Rapid Communications

Note on Wess-Zumino-Witten models and quasiuniversality in 2+1 dimensions

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We suggest the possibility that the two-dimensional $\mathrm{SU}(2)_k$ Wess-Zumino-Witten (WZW) theory, which has global SO(4) symmetry, can be continued to $2+\epsilon$ dimensions by enlarging the symmetry to $\mathrm{SO}(4+\epsilon)$. This is motivated by the three-dimensional sigma model with SO(5) symmetry and a WZW term, which is relevant to deconfined criticality. If such a continuation exists, the structure of the renormalization group flows at small ϵ may be fixed by assuming analyticity in ϵ. This leads to the conjecture that the WZW fixed point annihilates with a new, unstable fixed point at a critical dimensionality $d_c > 2$. We suggest that $d_c < 3$ for all k, and we compute d_c in the limit of large k. The flows support the conjecture that the deconfined phase transition in SU(2) magnets is a "pseudocritical" point with approximate SO(5), controlled by a fixed point slightly outside the physical parameter space.

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This Rapid Communication makes a conjecture about renormalization group (RG) flows in nonlinear sigma models (NLσMs) with Wess-Zumino-Witten (WZW) terms in $2+\epsilon$ dimensions. It is speculative, since we do not provide a concrete definition of these models in noninteger dimensions. But we point out that assuming the existence of such a continuation in ϵ leads to interesting conclusions. The WZW fixed point survives up to a critical ϵ, at which it annihilates with a new, unstable fixed point that did not exist in two dimensions (2D). This critical ϵ_c can be calculated easily only at large k, where k is the WZW level, but we conjecture that for all k the annihilation occurs in between 2D and 3D. Our motivation is the case $\epsilon = 1$, which is the SO(5)-symmetric NLσM for a five-component unit vector, in 3D. This is a useful effective field theory for various interesting phase transitions [1–3] that show numerical evidence of emergent SO(5) [4–8]. The scenario obtained here supports, and gives another way of thinking about, the "quasiuniversal" or "pseudocritical" RG flows conjectured previously for these models [9,10], since the fixed point annihilation at $d_c \lesssim 3$ suggested by this calculation provides a mechanism for slow RG flows in $d = 3$. We return to this at the end.

The Euclidean action for the $\mathrm{SU}(2)_k$ WZW model in 2D, in terms of an SU(2) matrix $g(x_1, x_2)$, is [11–15]

$$S = \frac{1}{2\lambda^2}\int d^2x\, \mathrm{Tr}(\partial_\mu g^{-1})(\partial_\mu g) + ik\,\Gamma. \tag{1}$$

Γ is the WZW term, written in terms of an extension $g(x_1, x_2, x_3)$ of the field to a fictitious 3D "bulk" as $\Gamma = \frac{\epsilon_{\mu\nu\lambda}}{12\pi}\int d^3x\, \mathrm{Tr}(g^{-1}\partial_\mu g)(g^{-1}\partial_\nu g)(g^{-1}\partial_\lambda g)$. The field lives on the sphere S^3, and can be written as a four-component unit vector Φ using the Pauli matrices: $g = \Phi_0\mathbb{I} + i\sum_{a=1}^{3}\Phi_a\sigma^a$. Therefore this is also the standard O(4) sigma model, with the addition of the WZW term, which reduces the internal symmetry to $\mathrm{SO}(4) = [\mathrm{SU}(2)_\mathrm{L}\times\mathrm{SU}(2)_\mathrm{R}]/\mathbb{Z}_2$. For a given $k \in \mathbb{Z}$, the theory has an unstable, trivial fixed point at $\lambda^2 = 0$, and a stable, nontrivial one at $\lambda_*^2 = 4\pi/|k|$ [11,15].

The construction generalizes to d dimensions, giving the NLσM for a $(d+2)$-component "spin," with a WZW term and $\mathrm{SO}(d+2)$ symmetry (see, e.g., Ref. [16]),

$$S_d = \frac{1}{\lambda^2}\int(\partial\Phi)^2 + \frac{2\pi ik\epsilon_{a_1\cdots a_{d+2}}}{\mathrm{area}(S^{d+1})}\int \Phi_{a_1}\partial_{x_1}\Phi_{a_2}\cdots\partial_u\Phi_{a_{d+2}}. \tag{2}$$

The most interesting case for us in the above hierarchy of theories is S_3, the SO(5) sigma model in $d = 3$. In $d = 1$ the standard kinetic term is irrelevant at low energies, and dropping it leaves the usual coherent-state path integral for a spin of size $k/2$ [17]. The $d = 0$ case is an integral: Writing $\Phi_0 + i\Phi_1 = e^{i\theta}$, the action is $S_0 = ik\theta$, and the "correlator" is $\langle e^{im\theta}\rangle = \delta_{m,k}$.

These theories, often with symmetry-breaking anisotropy terms, have many applications to critical phenomena. These applications can usually be understood heuristically from the fact that S_ℓ is the effective theory on an appropriate ℓ-dimensional defect (built by fixing the configuration of $d-\ell$ components of Φ) in the d-dimensional theory S_d. For example, we may construct a hedgehoglike configuration for d components of Φ. The effective theory at this defect is S_0 for the remaining two components. The above expression for $\langle e^{im\theta}\rangle$ then shows that such defects are forbidden except at the loci of insertions of $e^{i\theta(x)}$. This is connected to the fact that an anisotropic version of S_3 describes the 3D O(3) model with hedgehog defects forbidden [2,6,18–20].

Motivated by this hierarchy of field theories, let us entertain the possibility that the fixed points present in 2D can be tracked to $2+\epsilon$ dimensions. Whether this can be made precise is less clear than in the case without a WZW term, where the $2+\epsilon$ expansion is standard, because the structure of the topological term depends on the dimensionality [21].

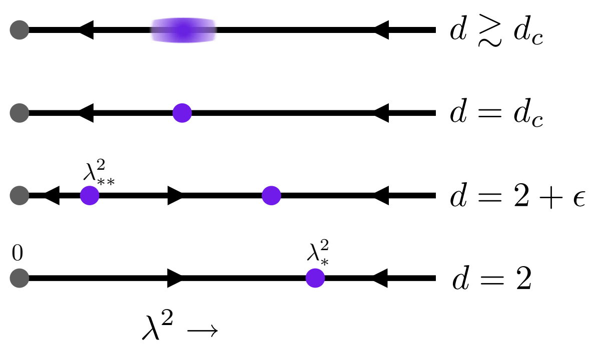

FIG. 1. Topology of flows, as a function of dimension. The smudge ($d \gtrsim d_c$) indicates slow RG flow, without a fixed point.

Nevertheless, if we assume the continuation exists, the flows at small ϵ can be fixed very simply using known results in 2D and assuming analyticity of the RG equations in ϵ. This is inspired by the treatment of the O(n) model close to $n = d = 2$ in Ref. [22].

In two dimensions the one-loop beta function is [11]

$$\frac{d\lambda^2}{d\ln L} = \frac{\lambda^4}{2\pi}\left[1 - \left(\frac{\lambda^2 k}{4\pi}\right)^2\right]. \tag{3}$$

The one-loop approximation is justified at large $|k|$ because the fixed point is at $\lambda^2 = \mathcal{O}(k^{-1})$, so that the entire action is multiplied by a large parameter of order k [11]. For k of order 1 we should use an unknown exact β function, but with the same topology of flows. We write this schematically as

$$\frac{d\lambda^2}{d\ln L} = \beta_k^{(0)}(\lambda^2). \tag{4}$$

We now go to $d = 2 + \epsilon$, assuming the RG equations are analytic in ϵ,

$$\frac{d\lambda^2}{d\ln L} = \beta_k^{(0)}(\lambda^2) + \epsilon\, \beta_k^{(1)}(\lambda^2) + \mathcal{O}(\epsilon)^2. \tag{5}$$

In the limit of small λ^2 we have, trivially,

$$\beta_k^{(0)}(\lambda^2) = \frac{\lambda^4}{2\pi} + \mathcal{O}(\lambda^6), \quad \beta_k^{(1)}(\lambda^2) = -\lambda^2 + \mathcal{O}(\lambda^4). \tag{6}$$

This is already enough to fix the topology of the RG flows when ϵ is small: see Fig. 1, third panel. At $\epsilon = 0$ we have a marginally unstable fixed point at $\lambda^2 = 0$ and a stable one at λ^2_*. The latter remains stable and isolated for small ϵ [but, if the signs predicted by the perturbative expressions are valid, it shifts towards the origin by $\mathcal{O}(\epsilon)$, and its irrelevant RG eigenvalue moves slightly towards zero]. In contrast, the perturbation splits the fixed point at $\lambda^2 = 0$ into a stable fixed point at $\lambda^2 = 0$ and an *unstable* fixed point at $\lambda^2_{**} \simeq 2\pi\epsilon$. This splitting in the vicinity of $\lambda^2 = 0$ is similar to the O(N) NLσM without a WZW term; in both cases the unstable fixed point governs a transition between phases with broken/unbroken symmetry. Here, however, the universality class of the fixed point at λ^2_{**} is different, as is that of the unbroken phase.

The likely situation is that, at some $d_c(k)$, the unstable fixed point which is moving away from the origin collides and annihilates with the stable fixed point which is moving towards the origin—so that in high dimensions there is no fixed point for real λ^2. At $d = d_c(k)$ we have a marginally stable fixed point (Fig. 1).

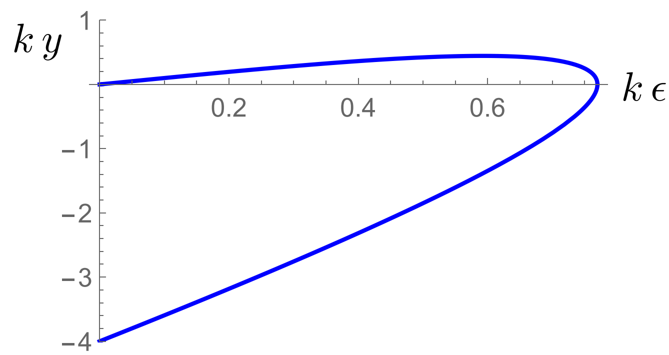

FIG. 2. RG eigenvalues y for stable (lower branch) and unstable (upper) fixed points as a function of ϵ at large k.

We can be more concrete when k is large. Consider the scaling $k \gg 1$ with ϵk of order 1. The relevant regime is where λ^2 is of order ϵ. The leading terms are then

$$\frac{d\lambda^2}{d\ln L} = -\epsilon\lambda^2 + \frac{\lambda^4}{2\pi}\left[1 - \left(\frac{\lambda^2 k}{4\pi}\right)^2\right]. \tag{7}$$

We see that the annihilation described above indeed occurs, and the critical dimensionality is

$$d_c(k) = 2 + \frac{4}{3\sqrt{3} \times k}. \tag{8}$$

Figure 2 shows the RG eigenvalues of the stable and unstable fixed points for $d < d_c$.

When $d \gtrsim d_c$ we have pseudocritical RG flows. Slow flow for $\lambda^2 \sim \frac{4\pi}{\sqrt{3}k}$, where the flows are approximately

$$\frac{d\delta\lambda^2}{d\ln L} \simeq -\frac{4\pi(d - d_c)}{\sqrt{3}k} - \frac{(\delta\lambda^2)^2}{2\pi}, \tag{9}$$

yields the exponentially large correlation length $\xi \sim \exp\frac{3^{1/4}\pi\sqrt{k}}{\sqrt{2(d-d_c)}}$, as in other theories with a fixed point annihilation [9,23–30]. Reference [10] argued that in such a situation, expanding the RG equations for irrelevant couplings in $d - d_c$ shows that quasiuniversality (independence of UV couplings) holds on long scales, to exponentially good precision in $[d - d_c]^{-1/2}$, despite the fact that λ^2 drifts: Different microscopic models travel along the same quasiuniversal flow line in theory space. For $d \gtrsim d_c$ we also have complex, SO($d + 2$)-symmetric fixed points with $\mathrm{Im}\,\lambda^2 \propto \sqrt{d - d_c}$. Complex fixed points have been explored recently in Refs. [31–35].

In the context of deconfined criticality we are interested in 3D models that in the UV have a smaller symmetry than SO(5). If d_c is close enough to 3 to give a large ξ in 3D, and assuming that the four-index symmetric tensor of SO($4 + \epsilon$) is irrelevant at d_c [4,10] (this is the case at large k, where scaling dimensions are close to those in 2D), then the above flows will lead to a pseudocritical phase transition with approximate emergent SO(5), by the scenario discussed in Refs. [10,36]. This scenario is consistent with simulations, and, since it does not require a unitary 3D fixed point, with conformal bootstrap [37–41]. It is also consistent with what we know about various dual gauge theories for deconfined criticality [9,10], including recent ϵ-expansion [42–45] and large N [46] results. The endpoint of the quasiuniversal flow line is the ordered phase ($\lambda^2 = 0$): In the application to deconfined criticality this means that at the very longest scales the emergent symmetry gets spontaneously broken, giving artificial SO(5) "Goldstone modes" with a very small mass [36,47].

Though speculative, the present lowest-order expansion supports this scenario. If a consistent framework for expanding to higher orders in ϵ [48] can be defined, then this would be one way to put the pseudocriticality scenario for SO(5) on firm ground. The above also suggests examining numerically the 3D models with $k > 1$ (or rather related sign-free lattice models which could be based on those relevant to the $k = 1$ case [6,7,9,49]), to test for pseudocriticality there.

We can consider other, related deformations of the WZW model. At the order to which we have worked, changing the dimension to $2 + \epsilon$ has the same effect on the RG flows as changing the power of momentum q in the kinetic term to $|q|^{2-\epsilon}$. This raises the question of whether we can study quasi-universality in the 3D model, while avoiding the WZW term in noninteger dimensions, by imposing a dispersion of the form $|q|^{3-\delta}$ with $\delta > 0$. It also raises the question of whether we can obtain pseudocriticality, fixed point annihilation, complex fixed points, etc., in the *one*-dimensional (0+1D) model with a WZ term, by taking a coupling that is long ranged [50,51] in time, $\sim|t - t'|^{-(2-\delta)}$, and varying δ. This model is relevant to the dynamics of a spin coupled to a bath [52–54]. We hope to return to these issues elsewhere.

Note added. Recently, I became aware of independent work by Ma and Wang reaching the same essential conclusions [55].

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