Anisotropic time-dependent London approach: Application to the ac response in the Meissner state

V. G. Kogan [*]
Ames Laboratory, DOE, Ames, Iowa 50011, USA

R. Prozorov [†]
Ames Laboratory, DOE, Ames, Iowa 50011, USA
and Department of Physics and Astronomy, Iowa State University, Ames, Iowa 50011, USA

(Received 14 September 2020; accepted 10 November 2020; published 23 November 2020)

The anisotropic London equations taking into account the normal currents are derived and applied to the problem of the surface impedance in the Meissner state of anisotropic materials. It is shown that the complex susceptibility of the anisotropic slab depends on the orientation of the applied microwave field relative to the crystal axes. In particular, the anisotropic sample in the microwave field is subject to a torque, unless the field is directed along one of the crystal principle axes.

DOI: 10.1103/PhysRevB.102.184514

I. INTRODUCTION

Its shortcomings notwithstanding, the approach based on London equations played—and still plays—a major role in describing magnetic properties of superconductors away from the critical temperature T_c, where it is, in fact, the only available and sufficiently simple technique for many practical applications. The physical reason for this success is in its ability to describe the Meissner effect, the major feature of superconductors at all temperatures. The anisotropic version of this approach [1,2] proved useful when strongly anisotropic high-T_c materials came to the fore. It was also realized that in time-dependent phenomena the normal dissipative currents due to normal excitations should be taken into account along with persistent currents [3,4]. In particular, normal currents influence superconductor behavior in microwaves absorption [3] and perturb the field distribution of moving vortices [5,6]. In this work, the anisotropic version of the time-dependent London equation is derived and applied to problems of surface impedance and magnetic susceptibility in a simple geometry.

Within the London approach, the current density consists, in general, of normal and superconducting parts:

$$\boldsymbol{J} = \sigma \boldsymbol{E} - \frac{c}{4\pi\lambda^2}\left(\boldsymbol{A} + \frac{\phi_0}{2\pi}\boldsymbol{\nabla}\theta\right), \quad (1)$$

where $\boldsymbol{E}$ is the electric field, λ is the penetration depth, $\boldsymbol{A}$ is the vector potential, θ is the phase, and ϕ_0 is the flux quantum. The conductivity σ of the quasiparticle flow is, in general, frequency dependent. If, however, the frequencies ω are bound by the inequality $\omega\tau_n \ll 1$, with τ_n being the scattering time for the normal excitations, one can consider σ to be a real ω-independent quantity. As always within the London approach, the order parameter is assumed to be constant in space.

In the absence of vortices, we have, by applying curl,

$$\text{curl}\,\text{curl}\boldsymbol{H} + \frac{1}{\lambda^2}\boldsymbol{H} = -\frac{4\pi\sigma}{c^2}\frac{\partial \boldsymbol{H}}{\partial t}. \quad (2)$$

These are, in fact, London equations corrected by the time-dependent right-hand side [5].

A. Surface impedance of the half-space isotropic sample

The surface impedance in isotropic superconductors was considered, e.g., by Clem and Coffey [3]. Equation (2) provides a simple and direct approach to this problem. Let a weak magnetic field $\boldsymbol{H} = H_0\hat{\boldsymbol{x}}e^{-i\omega t}$ be at the surface $z = 0$ of a superconducting half-space $z > 0$. Since the field is assumed to be weak, the order parameter is unperturbed, and we can use the London equation (2). The field is uniform in the plane (x, y) and depends only on z. We look for solutions of

$$-\frac{\partial^2 H_x}{\partial z^2} + \frac{1}{\lambda^2}H_x = -\frac{4\pi\sigma}{c^2}\frac{\partial H_x}{\partial t} \quad (3)$$

in the form $H_x(z)\,e^{-i\omega t}$ and obtain

$$H_x = H_0\,e^{-kz-i\omega t}, \quad k^2 = \frac{1}{\lambda^2} - \frac{2i}{\delta^2}, \quad (4)$$

where $\delta = c/\sqrt{2\pi\sigma\omega}$ is the quasiparticle-related skin depth.

The electric field is found from the Maxwell equation $\text{curl}\boldsymbol{E} = -\partial_t\boldsymbol{H}/c$: $E_y = (i\omega/ck)H_0\,e^{-kz-i\omega t}$, so that the surface impedance (see, e.g., [7])

$$\zeta = -\frac{E_y}{H_x}\Big|_{z=0} = -\frac{i\omega}{ck}. \quad (5)$$

If $\delta \gg \lambda$,

$$k \approx \frac{1}{\lambda}\left(1 - i\frac{\lambda^2}{\delta^2}\right), \quad (6)$$

*kogan@ameslab.gov
†prozorov@ameslab.gov

and

$$\zeta \approx \frac{\omega\lambda^3}{c\delta^2} - i\,\frac{\omega\lambda}{c}. \tag{7}$$

Thus, the dissipative part of the impedance is given by

$$\text{Re}\,\zeta \approx \frac{2\pi}{c^3}\omega^2\sigma\lambda^3. \tag{8}$$

The imaginary part of the impedance is not affected by the quasiparticle part of the current, (see, e.g., Ref. [7]); that is, it depends only on λ. It is worth noting that Eqs. (6) and (7) do not hold in the immediate vicinity of T_c, where λ diverges.

B. Susceptibility of a slab

It is instructive to consider Eq. (2) for a superconducting slab of thickness d in the applied ac field $H_x = H_0 e^{-i\omega t}$ parallel to the slab faces. The solution is

$$H_x = H_0 \frac{\cosh(kz)}{\cosh(kd/2)} e^{-i\omega t}, \tag{9}$$

with k from Eq. (4) and z counted from the slab middle. The electric field is

$$E_y = i\frac{\omega H_0}{ck}\,\frac{\sinh(kz)}{\cosh(kd/2)} e^{-i\omega t}, \tag{10}$$

and the surface impedance

$$\zeta = -\frac{E_y}{H_x}\Big|_{z=d/2} = -\frac{i\omega}{ck}\tanh\frac{kd}{2}. \tag{11}$$

A commonly measured quantity is the susceptibility, defined as the ratio of the average magnetization μ of the slab to the applied field:

$$\chi = \frac{\mu_x}{H_0} = \frac{1}{4\pi d}\int_{-d/2}^{d/2} \frac{H_x(z) - H_0}{H_0} dz$$
$$= -\frac{1}{4\pi} + \frac{1}{2\pi dk}\tanh\frac{kd}{2}. \tag{12}$$

Hence, we have a simple relation between the surface impedance and the slab susceptibility:

$$\chi + \frac{1}{4\pi} = \frac{ic}{2\pi d\omega}\,\zeta; \tag{13}$$

that is, the surface impedance is proportional to the deviation of the susceptibility from the Meissner value $-1/4\pi$.

Hence, for $\lambda \ll \delta$ one obtains, with the help of Eq. (6),

$$\chi + \frac{1}{4\pi} = \frac{\lambda}{2\pi d} + i\frac{\lambda^3}{2\pi d\delta^2}. \tag{14}$$

In the limit $\lambda \to \infty$, we obtain, for the normal metal slab,

$$\chi + \frac{1}{4\pi} = \frac{\delta}{4\pi d}(1+i)\tanh\frac{d(1-i)}{2\delta}. \tag{15}$$

II. ANISOTROPIC MATERIALS

In the absence of vortices, the order parameter can be taken to be real, so that the current equation becomes

$$J_k = \sigma_{kl}E_l - \frac{c}{4\pi}(\lambda^{-2})_{kl}A_l, \tag{16}$$

where σ_{kl} and $(\lambda^{-2})_{kl}$ are tensors of the conductivity due to normal excitations and of the inverse square of the penetration depth. As usual, summation is implied over double indices. Being interested in problems with no conversion of normal currents to supercurrents, we impose the conditions

$$\text{div}\boldsymbol{J}_n = \sigma_{kl}\frac{\partial E_l}{\partial x_k} = 0, \quad \text{div}\boldsymbol{J}_s = \lambda_{kl}^{-2}\frac{\partial A_l}{\partial x_k} = 0; \tag{17}$$

that is, the densities of normal excitations and of Cooper pairs are separately conserved. In particular, this implies a certain gauge for the vector potential.

In order to obtain an equation for the magnetic field exclusively, one has to isolate $\boldsymbol{E}$ and apply the Maxwell equation $\text{curl}\boldsymbol{E} = -\partial_t\boldsymbol{H}/c$. To this end, we multiply Eq. (16) by $\sigma_{sk}^{-1} = \rho_{sk}$, with ρ_{sk} being the resistivity tensor, and sum over k:

$$\rho_{sk}J_k = E_s - \frac{c}{4\pi}\rho_{sk}\lambda_{kl}^{-2}A_l. \tag{18}$$

In the following it is convenient to use the notation $\text{curl}_u\boldsymbol{E} = \epsilon_{uvs}\partial E_s/\partial x_v$, where ϵ_{uvs} is the Levi-Civita unit antisymmetric tensor: all components with an even number of transpositions from (xyz) are $+1$ or -1 for the odd ones and zero otherwise. Hence, applying $\epsilon_{uvs}\partial/\partial x_v$ to Eq. (18), one obtains anisotropic London equations for the magnetic field, the main result of this paper:

$$\frac{c}{4\pi}\rho_{sk}\epsilon_{uvs}\epsilon_{kmn}\frac{\partial^2 H_n}{\partial x_v\partial x_m} + \frac{\partial H_u}{c\,\partial t}$$
$$= -\frac{c}{4\pi}\rho_{sk}\lambda_{kl}^{-2}\epsilon_{uvs}\frac{\partial A_l}{\partial x_v}. \tag{19}$$

One can check that in the isotropic case this equation reduces to the time-dependent London equation (2). Another limit to check is the static anisotropic London equations [1]. In this case we have

$$\rho_{sk}\epsilon_{uvs}\left(\epsilon_{kmn}\frac{\partial^2 H_n}{\partial x_v\partial x_m} + \lambda_{kl}^{-2}\frac{\partial A_l}{\partial x_v}\right) = 0. \tag{20}$$

Clearly, this equation is satisfied if

$$\epsilon_{kmn}\frac{\partial H_n}{\partial x_m} + \lambda_{kl}^{-2}A_l = 0. \tag{21}$$

We now introduce a tensor $(\lambda^2)_{kl}$ inverse to $(\lambda^{-2})_{kl}$, multiply the last equation by $(\lambda^2)_{k\mu}$, and sum up over k:

$$\lambda_{k\mu}^2\epsilon_{kmn}\frac{\partial H_n}{\partial x_m} + A_\mu = 0. \tag{22}$$

Finally, we apply to this $\epsilon_{uv\mu}\partial/\partial x_v$ to replace $\text{curl}\boldsymbol{A}$ with $\boldsymbol{H}$ and obtain static anisotropic London equations [1].

A. Orthorhombic slab with plane faces *ab*

The cumbersome Eq. (19) is applicable in a coordinate system (x, y, z) oriented arbitrarily relative to the anisotropic sample. One, of course, can choose (x, y, z) to be the crystal frame (a, b, c) where ρ_{sk} and λ_{kl}^{-2} are diagonal. Consider a slab of thickness d of orthorhombic material with a, b (or x, y) plane faces; z is counted from the slab middle. Let the ac applied field $\boldsymbol{H}_0$ be parallel to x; the field inside the slab depends only on z.

Consider the first term in Eq. (19). Since v and m take only z values and $n = x$, it is readily seen that this term reduces to

$$-\frac{c}{4\pi}\rho_{yy}\frac{\partial^2 H_x}{\partial z^2}. \tag{23}$$

The term on the right of Eq. (19) can be treated similarly to obtain $c\rho_{yy}\lambda_{yy}^{-2}\partial_z A_y$. Hence,

$$-\frac{\partial^2 H_x}{\partial z^2} + \lambda_{yy}^{-2} H_x + \frac{4\pi}{c^2\rho_{yy}}\frac{\partial H_x}{\partial t} = 0. \tag{24}$$

This equation is equivalent to the isotropic Eq. (3) with the same solution (9) for the slab, but now

$$k_x^2 = \lambda_{yy}^{-2} - \frac{2i}{\delta_{yy}^2}, \qquad \delta_{yy}^2 = \frac{c^2}{2\pi\sigma_{yy}\omega}. \tag{25}$$

As expected, the decaying behavior of H_x is determined by characteristics of persistent and normal currents in the y direction.

Thus, the isotropic result for the susceptibility is directly translated to this situation. In particular, if $\lambda_{yy} \ll \delta_{yy}$, one obtains for component χ_{xx} of the susceptibility tensor

$$\chi_{xx} + \frac{1}{4\pi} = \frac{\lambda_{yy}}{2\pi d} + i\frac{\lambda_{yy}^3}{2\pi d\delta_{yy}^2}. \tag{26}$$

If the applied field is directed along y, the same argument leads to

$$\chi_{yy} + \frac{1}{4\pi} = \frac{\lambda_{xx}}{2\pi d} + i\frac{\lambda_{xx}^3}{2\pi d\delta_{xx}^2}. \tag{27}$$

These formulas cannot be used too close to T_c, where inequalities $\lambda \ll \delta$ and $\lambda \ll d$ are violated. The time-dependent London equation (24) can be solved without these preconditions (see the Appendix).

It is worth noting that the anisotropy of the penetration depth is related to the anisotropy of susceptibility:

$$\gamma_\lambda = \frac{\lambda_{xx}}{\lambda_{yy}} \approx \frac{\text{Re}\,\chi_{yy} + 1/4\pi}{\text{Re}\,\chi_{xx} + 1/4\pi}. \tag{28}$$

Taking the ratio of imaginary parts, we obtain

$$\frac{\text{Im}\,\chi_{yy}}{\text{Im}\,\chi_{xx}} \approx \frac{\delta_{yy}^2}{\delta_{xx}^2}\frac{\lambda_{xx}^3}{\lambda_{yy}^3} = \gamma_\sigma\gamma_\lambda^3, \tag{29}$$

where $\gamma_\sigma = \sigma_{xx}/\sigma_{yy}$.

1. Angular dependence of susceptibility

Let the applied field at the sample surface be at an angle φ with the a axis, $\boldsymbol{H} = H_0(\hat{\boldsymbol{x}}\cos\varphi + \hat{\boldsymbol{y}}\sin\varphi)e^{-i\omega t}$. Since London and Maxwell equations are linear, the solution is the superposition of two solutions for applied fields oriented along the principle directions:

$$\boldsymbol{H} = H_0\left[\hat{\boldsymbol{x}}\frac{\cos\varphi\cosh(k_x z)}{\cosh(k_x d/2)} + \hat{\boldsymbol{y}}\frac{\sin\varphi\cosh(k_y z)}{\cosh(k_y d/2)}\right], \tag{30}$$

where the factor $e^{-i\omega t}$ is omitted for brevity. It is worth noting that since the decay lengths for the magnetic field along $\hat{\boldsymbol{x}}$ (on the order of $1/k_x$) differ from $1/k_y$, the field in the (ab) plane rotates with increasing depth z. In this situation, the magnetic moment $\boldsymbol{\mu}$ will have not only the component parallel to the applied field, $\mu_\parallel$, but a perpendicular component as well.

One has for the electric field

$$\boldsymbol{E} = \frac{i\omega H_0}{c}\left[\frac{\hat{\boldsymbol{x}}\sin\varphi\sinh(k_y z)}{k_y\cosh(k_y d/2)} - \frac{\hat{\boldsymbol{y}}\,cos\varphi\sinh(k_x z)}{k_x\cosh(k_x d/2)}\right]. \tag{31}$$

The commonly measured susceptibility is defined as

$$\chi_\parallel = \frac{\mu_\parallel}{H_0} = \frac{\mu_x\cos\varphi + \mu_y\sin\varphi}{H_0} = \chi_{xx}\cos^2\varphi + \chi_{yy}\sin^2\varphi, \tag{32}$$

where χ_{xx} and χ_{yy} are given in Eqs. (26) and (27). This gives

$$\text{Re}\,\chi_\parallel = -\frac{1}{4\pi} + \frac{1}{2\pi d}(\lambda_{yy}\cos^2\varphi + \lambda_{xx}\sin^2\varphi),$$
$$\text{Im}\,\chi_\parallel = \frac{1}{2\pi d}\left(\frac{\lambda_{yy}^3}{\delta_{yy}^2}\cos^2\varphi + \frac{\lambda_{xx}^3}{\delta_{xx}^2}\sin^2\varphi\right). \tag{33}$$

2. Dissipation and torque

Given the fields at the surface $z = \pm d/2$, one evaluates the Pointing vector $\boldsymbol{S}$, i.e., the energy flux into the sample and the dissipation power [7]. One obtains, after straightforward algebra,

$$\overline{S}_z = -\frac{c}{8\pi}\text{Re}(\boldsymbol{E}\times\boldsymbol{H}_0^*)_{z=d/2} = \frac{\omega H_0^2}{8\pi}\frac{\lambda_{xx}^3}{\delta_{xx}^2}\left[\sin^2\varphi + \left(\frac{\lambda_{yy}}{\lambda xx}\right)^3\left(\frac{\delta_{xx}}{\delta yy}\right)^2\cos\varphi^2\right]. \tag{34}$$

Here, $\overline{S}_z$ denotes the time average over the period $2\pi/\omega$. If the parameter

$$p = \left(\frac{\lambda_{yy}}{\lambda xx}\right)^3\left(\frac{\delta_{xx}}{\delta yy}\right)^2 > 1, \tag{35}$$

$\cos^2\varphi$ dominates, and the dissipation has a minimum at $\varphi = \pi/2$, i.e., for $\boldsymbol{H}_0$ directed along y. If $p < 1$, the dissipation is minimal for the field $\boldsymbol{H}_0$ directed along x. Since the system prefers the state with minimum dissipation, one expects a torque for $0 < \varphi < \pi/2$ that acts to rotate the sample to this state.

This conclusion is confirmed by calculating the torque $\boldsymbol{\tau}$ averaged over the ac period:

$$\overline{\tau_z} = \tfrac{1}{2}\text{Re}(\boldsymbol{\mu}\times\boldsymbol{H}_0^*) = \tfrac{1}{2}\text{Re}(\mu_x H_{0y}^* - \mu_y H_{0x}^*), \tag{36}$$

where $\mu_x = \chi_{xx}H_0\cos\varphi$ and $\mu_y = \chi_{yy}H_0\sin\varphi$. We obtain, with the help of Eqs. (26) and (27),

$$\overline{\tau_z} = \frac{H_0^2}{8\pi d}(\lambda_{yy} - \lambda_{xx})\sin 2\varphi. \tag{37}$$

III. DISCUSSION

Anisotropic London equations taking into account normal currents are derived and applied for the evaluation of the surface impedance and susceptibility χ for a simple geometry in which sample surfaces coincide with the ab planes of the orthorhombic crystal. In principle, applying the ac field along the a and b crystal axes, one can extract both χ_{aa} and χ_{bb} of the susceptibility tensor.

In the usual situation where the penetration depth is small relative to the skin depth, the deviation of the real part of the susceptibility from Meissner's $-1/4\pi$ depends only on λ, so that the ratio of these deviations for the two principle directions gives the anisotropy parameter $\gamma_\lambda = \lambda_{aa}/\lambda_{bb}$, Eq. (28). Hence, γ_λ can, in principle, be extracted from microwave susceptibility data. The behavior of γ_λ with temperature is of intense interest in studies of new materials, and it remains to be seen whether or not experimental complications related to the finite size of actual samples can be overcome [8].

Given the slab geometry we consider in this paper, thick anisotropic films seem to be the best to check our formulas. The strongly anisotropic properties of cuprates make them good candidates for such measurements. One can find plenty of information on these possibilities in Ref. [9].

There are many ways to measure surface impedance. Commonly, the sample is placed inside a coil (for radio-frequency measurements) or in a microwave cavity or on a stripline resonator (for microwave frequencies of 1–100 GHz). The input ac frequency is tuned close to a specific resonant mode for which directions and amplitudes of electric and magnetic fields at the sample location are known. The sample is positioned so that the fields and currents are along the principal axes or at a known angle. The real and imaginary parts of the measured signal allow the evaluation of the complex impedance. Contemporary resonators approach sensitivities of 0.1 part per billion, which translates to a better than angstrom resolution of the London penetration depth for millimieter-sized crystals.

While deep in the superconducting state the contribution of normal quasiparticles to susceptibility is much smaller than the Meissner contribution by a factor $\sim\lambda^2/\delta^2$ [see Eq. (14)], only the latter is frequency dependent via the skin depth $\delta(\omega)$. Therefore, one can measure the response as a function of ω and extract the ω-dependent part. In fact, Eq. (14) can be written as $\chi + 1/4\pi = A + iB\omega$ with ω-independent A, B. Therefore, the derivative of the response with respect to frequency will provide the imaginary part of χ.

Another quantity which can be extracted from the susceptibility data is the conductivity of normal excitations σ. It coincides with the normal state conductivity near T_c (for gapless superconductors for all temperatures). However, experimentally, little is known about this conductivity away from T_c. Still, this quantity is of interest, in particular, given recent theoretical work of Smith *et al.* stating that the conductivity can be strongly enhanced due to inelastic scattering [10]. The anisotropy of σ can, in principle, be extracted from the ratio of imaginary parts of susceptibility and the anisotropy of the penetration depth, Eq. (29).

It should be noted that we applied the general equation (19) to an infinite slab. In experiments, one deals with finite samples. In this case, magnetic susceptibility measured in the applied field along, say, the b axis in addition to λ_{xx} will also depend on λ_{zz}. What is worse, the sample shape will give an extra angular modulation when the angle φ of the applied field direction is swept. These and other difficulties which may arise in measurements of the susceptibility of anisotropic samples and possible ways to overcome them are discussed elsewhere [8].

To summarize, the anisotropic London equations taking into account the normal currents were derived and applied to the problem of the surface impedance and magnetic susceptibility in the Meissner state of anisotropic superconductors. It was shown that the anisotropy of the London penetration depth can be expressed in terms of the anisotropy of the susceptibility. It was also shown that the anisotropic sample in the ac field is subject to a torque, unless the field is directed along one of the crystal principle axes.

ACKNOWLEDGMENTS

This work was supported by the U.S. Department of Energy (DOE), Office of Science, Basic Energy Sciences, Materials Science and Engineering Division. Ames Laboratory is operated for the U.S. DOE by Iowa State University under Contract No. DE-AC02-07CH11358.

APPENDIX

The slab susceptibility can be obtained by solving the time-dependent London equation, Eq. (3), for any material parameters not restricted by $\lambda \ll \delta$. We show that in the isotropic case, the generalization to orthorhombic slab is straightforward. From

$$k = \frac{1}{\lambda}\sqrt{1 - i\eta}, \qquad \eta = \frac{2\lambda^2}{\delta^2,} \tag{A1}$$

one readily obtains

$$\mathrm{Re}\, k = \frac{(1+\sqrt{1+\eta^2})^{1/2}}{\sqrt{2}\lambda} = \frac{1}{\lambda}\cosh\frac{u}{2},$$
$$\mathrm{Im}\, k = -\frac{\eta}{\lambda\sqrt{2}(1+\sqrt{1+\eta^2})^{1/2}} = \frac{1}{\lambda}\sinh\frac{u}{2}, \tag{A2}$$

where we introduced a new variable via $\eta = \sinh u$. Using Eq. (12), we obtain

$$4\pi\chi + 1 = \frac{2\lambda/d}{\cosh(u/2) - i\sinh(u/2)} \times \tanh\frac{\cosh(u/2) - i\sinh(u/2)}{2\lambda/d}. \tag{A3}$$

This result holds for any relations between λ, δ, and d. In particular, for the standard situation $\lambda \ll d$ and $\lambda \ll \delta$ we have $u \approx 2\lambda^2/\delta^2 \ll 1$ and

$$4\pi\chi + 1 \approx \frac{2\lambda}{d}\frac{1}{1 - i\lambda^2/\delta^2}, \tag{A4}$$

which coincides with Eqs. (26) and (27).

If $\lambda \to \infty$ (normal metal), $u = \sinh^{-1}(2\lambda^2/\delta^2) \approx \ln(4\lambda^2/\delta^2)$, which leads to

$$\cosh\frac{u}{2} \approx \sinh\frac{u}{2} \approx \frac{\lambda}{\delta} \tag{A5}$$

and to

$$4\pi\chi + 1 = \frac{\delta}{d}(1+i). \tag{A6}$$

This holds indeed for the susceptibility of an isotropic slab of a normal metal, Eq. (15), provided $d/\delta \gg 1$ and tanh is very close to 1.

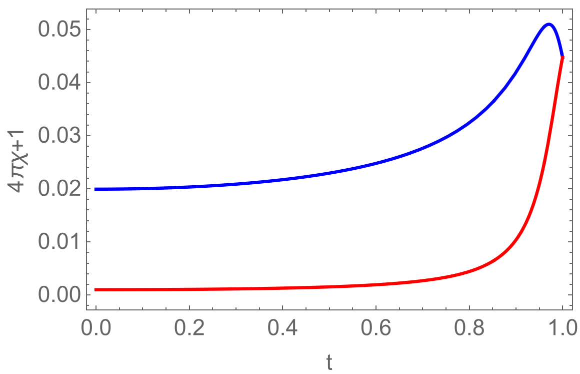

FIG. 1. Temperature dependences of the real (blue) and imaginary (red) parts of $4\pi\chi+1$ for $\eta_0=0.1$ and $r_0=0.02$ obtained numerically from Eq. (A3) assuming $\lambda=\lambda_0/\sqrt{1-t^2}$ for gapless materials.

One can proceed analytically evaluating the real and imaginary parts of χ of Eq (A3). But, eventually, we will have to resort to numerical evaluation anyway, so we can do this already at this point.

Since gapless materials are likely to have large λ, we use the temperature dependence $\lambda^2=\lambda_0^2/(1-t^2)$ with the reduced temperature $t=T/T_c$ [11,12]. Thus, the temperature enters via

$$\frac{2\lambda}{d}=\frac{r_0}{\sqrt{1-t^2}},\qquad u=\sinh^{-1}\frac{\eta_0}{1-t^2}. \tag{A7}$$

Hence, there are two dimensionless parameters, $r_0=2\lambda_0/d$ and $\eta_0=2\lambda_0^2/\delta^2$, that define the susceptibility of Eq. (A3). An example of numerical evaluation of real and imaginary parts of $\chi(T)$ is shown in Fig. 1. As is seen from Eq. (A6) for $d\gg\delta$, the curves must meet at $t=1$ where the slab becomes normal and the real and imaginary parts are equal. The ratio d/δ does not enter Eq. (A3) explicitly but can be expressed in terms of $\eta_0=2\lambda_0^2/\delta^2$ and $r_0=2\lambda_0/d$. Hence, for $\eta_0=0.1$ and $r_0=0.02$ chosen for this graph we have

$$\frac{\delta}{d}=\frac{r_0}{\sqrt{2\eta_0}}=0.0447. \tag{A8}$$

That is, in fact, the value of $4\pi\chi+1$ at $t=1$ shown in Fig. 1. It can be checked that the maximum position in $\mathrm{Re}\,\chi(t)$ corresponds to the temperature at which $\lambda(t)\approx\delta$, i.e., to $t=\sqrt{1-\eta_0/2}$.

[1] V. G. Kogan, Phys. Rev. B **24**, 1572 (1981).
[2] A. V. Balatskii, L. I. Burlachkov, and L. P. Gor'kov, Zh. Eksp. Teor. Fiz. **90**, 1478 (1986) [Sov. Phys. JETP **63**, 866 (1986)].
[3] J. R. Clem and M. W. Coffey, Phys. Rev. B. **46**, 14662 (1992).
[4] G. Blatter, M. V. Feigelman, V. B. Geshkenbein, A. I. Larkin, and V. M. Vinokur, Rev. Mod. Phys. **66**, 1125 (1994).
[5] V. G. Kogan, Phys. Rev. B **97**, 094510 (2018).
[6] V. G. Kogan and R. Prozorov, Phys. Rev. B **102**, 024506 (2020).
[7] L. D. Landau, E. M. Lifshitz, and L. P. Pitaevskii, *Elecrtodynamics of Continuous Media*, 2nd ed. (Elsevier, Amsterdam, 1984).
[8] Kyuil Cho, S. Teknowijoyo, E. Krenkel, M. A. Tanatar, N. D. Zhigadlo, V. G. Kogan, and R. Prozorov (unpublished).
[9] *YBCO and Related Systems, Their Coated Conductors, Thin Films, Vortex State and More on MgB_2*, edited by A. Narlikar, Studies of High Temperature Superconductors Vol. 41 (Nova Science Publishers, New York, 2003).
[10] M. Smith, A. V. Andreev, and B. Z. Spivak, Phys. Rev. B **101**, 134508 (2020).
[11] A. A. Abrikosov and L. P. Gor'kov, Zh. Eksp. Teor. Fiz. **39**, 1781 (1060) [Sov. Phys. JETP **12**, 1243 (1961)].
[12] V. G. Kogan, R. Prozorov, and C. Petrovic, J. Phys.: Condens. Matter **21**, 102204 (2009).