Rapid Communications

Topological disorder triggered by interaction-induced flattening of electron spectra in solids

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(Received 5 April 2020; revised 4 September 2020; accepted 16 September 2020; published 10 November 2020)

We address the intervention of classical-like behavior, well documented in experimental studies of strongly correlated electron systems of solids that emerge at temperatures T far below the Debye temperature T_D. We attribute this unexpected phenomenon to spontaneous rearrangement of the conventional Landau state beyond a critical point at which the *topological* stability of this state breaks down, leading to the formation of an *interaction-induced flat band* adjacent to the nominal Fermi surface. We demonstrate that beyond the critical point, the quasiparticle picture of such correlated Fermi systems still holds, since the damping of single-particle excitations remains small compared with the Fermi energy $T_F = p_F^2/2m_e$. A Pitaevskii-style equation for determination of the rearranged quasiparticle momentum distribution $n_*(\mathbf{p})$ is derived, which applies to explanation of the linear-in-T behavior of the resistivity $\rho(T)$ found experimentally.

DOI: 10.1103/PhysRevB.102.201108

Currently,"topological" has become one of the most commonly used terms in condensed-matter physics, surpassing "quantum critical point." It is sufficient to mention such collocations as topological order, topological transition, and topological insulator. On the other hand. over decades the mathematical literature has featured, along with more traditional types of chaotic behavior, relevant discussions of topological entropy (TE) and topological chaos, which exhibit *positive* entropy S [1–4] (see also the Supplemental Material (SM) [5] and sources [6–11] cited therein). In the present paper, addressing strongly correlated electron systems of solids including cuprates and graphene, we investigate possible existence of a finite entropy $S > 0$ at temperatures T much lower than the Debye value T_D identifying the boundary between classical and quantum regimes.

Seemingly, this option would be obviated by the Nernst theorem requiring $S(T)$ to vanish upon reaching $T = 0$. However, recent developments warrant a revision of this conventional stance. The first symptoms appeared in measurements [12,13] of the thermal expansion coefficient $\alpha(T) = -V^{-1}\partial V/\partial T = V^{-1}\partial S/\partial P$ of the strongly correlated heavy-fermion superconductor $CeCoIn_5$, which has a tiny critical value $T_c = 2.3\,K$ at which superconductivity terminates. Although experimental results are indeed consistent with obedience of the Nernst theorem requiring $\alpha(0) = 0$, it is nevertheless of paramount significance that at extremely low temperatures $T > T_c^+ = T_c + 0$, where the system is already in the normal state, experiment has established the perplexing behavior

$$\alpha(T) = \alpha_0 + \alpha_1 T. \tag{1}$$

The *nonzero* offset $\alpha_0 \simeq 0.5 \times 10^{-5}/K$ exceeds values found in ordinary metals at these temperatures by a huge factor of order $10^3 - 10^4$. This implies that an analogous *classical-like* offset S_0, associated with α_0 by the relation $\alpha_0 = \partial S_0/\partial P$, is present in the entropy itself—pointing unambiguously to the presence of disorder in the regime of extremely low $T > T_c^+ \ll T_D$.

Another experimental challenge is associated with the low-temperature, non-Fermi-liquid (NFL) behavior of the normal-state resistivity $\rho(T)$ of the same $CeCoIn_5$ metal at various pressures P, which, according to Fermi liquid (FL) theory, should obey the formula $\rho(T) = \rho_0 + A_2T^2$. Instead, at $P < P^* \simeq 2$ GPa, experiment [14] has revealed the *classical-like* strange-metal behavior

$$\rho(T) = \rho_0 + A_1 T, \tag{2}$$

shown in Fig. 1. It is as if classical physics already prevails at $T_c^+ < T \ll T_D$. This remarkable linear-in-T behavior of $\rho(T)$ is currently observed in diverse systems (see, e.g., Refs. [15–17]). In some cases, the slope A_1 experiences a noticeable jump [18] (see below).

Even more bizarre behavior has surfaced in recent studies [19] of the resistivity of twisted bilayer graphene (TBLG) as a function of twist angle θ, as depicted in Fig. 2. Profound variations of $A_1(\theta)$ are seen, especially toward the so-called magic angle θ_m, where the A_1 term increases by more than three orders of magnitude, as does the residual resistivity $\rho_0(\theta)$, echoing a tenfold variation of ρ_0 as a function of pressure P, as shown in Fig. 1. Since ρ_0 must be a parameter-independent quantity [20] if the impurity population remains unchanged, its documented behavior defies explanation within the standard FL approach.

Moreover, in high-temperature superconducting, overdoped copper oxides, where $T_c(x)$ terminates at critical doping

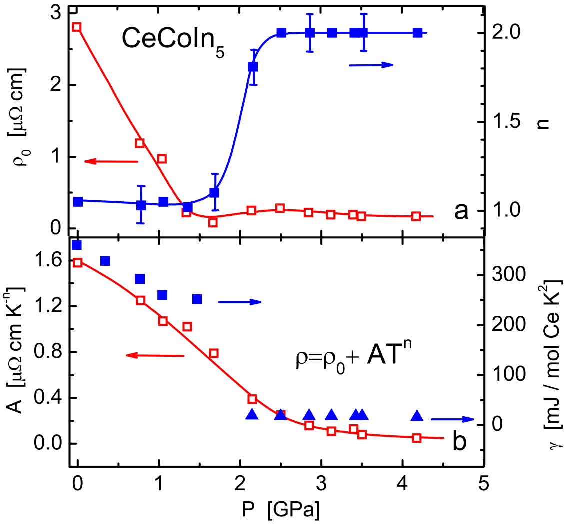

FIG. 1. Upper panel: Values of the residual resistivity ρ_0 (left axis, open squares) and the index n in the fit $\rho(T) = \rho_0 + AT^n$ (right axis, solid squares) versus pressure P. Bottom panel: Temperature coefficient of resistivity A (left panel, open squares) and specific-heat coefficient γ (right panel, solid squares and solid triangles).

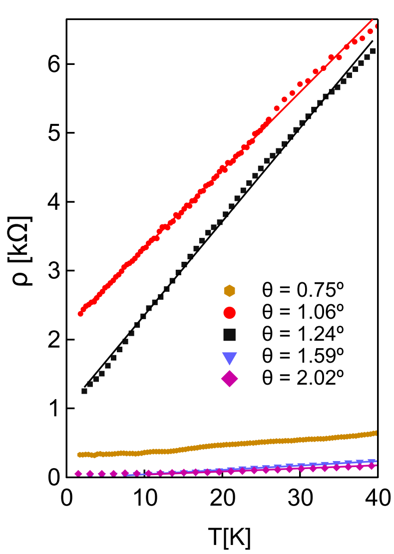

FIG. 2. NFL resistivity $\rho(T)$ measured in TBLG devices at different twist angles.

value x_c with nearly linear dependence on $x_c - x$ (see Fig. 3), the quite remarkable doping independence

$$A_1(x)/T_c(x) = \text{const}, \tag{3}$$

has been discovered [21,22], a feature shared with Bechgaard salts [23]. As emphasized in Ref. [22], this feature points to the presence of a hidden phase, emergent at x_c *simultaneously* with the superconducting state.

Explanation of the strange-metal behavior Eq. (2) observed ubiquitously at low T has become one of the most intensely debated theoretical problems of modern condensed-matter theory. Analysis of proposed scenarios in a recent review article [24] has concluded that none of these is capable of

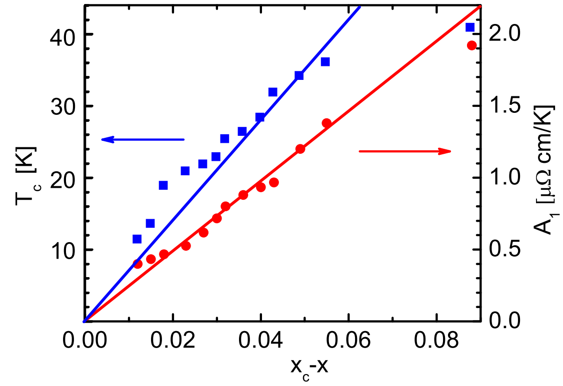

FIG. 3. Dependence of the factor A_1 in the resistivity $\rho(T)$ (red circles, right axis) and the critical temperature T_c (blue squares, left axis) of overdoped $La_{2-x}Sr_xCuO_4$ films on the doping x measured from its critical value $x_c = 0.26$ [21]. Red and blue lines show the best linear fits to the data, which support the conclusion that $A_1(x) \propto T_c(x)$, indicative of behavior inconsistent with conventional theory.

explaining all the relevant experimental findings. In particular, candidates based on a quantum-critical-point scenario fall short. As witnessed by the phase diagrams of $CeCoIn_5$, cuprates, and graphene, there are no appropriate ordered phases adjacent to the strange-metal region; the effects of associated quantum fluctuations are small.

In this situation, we turn to a different scenario, based on the formation of a fermion condensate (FC) [25–33]. Analogy with a boson condensate (BC) is evident in the respective densities of states $\rho_{\rm FC}(\varepsilon) = n_{\rm FC}\delta(\varepsilon)$ [25] and $\rho_{\rm BC}(\varepsilon) = n_{\rm BC}\delta(\varepsilon)$, where $n_{\rm FC}$ and $n_{\rm BC}$ are the FC and BC densities. To be more specific, the essence of the phenomenon of fermion condensation lies in a *swelling of the Fermi surface*, i.e., in emergence of an *interaction-induced* flat portion $\epsilon(\mathbf{p}) = 0$ of the single-particle spectrum $\epsilon(\mathbf{p})$ that occupies a region $\mathbf{p} \in \Omega$ where the real quasiparticle momentum distribution [hereafter denoted $n_*(\mathbf{p})$] departs drastically from the Landau step $n_L(\mathbf{p}) = \theta(-\epsilon(\mathbf{p}))$.

The trigger for such a profound rearrangement of the Landau state lies in violation of its necessary stability condition (NSC), which requires positivity of the change $\delta E = \sum_{\mathbf{p}} \epsilon(\mathbf{p})\delta n_L(\mathbf{p})$ of the ground-state energy E under *any variation* of the $n_L(\mathbf{p})$ compatible with the Pauli principle [28]. In Landau theory with $\epsilon(p) = v_F(p - p_F)$, this NSC is known to hold as long as the Fermi velocity v_F remains positive. Beyond a critical point where it breaks down, the Fermi surface becomes multiconnected. This aspect is a typical *topological* signature. Accordingly, the word topological in the term topological chaos has a twofold meaning, such that the associated bifurcation point can be called a topological critical point (TCP). Frequently, as in a neck-distortion problem addressed by I. M. Lifshitz in his seminal article [34], the corresponding topological rearrangement of the Fermi surface is unique. However, this is not the case in dealing with the TBLG problem, where nearly-flat-band solutions are found, a distinctive feature of those being related to the *passage* of the Fermi velocity through zero at the first magic twist angle $\theta_m^{(1)}$ [35–38]. A variety of options for violation of the topological stability of the TBLG Landau state then arise. In contrast, within the FC scenario, introduction of $e-e$ interactions leads to the advent of interaction-induced flat bands which replace the nearly flat bands found in Refs. [35–38]. Technically, this procedure is reminiscent of the Maxwell construction in statistical physics, where the isotherm in the Van der Waals pressure-volume phase diagram is in reality replaced by a horizontal line. An analogous situation is inherent in cuprates and other strongly correlated electron systems of solids. Importantly, in the familiar temperature-doping phase diagram, it is the TCP that separates the well-understood FL behavior from the behavior associated with topological chaos, which is responsible for the strange-metal regime.

We begin analysis with the reminder that in superconducting alloys that obey Abrikosov-Gor'kov theory [39,40], the damping γ acquires a *finite* value due to impurity-induced scattering, implying failure of the basic postulate $\gamma/\epsilon(\mathbf{p})<1$ of Landau theory. Nonetheless, the FL quasiparticle formalism, in which the pole part G_q of the single-particle Green's function $G=(\epsilon-\epsilon_p^0-\Sigma)^{-1}$ has the form

$$G_q(\mathbf{p},\varepsilon)=\frac{1-n_L(\mathbf{p})}{\varepsilon-\epsilon(\mathbf{p})+i\gamma}+\frac{n_L(\mathbf{p})}{\varepsilon-\epsilon(\mathbf{p})-i\gamma}, \qquad (4)$$

is still applicable [41].

Beyond the TCP where interaction-induced flat bands emerge, further alteration of the pole part G_q occurs, its form becoming [25–29]

$$G_q(\mathbf{p},\varepsilon)=\frac{1-n_*(\mathbf{p})}{\varepsilon-\epsilon(\mathbf{p})+i\gamma(\varepsilon)}+\frac{n_*(\mathbf{p})}{\varepsilon-\epsilon(\mathbf{p})-i\gamma(\varepsilon)}, \qquad (5)$$

with $\gamma>0$ and occupation numbers $0<n_*(\mathbf{p})<1$ characterizing the FC. Their difference from $n_L(\mathbf{p})$, which resides solely in the Ω region, is to be determined through solution of a nonlinear integral Landau-Pitaevskii style equation (cf. Refs. [42–44]) of the theory of fermion condensation, viz.,

$$\frac{\partial\epsilon(\mathbf{p})}{\partial\mathbf{p}}=\frac{\partial\epsilon_0(\mathbf{p})}{\partial\mathbf{p}}+2\int f(\mathbf{p},\mathbf{p}_1)\frac{\partial n_*(\mathbf{p}_1)}{\partial\mathbf{p}_1}\frac{d^3\mathbf{p}}{(2\pi)^3}. \qquad (6)$$

Here $f(\mathbf{p},\mathbf{p}_1)$ is the spin-independent part of the Landau interaction function. The free term includes all contributions to the group velocity that remain in the $f=0$ limit.

A salient feature of the $T=0$ FC solutions is the *identical vanishing* of the dispersion of the spectrum $\epsilon(\mathbf{p})$ in the Ω region. At $T>0$, the FC spectrum acquires a small dispersion, linear in T [27]:

$$\epsilon(\mathbf{p},T)=T\ln\frac{1-n_*(\mathbf{p})}{n_*(\mathbf{p})}, \quad \mathbf{p}\in\Omega. \qquad (7)$$

Experimental verification of this effect through ARPES measurements is crucial for substantiation of the FC concept under consideration.

Equation (6) is derived from the formal relation $\delta\Sigma=(\mathcal{U}\delta G)$ [with $\delta G(p,\varepsilon)=G(\mathbf{p}-e\mathbf{A},\varepsilon)-G(\mathbf{p},\varepsilon)$] of variational many-body theory for the self-energy in terms of the subset of Feynman diagrams $\mathcal{U}$ of the two-particle scattering amplitude that are irreducible in the particle-hole channel, hence regular near the Fermi surface. Assuming gauge invariance of the theory, one finds [44,45]

$$-\frac{\partial G^{-1}(\mathbf{p},\varepsilon)}{\partial\mathbf{p}}=\frac{\mathbf{p}}{m_e}-\left(\mathcal{U}(\mathbf{p},\varepsilon;\mathbf{k},\omega)\frac{\partial G(\mathbf{k},\omega)}{\partial\mathbf{k}}\right). \qquad (8)$$

The round brackets in this equation imply integration and summation over intermediate momenta and spins with a proper normalization factor. Implementation of a slightly refined universal quantitative procedure [30,31] for renormalization of this equation allows it, irrespective of correlations, to be recast in closed form, as if one were dealing with a *gas of interacting quasiparticles*. [The word gas is appropriate, since Eq. (6) contains only the single phenomenological amplitude f of quasiparticle-pair collisions.] A salient feature of this procedure is that Eq. (6) holds *both* in conventional FLs *and* in electron systems of solids moving in the periodic external field of the crystal lattice. This follows because *solely* gauge invariance was assumed in its derivation, which therefore holds for crystal structures as well. In short, the widespread impression that the FL approach is inapplicable to crystal structures is groundless. We emphasize once more that the FL renormalization procedure works properly *irrespective of the magnitude of the ratio* $\gamma/\epsilon(\mathbf{p})$ (see Sec. 2 of the SM [5] for specifics, and especially references [39,41–43,46]).

We are now in a position to consider the connection between the customary iterative procedure for solving the basic FC Eq. (6) and the topological chaos problem addressed in many mathematical articles (see, especially, Refs. [1,4]). In the standard iterative scheme, the jth iteration $n^{(j)}(\mathbf{p})$, with $j=0,1,2,\ldots$, is inserted into the right side of Eq. (6) to generate the next iteration of the single-particle spectrum, $\epsilon^{(j+1)}(\mathbf{p})$, and this process is repeated indefinitely to finally yield a convergent result whose TE is equal to 0. However, beyond the TCP, such a procedure fails, since the iterations $n^{(j)}(\mathbf{p})$ then undergo chaotic jumps from 0 to 1 and vice versa, generating noise, identified with some TE. To evaluate the spectrum quantitatively, in Ref. [31] the iterative discrete-time map was reconstructed in such a way that the discrete time t_j replaces the iteration number j. Subsequent time averaging of relevant quantities, adapted from formulas of classical theory, allows one to find a specific self-consistent solution. Its prominent feature is the development of an interaction-induced flat portion in the single-particle spectrum $\epsilon(\mathbf{p})$ that embraces the nominal Fermi surface (for exemplification, see Sec. 1 of the SM [5]). Another distinctive signature of the set of specific FC solutions of Eq. (6) lies in the occurrence of a nonzero entropy excess S_*, emergent upon their substitution into the familiar combinatoric formula for evaluation of the entropy. This yields [25,31,32,47]

$$S_*=-2\sum_{\mathbf{p}}[n_*(\mathbf{p})\ln n_*(\mathbf{p})+(1-n_*(\mathbf{p}))\ln(1-n_*(\mathbf{p}))], \qquad (9)$$

where summation is running over the FC region, and, in turn, a NFL nonzero value α_* of the coefficient of thermal expansion. Because the presence of a nonzero S_* would contradict the Nernst theorem $S(T=0)=0$ if it survived to $T=0$, the FC must inevitably disappear [25,32,47] at some very low T. One well-elaborated scenario for this metamorphosis is associated with the occurrence of phase transitions, such as the BCS superconducting transition emergent in the case of attraction forces in the Cooper channel, or an antiferromagnetic transition, typically replacing the superconducting phase in external magnetic fields H exceeding the critical field H_{c2}.

At $H < H_{c2}$, a nonzero BCS gap $\Delta(0)$ in the single-particle spectrum $E(\mathbf{p}) = \sqrt{\epsilon^2(\mathbf{p}) + \Delta^2}$ does provide for nullification of $S(T = 0)$. This scenario applies in systems that host a FC as well, opening a specific route to high-T_c superconductivity [25,48,49]. Indeed, consider the BCS equation for determining T_c:

$$D(\mathbf{p}) = -2 \int \mathcal{V}(\mathbf{p}, \mathbf{p}_1) \frac{\tanh \frac{\epsilon(\mathbf{p}_1, T_c)}{2T_c}}{2\epsilon(\mathbf{p}_1, T_c)} D(\mathbf{p}_1) dv_1. \qquad (10)$$

Here $D(\mathbf{p}) = \Delta(\mathbf{p}, T \to T_c)/\sqrt{T_c - T}$ plays the role of an eigenfunction of this linear integral equation, while $\mathcal{V}(\mathbf{p}, \mathbf{p}_1)$ is the block of Feynman diagrams for the two-particle scattering amplitude that are irreducible in the Cooper channel. Upon insertion of Eq. (7) into this equation and straightforward momentum integration over the FC region, one arrives at a non-BCS *linear* relation

$$T_c(x) = c(x)\, T_F, \qquad (11)$$

where $c(x) = \lambda\, \eta(x)$, with λ denoting the effective pairing constant and $\eta(x)$ the FC density. This behavior is in accord with the experimental Uemura plot [38,50].

As discussed above, the entropy excess $S_* \propto \eta$ comes into play at temperatures $T_c^+ < T \ll T_D$ so as to invoke a T-independent term α_0 in the coefficient of thermal expansion, which, in that regime, serves as a signature of fermion condensation [32,47]. Accordingly, execution of extensive low-T measurements of the thermal expansion coefficients in candidate materials would, in principle, provide means (i) to distinguish between flat bands that do not entail excess entropy S_* and the interaction-induced exemplars, and (ii) to create a database of systems that exhibit pronounced NFL properties, in aid of searches for new exotic superconductors.

Very recently, the FC scenario has gained tentative support from ARPES measurements performed in monolayer graphene intercalated by Gd, which have revealed the presence of a flat portion in the single-particle spectrum [51]. However, verification of the correspondence between the flat bands detected in the bilayer system TBLG [19,35,36,38,52–54] and the interaction-driven variety considered here requires a concerted analysis of kinetic properties, especially of comprehensive experimental data on the low T resistivity $\rho(T) = \rho_0 + A_1 T + A_2 T^2$.

Numerous theoretical studies of the NFL behavior of $\rho(T)$ based on the FC concept have been performed. Directing the reader to Refs. [47,55,56] for details, we summarize their pertinent results in the relation

$$\rho_0(x, P, \theta) = \rho_i + a_0 \eta^2(x, P, \theta),\ A_1(x, P, \theta) = a_1 \eta(x, P, \theta), \qquad (12)$$

where ρ_i is the impurity-induced part of ρ and a_0, a_1 are factors independent of input parameters. This expression properly explains the data shown in Figs. 1 and 2. Indeed, we see that in systems having a FC, the residual resistivity ρ_0 depends critically on the FC density η, which changes under variation of input parameters such as doping x, pressure P, and twist angle θ—an effect that is missing in the overwhelming majority of extant scenarios for the NFL behavior of the resistivity $\rho(T)$. Comparison of Eq. (12) with Eq. (11) shows that the theoretical ratio $A_1(x)/T_c(x)$ is indeed doping independent, in agreement with the challenging experimental results shown in Fig. 3. Moreover, assuming that the FC parameter $\eta(x)$ varies linearly with $x_c - x$, which is compatible with model numerical calculations based on Eq. (6), the corresponding result obtained from Eq. (12) is consistent with available experimental data [56].

Turning to the issue of classical-like Planck dissipation [57], we observe that such a feature is inherent in systems that possess a specific collective mode, transverse zero-sound (TZS), which enters provided $m^*/m_e > 6$ [58]. (Notably, in LSCO this ratio exceeds 10 [59], while in $CeCoIn_5$ it is of order 10^2 [60]). In the common case where the Fermi surface is multiconnected, some branches of the TZS mode turn out to be damped, thereby ensuring the occurrence of a linear-in-T term in the resistivity $\rho(T)$. This is broadly analogous to the situation that arises for electron-phonon scattering in solids in the classical limit $T > T_D$. (See also Sec. 3 of the SM [5] and Refs. [58,60–62].) As a result, FC theory predicts that a break will occur in the straight line $\rho(T) = \rho_0 + A_1 T$ at some characteristic Debye temperature $T_{\rm TZS}$ [55,63], in agreement with experimental data on $Sr_3Ru_2O_7$ [60]. However, in the case $T_{\rm TZS} < T_c$, often inherent in exotic superconductors such as $CeCoIn_5$ [14] and TBLG (M4 device) [54], this break disappears and the behavior of $\rho(T)$ is fully reminiscent of that in classical physics.

In contrast, a current scenario of Patel and Sachdev (PS) for Planckian dissipation [64] attributes the NFL behavior Eq. (2) of $\rho(T)$ to the presence of a significant random component in the amplitude of the interaction between quasiparticles near the Fermi surface. However, such a mechanism is hardly relevant to the physics of cuprates. Indeed, in their phase diagrams, the strange-metal regions located above respective high-T_c domains are commonly adjacent to the familiar FL ones, whose properties obey Landau FL theory, in which the interaction amplitudes are free from random components. Hence the PS model can be a toy model at best. Otherwise, boundaries of the high-T_c regions must *simultaneously* be points of phase transitions between FL phases and phases with random behavior of the interaction amplitudes, which is unlikely.

In conclusion, we have demonstrated that the concept of topological chaos is capable of explaining the NFL, classical-like behavior of strongly correlated electron systems that is emergent at temperatures T far below the Debye value T_D, where such behavior hitherto seemed impossible. The origin of the topological chaos, especially well pronounced in graphene, is shown to be associated with the presence of interaction-induced flat bands. The theoretical predictions are consistent with experimental findings, as documented in Figs. 1–3.

We are grateful to P. Esquinazi, P. Gegenwart, M. Greven, E. Henriksen, Ya. Kopelevich, S. Kravchenko, Z. Nussinov, F. Steglich, V. Shaginyan, J. D. Thompson, and G. Volovik for fruitful discussions. V.A.K. and J.W.C. acknowledge financial support from the McDonnell Center for the Space Sciences. The authors thank J. D. Thompson for permission to present data published in Ref. [14] and A. F. Young for permission to present data published as Fig. 3i of Ref. [19] and providing the corresponding file.

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