Rapid Communications

Universal properties of anyon braiding on one-dimensional wire networks

Tomasz Maciążek [1,2,*] and Byung Hee An [3]
[1] School of Mathematics, University of Bristol, Fry Building, Woodland Road, Bristol BS8 1UG, United Kingdom
[2] Center for Theoretical Physics, Polish Academy of Sciences, Al. Lotników 32/46, 02-668 Warszawa, Poland
[3] Department of Mathematics Education, Kyungpook National University, 80 Daehakro, 41566, Daegu, South Korea

(Received 3 July 2020; accepted 9 November 2020; published 23 November 2020)

We demonstrate that anyons on wire networks have fundamentally different braiding properties than anyons in two dimensions (2D). Our analysis reveals an unexpectedly wide variety of possible non-Abelian braiding behaviors on networks. The character of braiding depends on the topological invariant called the connectedness of the network. As one of our most striking consequences, particles on modular networks can change their statistical properties when moving between different modules. However, sufficiently highly connected networks already reproduce the braiding properties of 2D systems. Our analysis is fully topological and independent on the physical model of anyons.

DOI: 10.1103/PhysRevB.102.201407

Introduction. Studies of anyon braiding on one-dimensional (1D) wire networks are at the forefront of research of architectures for topological quantum computers. Such a computer would perform its tasks using topological states of matter (describing anyons) that are intrinsically robust against different types of noise and decoherence [1]. Anyons arise in quantum systems that are effectively one or two dimensional. The braiding of anyons transforms a state of the corresponding quantum system by a unitary operator which is a topological quantum gate. A robust realization of controlled braiding of anyons is one of the major challenges in this field. Recently developed experimental and theoretical proposals address this challenge by exploring the possibility of the braiding of anyons on junctions of one-dimensional wire networks [2,3]. Such networks are believed to provide a platform for engineering anyonic braiding most easily.

This Rapid Communication shows that the braiding of anyons on networks provides a wider range of possibilities for the resulting topological quantum operations in comparison to 2D architectures. This suggests that there may exist quantum systems where computational universality can be accomplished more easily than in currently known proposals. Our work also provides a mathematical justification for the fact that braiding rules in 2D are compatible with braiding rules on 1D networks. Our purpose here is to describe the above results, whose mathematical details will be spelled out elsewhere [4].

Of particular importance in this context is Kitaev's superconducting chain that supports Majorana edge modes. Such a chain can be modeled as semiconductor nanowires coupled to superconductors [3] as well as in other solid state [5–8] and photonic systems [9]. The braiding of edge modes is then realized by coupling the endpoints of wires so that they form a network or, in the simplest case, a trijunction [3]. Importantly, Majorana edge modes braid in a non-Abelian way, making them useful for quantum computation. However, the set of gates obtained by the braiding of Majorana fermions is never universal and in order to realize universal quantum computation one has to pursue certain additional strategies [10,11]. This proposal has been recognized as one of the most robust candidates for an architecture of a topological quantum computer. Experimental proposals of the above-mentioned trijunction have been made so far including photonic systems [9] and Josephson junctions [12,13]. We also mention in this context effective hopping models for anyons that have been studied in Ref. [14] and that have led to the classification of Abelian quantum statistics on networks [15].

Despite the significant interest in problems related to the braiding of anyons on networks, relatively little is known about their topological braiding properties. In this Rapid Communication, we fill this gap by studying relations coming from continuous deformations of paths corresponding to the braiding of anyons on a network. Because quantum statistics is a topological property, any physical model that supports anyonic braiding on a network has to respect such relations. In other words, any topological transformation of the quantum system related to an exchange of anyons remains invariant under a continuous deformation of the corresponding braid [16–19]. In the standard 2D setting, an example of such a relation is shown in Fig. 1. It relates two ways of exchanging a triplet of anyons. To see this, consider first the so-called simple braid from Fig. 2(a) that exchanges two neighboring anyons. Figure 2(a) also explains the origin of the term *braiding* as the world lines of anyons forming braids in space-time.

Let us denote such a simple braid that exchanges ith and $(i+1)$th anyons by σ_i. Any exchange of anyons in 2D can be written as a composition of simple braids. More formally, simple braids generate the planar braid group. However, they do not generate the braid group freely, as they are subject to the following braid relation: $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$. This can

*kk19347@bristol.ac.uk

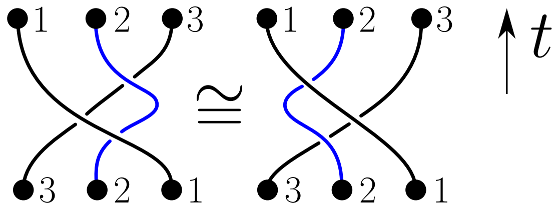

FIG. 1. The braiding relation for anyons in 2D.

be seen in Fig. 1—both braids differ by a deformation of the middle world line (blue line in Fig. 1).

Simple braids also satisfy a commutative relation where exchanges of disjoint sets of anyons commute with each other, $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|j - i| \geqslant 2$.

Braiding on junctions. In order to see if the braid relation is satisfied by braids on the trijunction, we first define the network counterpart of the simple braid. It is shown in Fig. 2(b)—the trijunction stretched in a time interval makes up three rectangles. Anyon 1 is first transported to the right branch (the bottom branch in the picture) of the trijunction, then anyon 2 travels to the left branch, making space for anyon 1 to go back to the original initial position of anyon 2. The exchange is completed by the return of anyon 2 from the right branch to the original initial position of anyon 1. In order to track all the moves of anyons on an arbitrary junction, we set up the following notation. For a d junction (d incident branches), we fix the initial position of anyons to align one after another on a fixed branch. Having drawn the junction on a plane, we enumerate the remaining branches in a clockwise fashion by labels from 1 to $d - 1$. The exchange of ith and $(i + 1)$th anyons will be unambiguously encoded by a sequence of integers $\mathbf{a} := (a_1, a_2, \dots, a_{i+1})$ with $1 \leqslant a_j \leqslant d - 1$ and $a_i \neq a_{i+1}$. Elements of $\mathbf{a}$ denote (i) labels of branches where first $(i - 1)$ anyons were distributed—these are $a_1, \dots, a_{i-1}$; and (ii) labels of branches where anyon i and $(i + 1)$ exchange—these are a_i and a_{i+1}. Note that swapping the order of a_i and a_{i+1} reverses the direction of the exchange. Going back to the concrete example of two particles on a trijunction from Fig. 2(b), the depicted braid would be denoted by $\sigma_1^{(2,1)}$, i.e., $\mathbf{a} = (2, 1)$.

In order to visualize the counterparts of braids in the braid relation from Fig. 1, we need to define the counterpart of σ_2—the simple braid exchanging anyons 2 and 3. To this end, anyon 1 has to be moved to the right or left branch of the junction so that anyons 2 and 3 can carry on and exchange as in Fig. 2(b). Let us choose the braid where anyon 1 moves to the right branch, which we denote $\sigma_2^{(2,2,1)}$. The composition $\sigma_1^{(2,1)}\sigma_2^{(2,2,1)}\sigma_1^{(2,1)}$ is shown in Fig. 3.

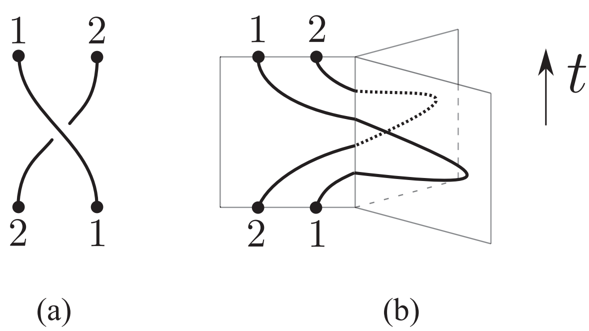

FIG. 2. Simple braid on the plane vs simple braid on a trijunction.

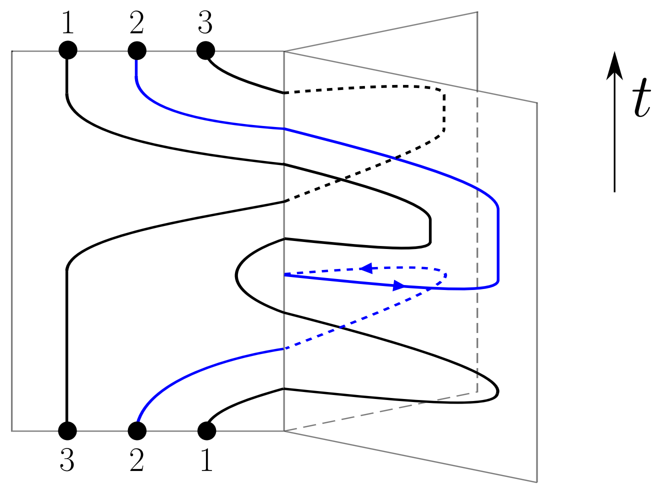

FIG. 3. Braid $\sigma_1^{(2,1)}\sigma_2^{(2,2,1)}\sigma_1^{(2,1)}$ on a trijunction. The depicted braid has been deformed to simplify the picture.

Strikingly, the world line of the middle anyon (blue line in Fig. 3) is now blocked by world lines of the other particles and it cannot be deformed freely. Consequently, there is no braiding relation on a trijunction. In fact, the three-particle braid group of the trijunction is freely generated by $\sigma_1^{(2,1)}$, $\sigma_2^{(2,2,1)}$, and $\sigma_2^{(1,2,1)}$. However, when one considers a bigger junction or a larger number of anyons, some relations appear. Their precise form is as follows.

(1) For $n \geqslant 4$, pseudocommutative relations appear. For $j - i \geqslant 2$,

$$\sigma_j^{a_1\cdots a_{j+1}}\sigma_i^{a_1\cdots a_{i+1}} = \sigma_i^{a_1\cdots a_{i+1}}\sigma_j^{a_1\cdots a_{i-1}a_{i+1}a_ia_{i+2}\cdots a_{j+1}}. \quad (1)$$

(2) For $d \geqslant 4$ and $n \geqslant 3$, pseudobraid relations appear. For $1 \leqslant i \leqslant n - 2$,

$$\sigma_{i+1}^{a_1\cdots a_{i-1}a_ia_{i+1}a_{i+2}}\sigma_i^{a_1\cdots a_{i-1}a_ia_{i+2}}\sigma_{i+1}^{a_1\cdots a_{i-1}a_{i+2}a_ia_{i+1}} = \sigma_i^{a_1\cdots a_{i-1}a_ia_{i+1}}\sigma_{i+1}^{a_1\cdots a_{i-1}a_{i+1}a_ia_{i+2}}\sigma_i^{a_1\cdots a_{i-1}a_{i+1}a_{i+2}}. \quad (2)$$

Let us emphasize that the braid group of a d junction has more generators than the planar braid group and hence it imposes fewer topological constraints on the unitary braiding operators that are assigned to simple braids in a physical model. This is perhaps most striking in the case of three anyons on a trijunction where we had three generators and no relations between them.

General (planar) network architectures. In order to relate braiding relations for anyons on general networks with the braiding of anyons in 2D, we first have to consider a different presentation of the planar braid group. Namely, we will consider the total braid δ which is a product of all simple braids, $\delta := \sigma_1\sigma_2\cdots\sigma_{n-1}$. Braid δ corresponds to the move where the first anyon exchanges consecutively with all anyons. Using 2D braiding relations one can show that any simple braid can be expressed by σ_1 and δ as $\sigma_i = \delta^{i-1}\sigma_1\delta^{1-i}$ [20].

We will next show how the above relations are recovered on networks. To this end, we fix a spanning tree of our network T which is a connected tree that contains all vertices of the

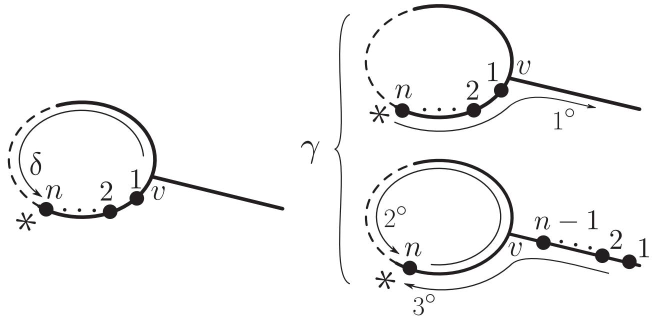

FIG. 4. (a) Total braid δ on a lollipop network. (b) One-particle move γ on a lollipop network. The rooted spanning tree with root $*$ is drawn by solid lines.

network. Moreover, we choose the root of T to be a vertex of degree two that lies on the boundary of the network. The initial configuration of anyons is such that the anyons are assembled on the edge of T which is incident to the root. The above choice of a spanning tree unambiguously defines all possible exchanges on junctions. To see this, note that for every essential vertex v of the network (i.e., the vertex at which three or more edges are incident), we have a unique path in T, denoted by $[v, *]$, that connects this vertex with the root. Such a path implies labeling of branches of the junction at v with branch 0 being the one that is contained in $[v, *]$ and the remaining branches labeled clockwise as described in the previous section. Consequently, simple braids at v will be denoted by an additional superscript, v. The counterpart of total braid δ is realized by utilizing a loop containing the root of T (effectively considering a lollipop-shaped subnetwork; see Fig. 4)—anyon 1 is transported along the loop to the end of the line. It is straightforward to check that up to some backtracking moves, in the lollipop setting from Fig. 4 we have

$$\sigma_i^{v;(1,\dots,1,2,1)} = \delta^{i-1}\sigma_1^{v;(2,1)}\delta^{1-i}. \tag{3}$$

Moreover, braid δ can be expressed in terms of simple braids at the junction of the lollipop and a one-particle move γ defined as the move where anyons 1 through $n-1$ are transported to branch 2 of the junction and anyon n travels alone around the lollipop loop. The precise relation reads

$$\gamma = \sigma_{n-1}^{v;(2,\dots,2,2,1)}\cdots\sigma_1^{v;(2,1)}\delta. \tag{4}$$

Let us pause for a moment to analyze the role of one-particle moves. Such moves do not describe any exchange, hence assigning unitary operators to these moves can only come from the existence of some external gauge fields puncturing the plane where the considered network is confined. For instance, the presence of a deltalike magnetic flux flowing perpendicularly through the middle of the lollipop loop would result with multiplication of the anyonic wave function by a phase factor due to the Aharonov-Bohm effect. From now on, we will always assume that there are no such external gauge fields present in the system. Consequently, we will equate all one-particle loops to identities.

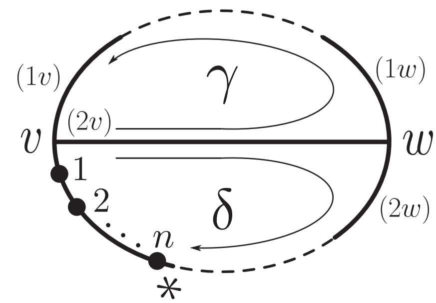

FIG. 5. A Θ network with the rooted spanning tree marked by solid lines. Arrows symbolize the total braid δ and one-particle loop γ as described in the main text.

By putting γ to identity, we obtain $\delta = \sigma_1^{v;(1,2)}\cdots\sigma_{n-1}^{v;(2,\dots,2,1,2)}$. Note that at this point we have almost recovered the presentation of the planar braid group that we considered at the beginning of this section. The only difference is that the expression for δ involves different simple braids than expression (3). As we show in the next section, this problem disappears for a wide class of networks that are sufficiently connected.

We say that a network is k connected when any two of its essential vertices can be connected by at least k paths that are mutually internally disjoint. By Menger's theorem [21], this is equivalent to the fact that after removing at most $k-1$ vertices, the network remains connected.

Braiding on 2-connected networks. The key feature of 2-connected networks that simplifies their braid groups is that for every trijunction in the network we can find suitable lollipop subnetworks that allow us to reduce the number of generators. In particular, if v is the first essential vertex from the root $*$ (i.e., there are no essential vertices on path $[v, *]$, as in Fig. 4), then for every branch a_1 at junction v, there exists a path connecting v and $*$ that contains branch a_1 and is independent of $[v, *]$. Consequently, we have a lollipop where, for any $\mathbf{a} = (a_1, a_2, \dots, a_{i+1})$, we obtain

$$\sigma_i^{v;\mathbf{a}} = \delta\sigma_{i-1}^{v;\mathbf{a}'}\delta^{-1}, \tag{5}$$

where $\mathbf{a}' = (a_2, \dots, a_{i+1})$. The above expression allows us to inductively reduce any simple braid at v to a braid of the form $\delta^{i-1}\sigma_1^{v;(a,b)}\delta^{1-i}$ with $a > b$. This in turn means that for simple braids taking place at a fixed trijunction spanned on branches (a, b) at vertex v, we indeed obtain a set of 2D braiding relations. However, braids at different trijunctions are still *a priori* independent of each other. One can show that a similar situation concerns simple braids at junctions that are further away from the root.

To recapitulate, 2-connected networks indeed support genuine 2D braiding relations. However, the relations are valid only within certain sets of braids that are restricted to fixed trijunctions of the network. This is strikingly different from the 2D anyon braiding where braiding is ruled by only one type of simple braids. Note that this feature of braiding can be utilized to design networks consisting of different modules where quantum statistics can be changed when moving anyons from module to module. An example of such a modular network is shown in Fig. 6.

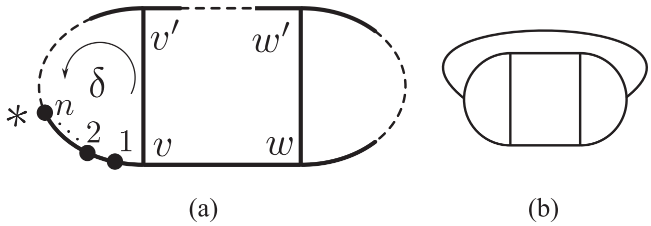

FIG. 6. (a) A modular network with two modules containing junctions v, v' and w, w', respectively. (b) Network from point (a) modified by adding one edge that makes it 3-connected.

Braiding on 3-connected networks. In contrast to the great complexity of braiding scenarios outlined so far for 1- and 2-connected networks, 3-connected networks bring a tremendous simplification. The key is to consider the so-called Θ network that consists of two essential vertices connected by three edges (Fig. 5). By definition, every two essential vertices in a 3-connected network can be connected by three independent paths that form a Θ subgraph. The key property of anyon braiding on a Θ network is that it identifies simple braids on different trijunctions. This in turn means that the 2D braiding relations are recovered. Similarly, by considering suitable Θ subnetworks of a general 3-connected network, one can show that the above identification recovers the planar braid group. Let us see explicitly how it happens for a Θ network from Fig. 5. Denote by γ the one-particle loop where the first anyon travels around the upper loop in Fig. 5. Moreover, denote by γ' a move which involves anyons 1 and 2 where (i) anyon 1 travels to branch $(2w)$ through the solid edge $[v, w]$, (ii) anyon 2 travels around the upper loop, and (iii) anyon 1 goes back along the tree from branch $(2w)$. Up to some backtracking moves, we have the following relations,

$$\delta\gamma = \gamma'\delta, \quad \sigma_1^{w;(2,1)}\gamma = \gamma'\sigma_1^{v;(2,1)}. \tag{6}$$

Because γ is a one-particle move, according to our assumption about the nonexistence of external fields, we put it to identity. Then, the left relation in (6) implies that γ' is an identity as well. This in turn applied to the right relation yields $\sigma_1^{w;(2,1)} = \sigma_1^{v;(2,1)}$.

To sum up, relations (4) and (5) for lollipops together with relation (6) for Θ subnetworks enabled us to identify the *a priori* complicated braiding relations on networks with the well-known 2D braiding when the considered network is 3-connected.

Example: A modular 2-connected network. A simple realization of a modular network that admits different non-Abelian quantum statistics in different modules is shown in Fig. 6.

Because one can span a Θ connection between trijunctions at v and v', simple braids at these junctions are identified with each other. Similarly, one identifies simple braids at trijunctions at w and w'. However, simple braids at v and w are independent of each other. Moreover, appropriate lollipop relations ensure that $\sigma_i^{v;\mathbf{a}} = \delta^{i-1}\sigma_1^{v;(2,1)}\delta^{1-i}$ and $\sigma_i^{w;\mathbf{a}} = \delta^{i-1}\sigma_1^{w;(2,1)}\delta^{1-i}$ for any $\mathbf{a}$, hence braiding is independent of the distribution of anyons. There are no topological constraints that would forbid utilizing simple braids at v and w in such a way that they would realize different topological quantum gates $\sigma_i^{v;\mathbf{a}} \to U_i$, $\sigma_i^{w;\mathbf{a}} \to V_i$. In such a system, the braiding of anyons i and $(i+1)$ at junctions v or v' would realize gate U_i, while braiding at junctions w or w' would realize gate V_i. This proposal shows that conducting quantum computations on a topological quantum computer based on such a modular network would be a relatively easy task provided that one could manipulate anyons efficiently.

Finally, let us remark that the above desired features of the modular network are lost when one adds just a single edge to the network in the way shown in Fig. 6(b). By a visual inspection, one can check that network from Fig. 6(b) is now 3-connected. Analogous properties of quantum statistics on networks have been observed for Abelian anyons [15,22].

Relation to the braiding of Majorana fermions. Let us consider Majorana fermions in quantum wires modeled by a Kitaev superconducting chain of spinless fermions [23,24]. In a certain range of this model's Hamiltonian (called the topological region forming topological strings) there exists one zero-energy eigenmode d_0 which can be represented in terms of two Majorana edge modes γ_1 and γ_2, $d_0 = \gamma_1 + i\gamma_2$, that are localized at the beginning and at the end of the chain, respectively [24]. Because of the localization of Majorana modes, one can consider their braiding on chains that are coupled into trijunctions by adiabatically tuning parameters of the Hamiltonian in a local fashion [3]. Performing quantum computations with Majorana edge modes would require creating a network with multiple well-separated edge modes on it. It has been shown in Ref. [3] that the exchange of two edge modes γ_i and γ_{i+1} gives a quantum gate $U_i = \exp(\pi\gamma_i\gamma_{i+1}/4)$. Moreover, any one-particle move where just one Majorana fermion is being adiabatically transported results with the multiplication of the wave function by a global phase factor. Hence, in terms of anyon braiding, this model has the following properties: (i) All one-particle moves do not change the quantum state of the system (are effectively put to identity) and (ii) all simple braids are represented by the same quantum gate, i.e., $\sigma_i^{v;\mathbf{a}} \to U_i$ for any v and $\mathbf{a}$. Although our introduced braids ignore the existence of topological regions that connect pairs of Majorana edge modes, our approach can be adapted to take them into account. In particular, one can perform all braids on junctions, all lollipop moves, and Θ moves in a way that avoids self-intersections of the topological "strings" (see Supplemental Material [25]). Therefore, the braiding of Majorana fermions on any network is exactly the same as braiding in 2D and it seems not to exploit the full potential of modular networks outlined in previous sections. That said, this model shows that one can hope to find other physical models for anyon braiding on networks that would not be directly equivalent to 2D braiding [26,27].

The authors gratefully acknowledge the support of the American Institute of Mathematics (AIM) where this collaboration was initiated. We would like to thank A. Sawicki and J. Harrison for useful discussions during the workshop at AIM. T.M. would like to also thank N. Jones and J. Slingerland for discussions about Majorana fermions and physical aspects of anyon braiding and J. Robbins for helpful feedback concerning the manuscript. B.H.A. was supported by the National Research Foundation of Korea (NRF) grant funded by the Ministry of Science and ICT (MSIT) No. 2020R1A2C1A01003201. T.M. acknowledges the support of the Foundation for Polish Science (FNP), START programme.

[1] C. Nayak, S. H. Simon, A. Stern, M. Freedman, and S. Das Sarma, Non-Abelian anyons and topological quantum computation, Rev. Mod. Phys. **80**, 1083 (2008).

[2] J. Alicea, New directions in the pursuit of Majorana fermions in solid state systems, Rep. Prog. Phys. **75**, 076501 (2012).

[3] J. Alicea, Y. Oreg, G. Refael, F. von Oppen, and M. P. A. Fisher, Non-Abelian statistics and topological quantum information processing in 1D wire networks, Nat. Phys. **7**, 412 (2011).

[4] B. H. An and T. Maciazek, Geometric presentations of braid groups for particles on a graph, arXiv:2006.15256.

[5] A. Das, Y. Ronen, Y. Most, Y. Oreg, M. Heiblum, and H. Shtrikman, Zero-bias peaks and splitting in an Al-InAs nanowire topological superconductor as a signature of Majorana fermions, Nat. Phys. **8**, 887 (2012).

[6] V. Mourik, K. Zuo, S. M. Frolov, S. R. Plissard, E. P. A. M. Bakkers, and L. P. Kouwenhoven, Signatures of Majorana fermions in hybrid superconductor-semiconductor nanowire devices, Science **336**, 1003 (2012).

[7] L. P. Rokhinson, X. Liu, and J. K. Furdyna, The fractional a.c. Josephson effect in a semiconductor-superconductor nanowire as a signature of Majorana particles, Nat. Phys. **8**, 795 (2012).

[8] S. Nadj-Perge, I. K. Drozdov, J. Li, H. Chen, S. Jeon, J. Seo, A. H. MacDonald, B. A. Bernevig, and A. Yazdani, Topological matter. Observation of Majorana fermions in ferromagnetic atomic chains on a superconductor, Science **346**, 602 (2014).

[9] J.-S. Xu, K. Sun, Y.-J. Han, C.-F. Li, J. K. Pachos, and G.-C. Guo, Simulating the exchange of Majorana zero modes with a photonic system, Nat. Commun. **7**, 13194 (2016).

[10] T. Hyart, B. van Heck, I. C. Fulga, M. Burrello, A. R. Akhmerov, and C. W. J. Beenakker, Flux-controlled quantum computation with Majorana fermions, Phys. Rev. B **88**, 035121 (2013).

[11] T. E. O'Brien, P. Rożek, and A. R. Akhmerov, Majorana-Based Fermionic Quantum Computation, Phys. Rev. Lett. **120**, 220504 (2018).

[12] F. Pientka, A. Keselman, E. Berg, A. Yacoby, A. Stern, and B. I. Halperin, Topological Superconductivity in a Planar Josephson Junction, Phys. Rev. X **7**, 021032 (2017).

[13] A. Stern and E. Berg, Fractional Josephson Vortices and Braiding of Majorana Zero Modes in Planar Superconductor-Semiconductor Heterostructures, Phys. Rev. Lett. **122**, 107701 (2019).

[14] J. M. Harrison, J. P. Keating, and J. M. Robbins, Quantum statistics on graphs, Proc. R. Soc. A **467**, 212 (2011).

[15] J. M. Harrison, J. P. Keating, J. M. Robbins, and A. Sawicki, *n*-Particle quantum statistics on graphs, Commun. Math. Phys. **330**, 1293 (2014).

[16] J. M. Leinaas and J. Myrheim, On the theory of identical particles, Il Nuovo Cimento B **37** 1 (1977).

[17] J. M. Souriau, *Structure des Systémes Dynamiques* (Dunod, Paris, 1970).

[18] T. Einarsson, Fractional Statistics on a Torus, Phys. Rev. Lett. **64**, 1995 (1990).

[19] F. Wilczek, *Fractional Statistics and Anyon Superconductivity* (World Scientific, Singapore, 1990).

[20] K. Murasugi and B. Kurpita, *A Study of Braids*, Mathematics and its Applications Vol. 484 (Kluwer Academic, Dordrecht, 1999).

[21] K. Menger, Zur allgemeinen Kurventheorie, Fund. Math. **10**, 96 (1927).

[22] K. H. Ko and H. W. Park, Characteristics of graph braid groups, Discrete Comput. Geom. **48**, 4 (2012).

[23] F. von Oppen, Y. Peng, and F. Pientka, Topological superconducting phases in one dimension, in *Topological Aspects of Condensed Matter Physics*, edited by C. Chamon, M. O. Goerbig, R. Moessner, and L. F. Cugliandolo, Lecture Notes of the Les Houches Summer School, CIII, 2014 (Oxford University Press, Oxford, UK, 2017).

[24] A. Yu. Kitaev, Unpaired Majorana fermions in quantum wires, Phys.-Usp. **44**, 131 (2001).

[25] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevB.102.201407 for technical details on adapting our braiding relations to Majorana fermions paired by topological strings.

[26] T. Maciążek and A. Sawicki, Homology groups for particles on one-connected graphs, J. Math. Phys. **58**, 062103 (2017).

[27] T. Maciążek and A. Sawicki, Non-abelian quantum statistics on graphs, Commun. Math. Phys. **371**, 921 (2019).