). Unfortunately, we found some inconsistencies related to the treatment of the DKP propagator, in fact, the expression (13) in Ref. [1] is incorrect [8–10], therefore, the final result (6) is invalid. Below, we justify our statement.

We start considering that the interaction of the electromagnetic field with the long-wavelength (low-frequency) excitations of charge carriers on the Lieb lattice can be described by relativistic quantum electrodynamics for integer spin particles [8,11]. The action (8) in Ref. [1] is built upon the motion equation,

$$(i\hbar\not{\partial} - q\not{A} - m)\Psi = 0, \tag{7}$$

where m is the mass-gap parameter, q is the coupling parameter, and A_μ is the vector gauge potential,

$$\not{\partial} = \beta^\mu\frac{\partial}{\partial x^\mu}, \quad \not{A} = \beta^\mu A_\mu.$$

The resulting polarization tensor in the momentum representation is given by

$$i\Pi^{\mu\nu}(p) = +\frac{q^2}{\hbar}\int\frac{d^3k}{(2\pi)^3}\mathrm{Tr}[\beta^\mu G_\Psi(k-p)\beta^\nu G_\Psi(k)], \tag{8}$$

where

$$G_\Psi(k) = i\frac{1}{\beta^\mu k_\mu - m} \tag{9}$$

is the DKP-free (Feynman) propagator. The tensor polarization (8) is equivalent to Eq. (12) in Ref. [1], by changing $\mu \leftrightarrow \nu$. The vacuum polarization diagram we have considered is the one shown in Fig. 1, which allowed us the built $i\Pi^{\mu\nu}(p)$ following the same Feynman rules of the usual QED_{2+1}, except for the plus sign (+) in front, which is reminiscent

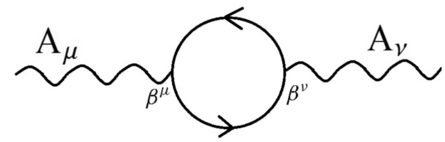

FIG. 1. Vacuum polarization diagram for the Hamiltonian given by Eq. (1).

of the bosonic nature of the DKP theory [8]. The standard procedure to calculate $i\Pi^{\mu\nu}(p)$ says that we must first evaluate the trace of β matrices, which implies that $G_\Psi(k)$ in (9) must be rewritten in such a way that these matrices appear in the numerator. Nevertheless, this process in DKP theory is more complicated (as compared with Dirac theory) due mainly to its algebra and because the β matrices are singulars ($\det[\beta] = 0$). As β^{-1} does not exist, it implies that some common identities are not valid anymore, for instance, $(\beta^\mu p_\mu)(\beta^\nu p_\nu)^{-1} = \mathbb{I}$. Because of the omission of this fact, the propagator in Ref. [1] was incorrectly constructed (see Eq. (13) in Ref. [1]). The correct form for the DKP propagator is also performed in Refs. [14–16] and is expressed as follows:

$$G_\Psi(k) = i\frac{1}{\beta^\mu k_\mu - m} = \frac{i}{m}\left[\frac{\not{k}(\not{k}+m)}{k^2 - m^2} - 1\right]. \tag{10}$$

It is straightforward to see that the propagator in (10) is strictly defined for massive particles as required for the DKP theory. In fact, Eq. (2) with $m = 0$ represents a different relativistic equation, the so-called Harish-Chandra equation [17], whose analysis we will leave aside here. So, we go to focus on the propagator in (10), alternatively rewritten as

$$\begin{aligned} G_\Psi(k) &= G_1(k) + G_2(k) \\ &= i\frac{(\not{k}+m)}{k^2 - m^2} + \frac{i}{m}\frac{(\not{k}^2 - k^2)}{k^2 - m^2}. \end{aligned} \tag{11}$$

Note that the first term on the right $G_1(k)$ coincides exactly with the propagator proposed by Menezes *et al.* (see Eq. (13) in Ref. [1]), which is used to determine the value of the Hall conductivity by considering the contribution from antisymmetric part of the polarization tensor $i\Pi^{\mu\nu} \sim \int d^3k\, \mathrm{Tr}[\beta^\mu G_1\beta^\nu G_1]$ and the second term $G_2(k)$ is absent. The point of this Comment is to demonstrate that, when $G_2(k)$ is inserted into (8), the term $\sim \int d^3k\, \mathrm{Tr}[\beta^\mu G_1\beta^\nu G_2 + \beta^\mu G_2\beta^\nu G_1 + \beta^\mu G_2\beta^\nu G_2]$ contributes a non-negligible value to the value of the Hall conductivity found in Ref. [1].

For this purpose, we rewrite the polarization tensor according to decomposition (11),

$$i\Pi^{\mu\nu}(p) = i\Pi^{\mu\nu}_{(1,1)} + i\Pi^{\mu\nu}_{(1,2)} + i\Pi^{\mu\nu}_{(2,1)} + i\Pi^{\mu\nu}_{(2,2)}, \tag{12}$$

where $i\Pi^{\mu\nu}_{(i,j)}$ are functions of momentum p, conveniently defined as

$$i\Pi^{\mu\nu}_{(1,1)} = \frac{q^2}{\hbar}\int\frac{d^3k}{(2\pi)^3}\mathrm{Tr}[\beta^\mu G_1(k-p)\beta^\nu G_1(k)],$$

$$i\Pi^{\mu\nu}_{(1,2)} = \frac{q^2}{\hbar}\int\frac{d^3k}{(2\pi)^3}\mathrm{Tr}[\beta^\mu G_1(k-p)\beta^\nu G_2(k)],$$

$$i\Pi^{\mu\nu}_{(2,1)} = \frac{q^2}{\hbar}\int \frac{d^3k}{(2\pi)^3}\mathrm{Tr}[\beta^\mu G_2(k-p)\beta^\nu G_1(k)],$$
$$i\Pi^{\mu\nu}_{(2,2)} = \frac{q^2}{\hbar}\int \frac{d^3k}{(2\pi)^3}\mathrm{Tr}[\beta^\mu G_2(k-p)\beta^\nu G_2(k)].$$

As previously mentioned, the result obtained in Ref. [1] can be reproduced by computing simply the antisymmetric part of $i\Pi^{\mu\nu}_{(1,1)}$, so, in that sense, it is convenient to divide Eq. (12) into two contributions: $i\Pi^{\mu\nu}_{(1,1)}$ and $i\Pi^{\mu\nu}_{(1,2)} + i\Pi^{\mu\nu}_{(2,1)} + i\Pi^{\mu\nu}_{(2,2)}$. As we are only interested in the topological part of each contribution, we focus our attention on the terms $\sim\epsilon^{\mu\nu\alpha}p_\alpha$, which come from combining an odd number of β matrices,

$$\mathrm{Tr}[\beta^\rho\beta^\sigma\beta^\theta] = i\epsilon^{\rho\sigma\theta}, \tag{13}$$

$$2\mathrm{Tr}[\beta^\rho\beta^\alpha\beta^\sigma\beta^\omega\beta^\theta] = ig^{\rho\alpha}\epsilon^{\sigma\omega\theta} + ig^{\alpha\sigma}\epsilon^{\omega\theta\rho} + ig^{\sigma\omega}\epsilon^{\theta\rho\alpha} + ig^{\omega\theta}\epsilon^{\rho\alpha\sigma} + ig^{\rho\omega}\epsilon^{\alpha\sigma\theta} + ig^{\alpha\theta}\epsilon^{\rho\sigma\omega}. \tag{14}$$

Thus, following the standard methods for QED calculations [8], we will compute the antisymmetric part of each contribution separately.

(1) Computing $i\Pi^{\mu\nu}_{AS(1,1)}$.

To determinate the contribution from this first term (which is used in Ref. [1] to get its main result), we start applying the trace property (13). After a few calculations, we obtain

$$i\Pi^{\mu\nu}_{AS(1,1)} = -\frac{imq^2}{\hbar}\int_0^1 dx\int\frac{d^3k}{(2\pi)^3}\frac{p_\alpha}{[k^2-\Delta^2]^2}\epsilon^{\mu\nu\alpha},$$

where we have used the Feynman parametrization procedure,

$$\frac{1}{[(k-p)^2-m^2][k^2-m^2]} = \int_0^1 dx\frac{1}{[k^2-\Delta^2]^2}, \tag{15}$$

together with the changes $k\to k+xp$ and $\Delta = m^2 - p^2x(1-x)$. These integrals are the massive one-loop Feynman integrals, widely studied in QED, and whose result we will use directly. Thereby we get

$$i\Pi^{\mu\nu}_{AS(1,1)} = \frac{m}{|m|}p_\alpha\epsilon^{\mu\nu\alpha}\frac{q^2}{4h}\int_0^1 dx\frac{1}{\sqrt{1-x(1-x)p^2/m^2}},$$
$$= \mathrm{sgn}(m)\frac{1}{4}\frac{q^2}{h}\epsilon^{\mu\nu\alpha}p_\alpha, \tag{16}$$

where in the last line we have considered the Chern-Simons regime ($m \gg p$). This is the result obtained by Menezes *et al.* in their paper as expected.

(2) Computing $i\Pi^{\mu\nu}_{AS(1,2)} + i\Pi^{\mu\nu}_{AS(2,1)} + i\Pi^{\mu\nu}_{AS(2,2)}$.

In this case, both (13) and (14) are required. A quick inspection allows us to demonstrate that $i\Pi^{\mu\nu}_{AS(2,2)} = 0$, i.e., this term does not have a antisymmetric part. On the other hand, the sum of the cross terms provides

$$i\Pi^{\mu\nu}_{AS(1,2)} + i\Pi^{\mu\nu}_{AS(2,1)} = -\frac{iq^2}{2m\hbar}\int_0^1 dx\int\frac{d^3k}{(2\pi)^3} \times\frac{1}{[(k-px)^2-\Delta^2]^2}\mathrm{Tr}[p],$$

where we use the parametrization (15) with $\Delta = m^2 - p^2x(1-x)$ and

$$\mathrm{Tr}[p] = 2\epsilon^{\nu\alpha\omega}p_\alpha k^\mu k_\omega - 2\epsilon^{\mu\alpha\omega}p_\alpha k^\nu k_\omega + 2\epsilon^{\mu\nu\alpha}k_\alpha(pk) - \epsilon^{\nu\alpha\omega}p_\alpha p^\mu k_\omega + \epsilon^{\mu\alpha\omega}p_\alpha p^\nu k_\omega - \epsilon^{\mu\nu\alpha}p_\alpha(pk) - \epsilon^{\mu\nu\alpha}k_\alpha p^2.$$

At this point, a regularization scheme is required. We use the Pauli-Villars regularization, which allows us to make the change $k\to k+px$, to then exclude the linear terms in k associated with odd integrals, and replace $k^\mu k^\nu$ by $g^{\mu\nu}k^2/3$ in the numerator. Thereby we obtain

$$i\Pi^{\mu\nu}_{\mathrm{AS}(1,2)} + i\Pi^{\mu\nu}_{\mathrm{AS}(2,1)} = \mathrm{sgn}(m)\frac{3}{4}\frac{q^2}{h}\epsilon^{\mu\nu\alpha}p_\alpha \times\int_0^1 dx\frac{3-2x(1-x)(p^2/m^2)}{3\sqrt{1-x(1-x)(p^2/m^2)}}.$$

In the Chern-Simons regime ($m \gg p$),

$$i\Pi^{\mu\nu}_{\mathrm{AS}(1,2)}(p) + i\Pi^{\mu\nu}_{\mathrm{AS}(2,1)}(p) = \mathrm{sgn}(m)\frac{3}{4}\frac{q^2}{h}\epsilon^{\mu\nu\alpha}p_\alpha. \tag{17}$$

The above expression represents the main result of this Comment. Note that it is three times larger than the one found in (16), which implies a important modification to the result found by Menezes *et al.* [1]. Thus, if we combine Eqs. (16) and (17) to find the full expression of the dynamically generated Chern-Simons term, we get

$$i\Pi^{\mu\nu}_{AS}(p) = i\Pi^{\mu\nu}_{AS(1,1)} + i\Pi^{\mu\nu}_{AS(1,2)} + i\Pi^{\mu\nu}_{AS(2,1)},$$
$$= \mathrm{sgn}(m)\frac{q^2}{h}\epsilon^{\mu\nu\theta}p_\theta. \tag{18}$$

The Hall conductivity can be obtained via the Kubo formula,

$$\sigma^{xy} = \lim_{\mathbf{p}\to 0,\, p_0\to 0}\frac{i\Pi^{xy}_{AS}}{p_0} = \mathrm{sgn}(m)\frac{q^2}{h}, \tag{19}$$

which is the result expected according to the literature [3,18]. Therefore, there is no such atypical Hall conductivity as reported by the authors.

We conclude by emphasizing that the results and conclusions presented is this Comment do not alter the others results shown in Ref. [1], concerning the obtaining of Landau levels in DKP theory and to the extension of the Jackiw-Rebbi approach for the DKP quasiparticles.

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