Exact local correlations in kicked chains

Boris Gutkin,[1] Petr Braun,[2] Maram Akila,[2,3] Daniel Waltner,[2] and Thomas Guhr[2]
[1]*Department of Applied Mathematics, Holon Institute of Technology, 58102 Holon, Israel*
[2]*Fakultät für Physik, Universität Duisburg-Essen, Lotharstraße 1, 47048 Duisburg, Germany*
[3]*Fraunhofer IAIS, Schloss Birlinghoven, 53757 Sankt Augustin, Germany*

(Received 27 May 2020; accepted 19 October 2020; published 11 November 2020)

We show that local correlators in a wide class of kicked chains can be calculated exactly at light-cone edges. Extending previous works on circuit lattice systems, the correlators between local operators are expressed through the expectation values of transfer matrices **T** with small dimensions. For dual-unitary kicked chains, with spatial-temporal symmetry of the dynamics, this provides a full characterization of local correlators. Furthermore, we identify a remarkable family of dual-unitary models where an explicit information on the spectrum of **T** is available. For this class of models we provide a closed analytical formula for the corresponding two-point correlators. The results are exemplified on the kicked Ising spin chain model.

DOI: 10.1103/PhysRevB.102.174307

I. INTRODUCTION

Spatially extended Hamiltonian systems with local interactions are paradigm systems in the field of many-body physics. On the experimental side, various aspects have become ever more amenable to direct measurement [1–4] while a recent burst of activities [5–10] greatly improved our theoretical understanding. In the context addressed here, the outstanding importance of these systems is rooted in their spatiotemporal correlation of local observables which describe, in an often generic manner, experimentally accessible features of interacting many-body systems such as spectral statistics or transport properties [2,11,12]. The wealth of available results, unfortunately, covers systems which are either dynamically too simple, such as free or integrable ones, or too low in dimension, such as cat or baker maps. It is thus of paramount interest to find representatives of those systems, on the one hand, capturing the full complexity and, on the other hand, allowing for analytical treatment.

In this work we consider a class of systems admitting a number of different dynamical descriptions [13]. The standard one corresponds to the system evolution with respect to time, induced by the system Hamiltonian. Alternatively, one can consider evolution along one of the spatial directions. In this *dual* approach the corresponding coordinate takes on the role of time. The resulting dynamical system is generically a non-Hamiltonian one [13–15]. However, in some special cases it might happen that the dual spatial evolution is a Hamiltonian one, as well. The representatives of such systems, referred to as *dual unitary*, can be found among coupled map lattices [16,17], kicked spin chains [18–21], circuit lattices [22–24], and continuous field theories [25].

Dual-unitary systems have recently attracted considerable attention [18–30] due to their intriguing properties. On the one hand, these models generically exhibit features of maximally chaotic many-body systems. In particular, their spectral statistics are well described by the Wigner-Dyson distribution. They are insusceptible to many-body localization effects even in the presence of strong disorder [18,20]. The entanglement has been shown to grow linearly with time and to saturate the maximum bound. On the other hand, dual-unitary models turned out to be amenable to exact analytical treatment. The growth of the entanglement entropy for kicked Ising spin chains (KICs) for certain types of initial states has been evaluated exactly in [19] and their entanglement spectrum was found to be trivial [23]. Furthermore, correlations of local operators in dual-unitary quantum circuit latices [22,31] can be expressed exactly in terms of small dimensional transfer operators.

So far, no full characterization of dual-unitary systems has been given. Although concrete examples of such models have been presented, there is no general prescription for their construction. In the present contribution we introduce a wide class of dual-unitary kicked chains (DUKCs) built upon a pair of $L \times L$ complex Hadamard matrices and study correlations between local operators. Importantly, these models are defined for arbitrary length of the chain, N, and the on-site Hilbert space dimension L. This allows, at least in principle, to look at both the thermodynamic limit, $N \to \infty$, and the semiclassical limit $L \to \infty$ (or combinations of them), which is important for quantum chaos studies. As shown in the body of the paper, the correlators of strictly local traceless operators vanish identically in DUKCs for sufficiently long chains. On the other hand, correlations between operators with finite support are, generically, nontrivial along the light-cone edges. In agreement with [22,31] such correlations can be expressed through the expectation values of a transfer matrix **T** whose dimension is determined by L rather than N.

In what follows, we identify within DUKCs a remarkable family of models, where explicit information on the spectrum of **T** is available. For this family of DUKCs we obtain a closed analytical formula for correlations between operators supported on two adjacent lattice sites.

 ©2020 American Physical Society

The paper is structured as follows. In Sec. II we describe the kicked chain (KC) model and establish conditions for dual unitarity. In Sec. III we obtain the formula for correlators between local operators in KCs along light-cone edges. They are expressed through the expectation values of transfer matrices $\mathbf{T}$ with small dimensions. In Sec. IV we introduce a special family of dual-unitary KCs, where exact information on the spectrum of $\mathbf{T}$ is available and correlators are evaluated explicitly along light-cone edges. In Sec. V the results are exemplified on the kicked Ising spin chain model. Finally, Sec. VI gives a summary of the results and an outlook for future work.

II. KICKED CHAINS

In this paper we consider cyclic chains of N locally interacting particles, periodically kicked with an on-site external potential. The system is governed by the Hamiltonian

$$H(t) = H_I + H_K \sum_{m=-\infty}^{+\infty} \delta(t-m), \tag{1}$$

with H_I and H_K being the interaction and kick parts, respectively. The corresponding Floquet time evolution is the product of the operators, $U_I = e^{-iH_I}$ and $U_K = e^{-iH_K}$, acting on the Hilbert space $\mathcal{H}^{\otimes N}$ of the dimension L^N, where $\mathcal{H} = \mathbb{C}^L$ is the local Hilbert space equipped with the basis $\{|s\rangle, s = 1, \ldots, L\}$. We require that H_I couples nearest-neighbor sites of the chain taking on a diagonal form in the product basis, $\{|\boldsymbol{s}\rangle = |s_1\rangle|s_2\rangle \cdots |s_N\rangle\}$. The respective evolution is fixed by a real function f_1,

$$\langle \boldsymbol{s}|U_I[f_1]|\boldsymbol{s}'\rangle = \delta(\boldsymbol{s},\boldsymbol{s}')e^{i\sum_{n=1}^{N} f_1(s_n,s_{n+1})}, \tag{2}$$

with $\delta(\boldsymbol{s},\boldsymbol{s}') = \prod_{i=1}^{N}\delta(s_i - s_i')$, and cyclic boundary condition $s_{N+1} \equiv s_1$. The second, kick part, is given by the tensor product

$$U_K[f_2] = \bigotimes_{i=1}^{N} u_2,\ \langle \boldsymbol{s}|U_K[f_2]|\boldsymbol{s}'\rangle = \prod_{i=1}^{N}\langle s_i|u_2|s_i'\rangle, \tag{3}$$

where u_2 is an $L \times L$ unitary matrix with the elements $e^{if_2(n,m)}/\sqrt{L}$ determined by a complex function f_2. Combining the two parts together we obtain the quantum evolution

$$U = U_I[f_1]U_K[f_2], \tag{4}$$

acting on the Hilbert space of dimension L^N.

In the same way, one constructs the dual evolution acting on the Hilbert space of dimension L^T by exchanging $N \leftrightarrow T$ and $f_1 \leftrightarrow f_2$:

$$\tilde{U} = U_I[f_2]U_K[f_1]. \tag{5}$$

The following remarkable duality relation [14,15] holds between their traces for any integers T and N:

$$\mathrm{Tr}\, U^T = \mathrm{Tr}\, \tilde{U}^N. \tag{6}$$

In contrast to the original evolution, $\tilde{U}$ is a nonunitary operator, in general. However, if

$$\langle n|u_1|m\rangle = \frac{e^{if_1(n,m)}}{\sqrt{L}}, \quad \langle n|u_2|m\rangle = \frac{e^{if_2(n,m)}}{\sqrt{L}}, \tag{7}$$

are $L \times L$ complex Hadamard matrices (i.e., unitary matrices for which matrix elements have the same absolute value), the dual operator, $\tilde{U}$, is unitary as well. We refer to such models as dual unitary. Note that in the dual-unitary case both f_1 and f_2 are real.

It is a natural question to ask how wide the class of DUKC models is. Each dual model is essentially built upon a pair of complex Hadamard matrices, u_1 and u_2 (up to the $1/\sqrt{L}$ factor). A generic family of complex Hadamard matrices can be constructed for each L by taking the unitary discrete Fourier transform (DFT) and multiplying it on both sides by diagonal unitary and permutation matrices. This exhausts all possible cases for $L = 2, 3, 5$. For a general L the classification of complex Hadamard matrices is an open problem [32]. It is worth noting that for each DUKC there exists an associated dual-unitary circuit lattice. The opposite, however, is not necessarily true. The precise connection between the two classes of models is discussed in Appendix A.

III. CORRELATIONS BETWEEN LOCAL OPERATORS

In what follows we consider correlations between local operators supported on pairs of adjacent sites of KCs. Specifically, let $(\mathbf{q}_1, \mathbf{q}_2)$ and $(\mathbf{q}_3, \mathbf{q}_4)$ be two pairs of matrices acting on the on-site Hilbert space $\mathcal{H}$. We define the corresponding many-body operators

$$\Sigma_{n_1} = \underbrace{\mathbb{1}\otimes\cdots\otimes\mathbb{1}}_{n_1-1} \otimes \mathbf{q}_1 \otimes \mathbf{q}_2 \otimes \underbrace{\mathbb{1}\otimes\cdots\otimes\mathbb{1}}_{N-n_1-1}, \tag{8}$$

$$\Sigma_{n_2} = \underbrace{\mathbb{1}\otimes\cdots\otimes\mathbb{1}}_{n_2-1} \otimes \mathbf{q}_3 \otimes \mathbf{q}_4 \otimes \underbrace{\mathbb{1}\otimes\cdots\otimes\mathbb{1}}_{N-n_2-1}, \tag{9}$$

supported at the sites n_1, n_1+1 and n_2, n_2+1 of the chain, respectively. The associated two-point correlator is given by

$$C(n,t) = L^{-N}\,\mathrm{Tr}\, U^t \Sigma_{n_1} U^{-t} \Sigma_{n_2}, \tag{10}$$

where we assume $n = n_2 - n_1 > 0$, $t > 0$. By translation symmetry of the model, we can set $n_1 = 1$ and $n_2 = n+1$ without loss of generality. Note that correlations between operators supported on single chain sites (referred to by us as *strictly local* operators) can be obtained from Eq. (10) by fixing $\mathbf{q}_1 = \mathbf{q}_4 = \mathbb{1}$.

A. Partition function representation

By inserting identities for the different times k,

$$C(n,t) = \frac{1}{L^N}\langle \boldsymbol{s}_{2t}|\Sigma_{n_1}|\boldsymbol{s}_1\rangle\langle \boldsymbol{s}_t|\Sigma_{n_2}|\boldsymbol{s}_{t+1}\rangle \times \prod_{k=1}^{t}\langle \boldsymbol{s}_k|U^\dagger|\boldsymbol{s}_{k+1}\rangle \prod_{k=t+1}^{2t}\langle \boldsymbol{s}_k|U|\boldsymbol{s}_{k+1}\rangle, \tag{11}$$

$|\boldsymbol{s}_k\rangle = |s_{1,k}\rangle\cdots|s_{N,k}\rangle$, $|\boldsymbol{s}_1\rangle \equiv |\boldsymbol{s}_{2t+1}\rangle$, the correlator (10) can be written in the form of the two-dimensional (2D) partition function,

$$C(n,t) = \frac{1}{L^{Nt}} \sum_{\{s_{mk}|(m,k)\in\mathcal{L}_1\}} e^{-i\mathcal{F}(\{s_{mk}\})} \times \left[\prod_{(m,k)\in\mathcal{L}_2}\delta(s_{mk}, s_{m,2t-k+1})\right] D\big(s_{n_1 1}, \ldots, s_{n_2 t}\big), \tag{12}$$

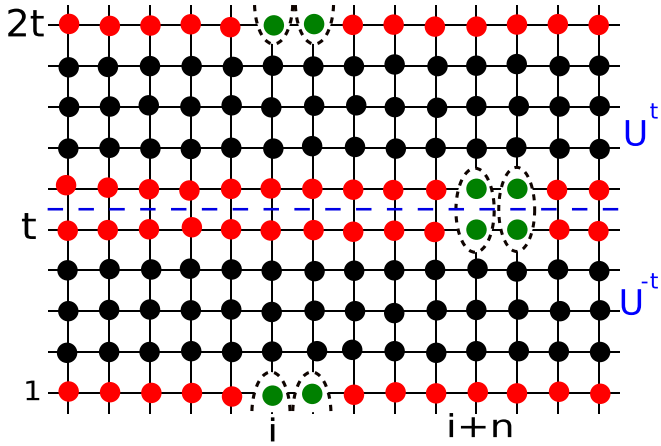

FIG. 1. The initial expression (12), where the sum runs over $N \times 2t$ variables $s_{m,k}$. Circles in red show (m,k) sites, where variables are paired by the condition $s_{m,k} = s_{m,2t-k+1}$. The green circles correspond to the location of the operators Σ_i, $U^{-t}\Sigma_{n+i}U^t$.

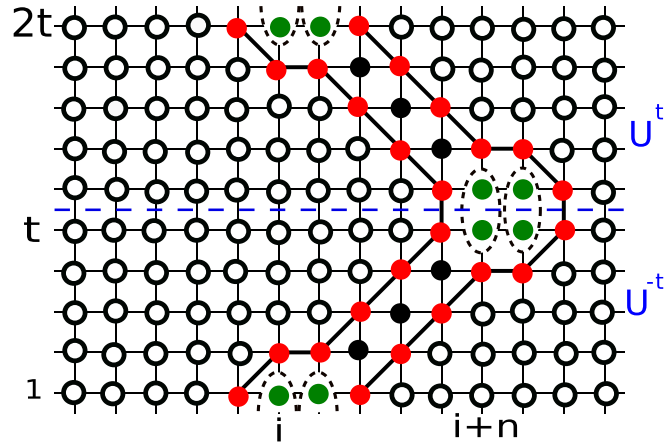

FIG. 2. Elimination of the summation variables in the partition function (12) representing the four-point correlator (10). The eliminated sites are shown by empty black circles. The remaining sum along the light-cone edge can be represented in the form of the expectation value (15) of the $L^2 \times L^2$ transfer operator $\mathbf{T}$.

where the last factor,

$$D = \langle s_{n_1,2t}|\mathbf{q}_1^c|s_{n_1,1}\rangle\langle s_{n_1,2t}|\mathbf{q}_2^c|s_{n_1,1}\rangle \times \langle s_{n_2,t}|\mathbf{q}_3|s_{n_2,t+1}\rangle\langle s_{n_2,t}|\mathbf{q}_4|s_{n_2,t+1}\rangle, \quad (13)$$

$\mathbf{q}_1^c = u_2\mathbf{q}_1u_2^\dagger$, $\mathbf{q}_2^c = u_2\mathbf{q}_2u_2^\dagger$, depends on the eight lattice sites, $\mathcal{L}_0 = \{(n_1,k),(n_1+1,k)|k=1,2t\} \cup \{(n_2,k),(n_2+1,k)|k=t,t+1\}$ corresponding to the location of the observables and the function $\mathcal{F}(\{s_{m,k}\})$ is given by

$$\mathcal{F} = \sum_{k=1}^{t}\sum_{m=1}^{N} f_1(s_{m,k}, s_{m+1,k}) - f_1(s_{m,k+t}, s_{m+1,k+t}) + f_2(s_{m+1,k}, s_{m,k}) - f_2(s_{m,k+t}, s_{m+1,k+t}). \quad (14)$$

The sum in Eq. (12) runs over $2t \times N$ sites of the lattice $\mathcal{L}_1 = \{(m,k)|k=1,\ldots,2t, m=1,\ldots,N\}$ while the product in Eq. (12) is, furthermore, restricted to the subset $\mathcal{L}_2 = \{(m,k)|k=1,t,t+1,2t,m=1,\ldots,N\} \setminus \mathcal{L}_0$.

B. Correlator evaluation

We consider now the correlator (10) at the light-cone border $n=t$. The initial expression (12) is illustrated in graphic form in Fig. 1. Here, the green and red circles mark the positions of points on a 2D lattice from the $\mathcal{L}_0$ and $\mathcal{L}_2$ sets, respectively. As we show in Appendix B, the summation variables $s_{m,k}$ can be excluded one by one by applying the contraction rules, depicted in Fig. 5. For $N > 2t$ the elimination of $s_{m,k}$ variables can be continued up to reaching the stage illustrated by Fig. 2, where the empty circles represent eliminated variables. The remaining summation variables (shown in red and black) are located along the one-dimensional strip only, which reduces the whole problem to calculation of a quasi-one-dimensional partition function. The resulting expression can be represented in the form of the expectation value

$$C_t \equiv C(t,t) = \langle\bar{\Phi}_{\mathbf{q}_1\mathbf{q}_2}|\mathbf{T}^{t-2}|\Phi_{\mathbf{q}_3\mathbf{q}_4}\rangle, \quad (15)$$

of the transfer operator $\mathbf{T}$,

$$\langle\nu\eta|\mathbf{T}|\eta'\nu'\rangle = \frac{1}{L^3}\left|\sum_{s=1}^{L} e^{i(f_1(\eta,s)+f_1(s,\nu')+f_2(\nu,s)+f_2(s,\eta'))}\right|^2, \quad (16)$$

acting on the small space $\mathcal{H}\otimes\mathcal{H}$. The left $\bar{\Phi}_{\mathbf{q}_1\mathbf{q}_2}$ and the right $\Phi_{\mathbf{q}_3\mathbf{q}_4}$ vectors are defined as

$$\langle\nu\eta|\Phi_{\mathbf{q}_3\mathbf{q}_4}\rangle = \frac{1}{L^3}\sum_{a,\bar{a},b=1}^{L}\Gamma^b_{a\bar{a}}\langle a|\mathbf{q}_3|\bar{a}\rangle\langle b|\mathbf{q}_4|b\rangle, \quad (17)$$

$$\langle\bar{\Phi}_{\mathbf{q}_1\mathbf{q}_2}|\eta\nu\rangle = \frac{1}{L^3}\sum_{a,\bar{a},b=1}^{L}\bar{\Gamma}^b_{a\bar{a}}\langle a|\mathbf{q}_2^c|\bar{a}\rangle\langle b|\mathbf{q}_1^c|b\rangle, \quad (18)$$

where

$$\Gamma^b_{a\bar{a}} = e^{i(f_1(\eta,\bar{a})-f_1(\eta,a)+f_2(\bar{a},\nu)-f_2^*(a,\nu)-f_1(a,b)+f_1(\bar{a},b))},$$
$$\bar{\Gamma}^b_{a\bar{a}} = e^{i(f_1(a,\nu)-f_1(\bar{a},\nu)+f_2(\eta,a)-f_2^*(\eta,\bar{a})+f_1(b,a)-f_1(b,\bar{a}))}.$$

It is easy to check that $\mathbf{T}$ is doubly stochastic; i.e., it satisfies

$$\sum_{\nu=1}^{L}\sum_{\eta=1}^{L}\langle\nu\eta|\mathbf{T}|\eta'\nu'\rangle = \sum_{\nu'=1}^{L}\sum_{\eta'=1}^{L}\langle\nu\eta|\mathbf{T}|\eta'\nu'\rangle = 1.$$

This implies that the spectrum of $\mathbf{T}$ is contained within the unit disk with the largest eigenvalue, $\mu_1 = 1$. The left (right) eigenvector corresponding to μ_1 is given by the choice $\mathbf{q}_3 = \mathbf{q}_4 = \mathbb{1}$ ($\mathbf{q}_1 = \mathbf{q}_2 = \mathbb{1}$). For typical system parameters the correlators between traceless observables decay exponentially with the rates determined by the second eigenvalue μ_2, $|\mu_2| \leqslant |\mu_1|$, of $\mathbf{T}$ having the largest absolute value after μ_1. Equation (15) can be also used to evaluate correlations between strictly local observables in KCs by setting $\mathbf{q}_1 = \mathbb{1}$ and $\mathbf{q}_4 = \mathbb{1}$. In a DUKC $\bar{\Phi}_{\mathbb{1}\mathbf{q}_2} = \Phi_{\mathbf{q}_3\mathbb{1}} = 0$ for traceless $\mathbf{q}_2$ and $\mathbf{q}_3$, implying vanishing correlations between strictly local operators. However, in a general KC these vectors do not vanish, generically leading to the nontrivial correlator C_t.

It is important to emphasize that Eq. (15) holds for any KC model (1) and does not require dual unitarity. (For similar observation on unitary circular lattices, see [31].) In essence, any KC of this type is solvable, insofar as local correlators are restricted to the light-cone edge. What makes the dual-unitary case special is that $C(n,t)$ is zero there for traceless $\mathbf{q}_i$'s if $n \neq t$ and $N > 2t$. As has been pointed out in [22], this can be understood in a simple intuitive way. Since the speed of information propagation in KCs (1) equals 1, the correlator of operators (8) and (9) with traceless $\mathbf{q}_i$'s must vanish outside of the light cone $|t| < |n|$, $n = n_2 - n_1$. By dual unitarity, a similar result holds for points within the light cone $|t| > |n|$, as well. This leaves the light-cone edges $|t| = |n|$ as the only possible places on the space-time lattice where nontrivial correlations might arise. Accordingly, for dual-unitary models we have

$$C(n,t) = \delta(n,t)C_t, \tag{19}$$

where C_t is given by Eq. (15).

IV. FULLY SOLVABLE MODEL

We recall that a DUKC is fully determined by the pair of complex Hadamard matrices, u_1 and u_2. The most straightforward way to realize a DUKC is to set $u_1 = \Lambda_1 F \Lambda_1'$ and $u_2 = \Lambda_2 F \Lambda_2'$, where F is an $L \times L$ unitary DFT and Λ_1, Λ_1', Λ_2, and Λ_2' are arbitrary unitary diagonal matrices with the elements $e^{i\lambda_1(m)}$, $e^{i\lambda_1'(m)}$, $e^{i\lambda_2(m)}$, and $e^{i\lambda_2'(m)}$, $m = 1, 2, \ldots, L$. In such a case we have

$$f_1(m,n) = -\frac{2\pi(m-1)(n-1)}{L} + \lambda_1(m) + \lambda_1'(n),$$
$$f_2(m,n) = -\frac{2\pi(m-1)(n-1)}{L} + \lambda_2(m) + \lambda_2'(n).$$

In what follows we refer to such models as Fourier transform chains (FTCs).

A. Eigenvalues

By Eq. (16) the elements of the transfer operator in the FTC take the form

$$\langle mn|\mathbf{T}|n'm'\rangle = \frac{1}{L^3}\left|\sum_{s=0}^{L-1} e^{\frac{2\pi i(m+n+m'+n'-4)s}{L} - i\mu(s+1)}\right|^2,$$

where $\mu(s) = \lambda_1(s) + \lambda_1'(s) + \lambda_2(s) + \lambda_2'(s)$. Since the matrix elements depend only on the combination $m+n+m'+n'$, $\mathbf{T}$ can be diagonalized by using $F \otimes F$ unitary transformation. The resulting spectrum of $\mathbf{T}$ is composed of L nontrivial eigenvalues supplemented by $L(L-1)$ eigenvalues equal to zero. Explicitly, the nontrivial part of the $\mathbf{T}$ spectrum is given by $\lfloor \frac{L-1}{2} \rfloor$ pairs of the eigenvalues $\mathbf{t}_m = -\mathbf{t}_{L-m} = |d_m|$, $m = 1, 2, \ldots, \lfloor \frac{L-1}{2} \rfloor$, with

$$d_m = \frac{1}{L}\sum_{s=0}^{L-1} e^{i\mu(1+s) - i\mu(1+(s+m) \bmod L)}, \tag{20}$$

and either one additional unpaired eigenvalue, $\mathbf{t}_0 = 1$, for odd L, or the two unpaired eigenvalues equal to $\mathbf{t}_0 = 1$, $\mathbf{t}_{L/2} = d_{L/2}$, for even L.

It is worth noting that for any quasiperiodic μ, such that

$$\mu(s+k) - \mu(s) \in 2\pi\mathbb{Z}, \quad s = 0, \ldots, L-1,$$

for some $m \neq 0$, the corresponding eigenvalues $\mathbf{t}_m$ and $\mathbf{t}_{L-m}$ have absolute values equal to 1. This in turn implies the existence of local operators with nondecaying two-point correlations. On the other hand, for a nonquasiperiodic μ all subleading eigenvalues satisfy $|\mathbf{t}_m| < 1$, $m \neq 0$. As a result, any correlator (15) between traceless operators in the corresponding FTC decays exponentially.

B. Eigenvectors

To construct the eigenvectors of $\mathbf{T}$ note that $\Phi_{\mathbf{ab}}$ and $\bar{\Phi}_{\mathbf{ab}}$ vectors are fixed by the choice of the local operator $\mathbf{a}$, and the diagonal part of $\mathbf{b}$ [see Eqs. (17) and (18)]. Given an integer m let $\mathbf{e}_m$ be the diagonal matrix with the elements

$$\langle s|\mathbf{e}_m|s'\rangle = \delta(s,s')e^{-i2\pi sm/L}, \quad s', s \in \{1, \ldots, L\}.$$

It is straightforward to see that for an arbitrary $\mathbf{a}$ and $\mathbf{b} = \mathbf{e}_m$ the corresponding vector $\Phi_{\mathbf{ae}_m}$ is an eigenvector of $\mathbf{T}^2$ with the eigenvalue $|d_m|^2$. The eigenvectors of $\mathbf{T}$ are, therefore, symmetric and antisymmetric combinations of $\Phi_{\mathbf{ae}_m}$ and $\Phi^*_{\mathbf{ae}_m}$ for $m = 0, 1, 2, \ldots, \lfloor L/2 \rfloor$:

$$\left|\Phi^s_{\mathbf{a},m}\right\rangle = e^{-i\phi_m/2}\left|\Phi_{\mathbf{ae}_m}\right\rangle + e^{i\phi_m/2}\left|\Phi^*_{\mathbf{ae}_m}\right\rangle,$$
$$\left|\Phi^a_{\mathbf{a},m}\right\rangle = e^{-i\phi_m/2}\left|\Phi_{\mathbf{ae}_m}\right\rangle - e^{i\phi_m/2}\left|\Phi^*_{\mathbf{ae}_m}\right\rangle, \tag{21}$$

$e^{i\phi_m} = \frac{d_m}{|d_m|}$. They correspond to the eigenvalues $\mathbf{t}_m$ and $\mathbf{t}_{L-m}$, respectively. Note that for $m = 0$ and $m = L/2$ (for even L) only the symmetric eigenvector exists.

C. Correlators

To obtain an explicit form of the correlator (15) we decompose the vectors $|\Phi_{\mathbf{q}_3\mathbf{q}_4}\rangle$ in the basis of the eigenstates. After application of $\mathbf{T}^{t-2}$ operators this yields (see Appendix C for more details)

$$C_t = \sum_{m=0}^{\lfloor L/2 \rfloor} (\mathbf{t}_m)^{t-2}\left(2 - \delta_{m,0} - \delta_{m,\frac{L}{2}}\right)\mathcal{C}_m, \tag{22}$$

where the coefficients $\mathcal{C}_m$ factorize in the products of four factors:

$$\mathcal{C}_m = \mathrm{Re}\left[e^{-i\phi}A^*_m(\mathbf{q}_4)A_m(\mathbf{q}_1^c)B^{(1)}_m(\mathbf{q}_3)B^{(2)}_m(\mathbf{q}_2^c)\right],$$
$$\mathcal{C}_m = \mathrm{Re}\left[A_m(\mathbf{q}_4)A_m(\mathbf{q}_1^c)B^{(1)}_m(\mathbf{q}_3)B^{(2)}_m(\mathbf{q}_2^c)\right], \tag{23}$$

for odd and even t, respectively. Here $A_m(\mathbf{q})$ are defined as DFTs of the diagonal elements of $\mathbf{q}$:

$$A_m(\mathbf{q}) = \frac{1}{L}\sum_{s=1}^{L} e^{i2\pi sm/L}\langle s|\mathbf{q}|s\rangle.$$

For the remaining factors one has

$$B^{(j)}_m(\mathbf{q}) = \frac{1}{L}\sum_{s=1}^{L} e^{i(\mu_j(s) - \mu_j(s^{(m)}))}\langle s|\mathbf{q}|s^{(m)}\rangle,$$

where $s^{(m)} = 1 + (s+m-1) \bmod L$, $\mu_1(s) = -\lambda_1(s) - \lambda_1'(s) - \lambda_2(s)$, and $\mu_2(s) = \lambda_1(s) + \lambda_1'(s) + \lambda_2'(s)$, respectively. For any real observable $\mathbf{q}$ the relations

$A_m(\mathbf{q}) = A^*_{L-m}(\mathbf{q})$ and $B^{(j)}_m(\mathbf{q}) = (B^{(j)}_m(\mathbf{q}))^*$, $j = 1, 2$, hold for all m. Furthermore, for traceless $\mathbf{q}$ all factors vanish at $m = 0$.

The above results can be straightforwardly extended to systems with spatial-temporal disorder, where the local functions f_1 and f_2 depend on the lattice sites. In such a case the transfer operator $\mathbf{T}^{t-2}$ in Eq. (15) is substituted with the product of local "gate" operators $\mathbf{T}_1\mathbf{T}_2\cdots\mathbf{T}_{t-2}$, where each $\mathbf{T}_i$ is determined by the functions f_1 and f_2 at the point (i, i) of the spatial-temporal lattice. For FTCs all matrices $\mathbf{T}_i$ are diagonalized by one and the same unitary transformation. As a result, the decay exponents of the correlators (10) in the disordered case are just given by the averages of the local exponents.

V. KIC MODEL

Below we illustrate our results on the example of the KIC model providing a minimal, $L = 2$, realization of the KC (1) and FTC model in the dual-unitary regime. The KIC evolution is governed by the Hamiltonians

$$H_I = \sum_{n=1}^{N} J\hat{\sigma}^z_n\hat{\sigma}^z_{n+1} + h\hat{\sigma}^z_n, \quad H_K = b\sum_{n=1}^{N}\hat{\sigma}^x_n, \tag{24}$$

$$\hat{\sigma}^\alpha_n = \underbrace{\mathbb{1}\otimes\cdots\otimes\mathbb{1}}_{n-1}\otimes\sigma^\alpha\otimes\underbrace{\mathbb{1}\otimes\cdots\otimes\mathbb{1}}_{N-n},$$

where $\hat{\sigma}^\alpha_1 = \hat{\sigma}^\alpha_{N+1}$ and σ^α, $\alpha = x, y, z$, are Pauli matrices.

A. Local operator correlators

Here we consider the correlator (10) with

$$\Sigma_{n_1} = \hat{\sigma}^\alpha_{n_1}\hat{\sigma}^\beta_{n_1+1}, \quad \Sigma_{n_2} = \hat{\sigma}^\gamma_{n_2}\hat{\sigma}^\delta_{n_2+1}.$$

For the sake of simplicity of exposition the parameters are set: $b = \pi/4$ and h and J are arbitrary. For this choice of parameters, Eq. (15) gives (see Appendix D) at $n = t$ and $N > 2t$

$$C_t = \mathcal{C}^{\gamma\delta}_{\alpha\beta}(\sin^2 2J\cos 2h)^t, \tag{25}$$

where the prefactors $\mathcal{C}^{\gamma\delta}_{\alpha\beta}$ depend on the operators $\mathbf{q}_1 = \sigma^\alpha$, $\mathbf{q}_2 = \sigma^\beta$, $\mathbf{q}_3 = \sigma^\gamma$, and $\mathbf{q}_4 = \sigma^\delta$. Specifically, $\mathcal{C}^{yz}_{yz} = 1$, $\mathcal{C}^{xz}_{yx} = \tan^2 2h$, $\mathcal{C}^{yz}_{yx} = \mathcal{C}^{xz}_{yz} = -\tan 2h$, and $\mathcal{C}^{\gamma\delta}_{\alpha\beta}$ are zero for all other spin combinations.

B. Strictly local operator correlators

As has been explained above, in the dual-unitary case all two-point correlators

$$C^{\alpha\beta}(n, t) = \frac{1}{2^N}\,\mathrm{Tr}\left(U^{-t}\hat{\sigma}^\alpha_{n+1}U^t\hat{\sigma}^\beta_1\right), \tag{26}$$

$\alpha, \beta \in \{x, y, z\}$ between local spin operators vanish identically for $t > 0$, $N > 2t$. For a general KIC, away from the self-dual regime, the correlators (26) are nonzero, in general, and can be evaluated at $n = t - 1$, $N > 2t$, by using Eq. (15). To this end we set $\mathbf{q}_1 = \mathbb{1}$, $\mathbf{q}_4 = \mathbb{1}$ and $\mathbf{q}_2 = \sigma^\alpha$, $\mathbf{q}_3 = \sigma^\beta$, which yields for $t > 1$

$$C^{\alpha\beta}(t-1, t) = \left\langle\bar{\Phi}_{\mathbb{1}\mathbf{q}_2}\middle|\mathbf{T}^{t-2}\middle|\Phi_{\mathbf{q}_3\mathbb{1}}\right\rangle. \tag{27}$$

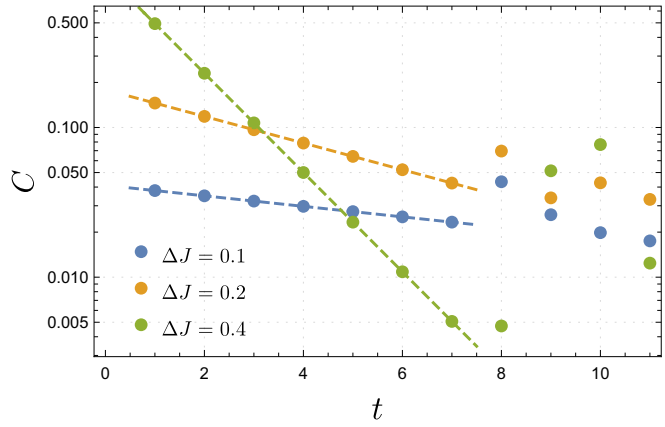

FIG. 3. Time dependence of $\frac{1}{2^N}\,\mathrm{Tr}\,(U^{-t}\sigma^x_t U^t\ \sigma^x_1)$ for $N = 14$ spins with generic values of J and h and $b = \pi/4$. Straight lines are determined by Eq. (28) with $h = 3.0$ and $\Delta J = \pi/4 - J = 0.1$, 0.2, and 0.4. The dots are obtained by direct numerical calculation of correlators for the same system parameters. Note perfect agreement with the analytic predictions for $2t \leqslant N$. For $2t > N$ Eq. (28) is no longer valid which can be clearly observed at the plot.

For $b = \pi/4$ and general J a straightforward evaluation of Eq. (27) leads to

$$C^{\alpha\beta}(t-1, t) = \mathcal{C}^{\alpha\beta}(\cos 2h\sin^2 2J)^t\cot^2 2J \tag{28}$$

with the coefficients given by

$$\mathcal{C}^{xx} = 1, \quad \mathcal{C}^{xy} = \mathcal{C}^{zx} = \tan 2h, \quad \mathcal{C}^{zy} = \tan^2 2h,$$

and by zeros for other α, β pairs. Note that for all $n \geqslant t$ the correlator $C^{\alpha\beta}(n, t)$ vanishes. For $n = t$ this result can be obtained by the substitution $\mathbf{q}_2 = \mathbb{1}$, $\mathbf{q}_4 = \mathbb{1}$ and $\mathbf{q}_1 = \sigma^\alpha$, $\mathbf{q}_3 = \sigma^\beta$ into Eq. (15). Since $\bar{\Phi}_{\mathbf{q}_1\mathbb{1}} = 0$, one gets immediately $C^{\alpha\beta}(t, t) = 0$. For a larger $n > t$, the same answer follows straightforwardly from the fact that the speed of information propagation in KIC is 1.

The correlators (25) and (28) decay exponentially with the rates $\cos 2h\sin^2 2J$ (see Fig. 3). Exceptions are the cases where $\frac{2J}{\pi} - \frac{1}{2} \in \mathbb{Z}$ and $\frac{2h}{\pi} \in \mathbb{Z}$, which correspond to well-known cases of the integrable classical 2D Ising spin model with complex parameters [33–35].

C. Dual-unitary KIC

The dual-unitary KIC is obtained when $J = b = \pi/4$. By Eqs. (19) and (25) we immediately have

$$C(n, t) = \delta(n, t)\mathcal{C}^{\gamma\delta}_{\alpha\beta}(\cos 2h)^t, \tag{29}$$

$n = n_2 - n_1 < N/2$. As we show in Appendix D, this result has a simple extension to the case of disordered chains,

$$C(n, t) = \delta(n, t)\,\mathcal{C}^{\delta\gamma}_{\alpha\beta}\prod_{i=n_1+1}^{n_2}\cos 2h_i, \tag{30}$$

where h_i is the value of the local magnetic field at the ith site of the chain. The prefactors $\mathcal{C}^{\delta\gamma}_{\alpha\beta}$ depend on the operators Σ_{n_1} and Σ_{n_2}. Specifically, $\mathcal{C}^{yz}_{yz} = 1$, $\mathcal{C}^{xz}_{yx} = \tan 2h_{n_1+1}\tan 2h_{n_2}$, $\mathcal{C}^{yz}_{yx} = -\tan 2h_{n_1+1}$, $\mathcal{C}^{xz}_{yz} = -\tan 2h_{n_2}$ and $\mathcal{C}^{\gamma\delta}_{\alpha\beta}$ are zero for all other spin combinations.

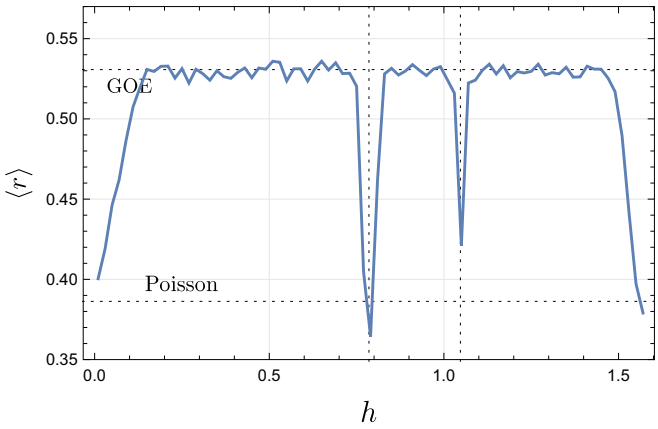

FIG. 4. The value of r for the spectrum of the dual-unitary KIC at $N = 15$ as a function of h averaged over all values in the separate (translation) symmetry sectors of $k = 1, \ldots, 7$. Note that the graph is symmetric under reflection $h \to \pi - h$. For most values of h the data for $\langle r \rangle$ fit the Gaussian orthogonal ensemble (GOE) prediction. The four dips at $h = 0, \pi/4, \pi/3$, and π correspond to spectral statistics characteristic of integrable systems.

The correlator (29) decays exponentially for any value of h except for the set of integrable points $h = \frac{1}{4}\pi k$, $k \in \mathbb{Z}$, where the subleading eigenvalue of $\mathbf{T}$ has absolute value 1. It is interesting to compare this behavior of correlators with the KIC spectral statistics. By the translation symmetry, the spectrum of the KIC evolution operator can be split into N uncorrelated subspectra $\{e^{i\theta_n^{(k)}}\}$, $k = 1, 2, \ldots, N$ [13]. In Fig. 4 we show the averaged ratio between three successive eigenphases from the same sector,

$$r = \frac{\min\left\{\theta_n^{(k)} - \theta_{n-1}^{(k)}, \theta_{n+1}^{(k)} - \theta_n^{(k)}\right\}}{\max\left\{\theta_n^{(k)} - \theta_{n-1}^{(k)}, \theta_{n+1}^{(k)} - \theta_n^{(k)}\right\}},$$

which is a well-established diagnostic for quantum chaos (see Refs. [36,37]). As can be seen in Fig. 4, for a generic value of h the disymmetrized spectrum of the dual-unitary KIC corresponds to a fully chaotic system. This is in agreement with the exponential decay of the correlator (25) on the light-cone border. There are, however, four special points on the h axis, $h = 0$, $h = \pi$, $h = \pi/4$, and $h = \pi/3$, where the KIC spectrum turns out to be "nonchaotic." The first three cases correspond to known integrable cases of KICs. The most intriguing is the last "integrable" case of $h = \pi/3$, which to the best of our knowledge has not been investigated so far. Here, despite Poissonian spectral statistics, the correlators decay exponentially, e.g., $C_{zy}^{zy} = (-2)^{-t}$, on the light-cone border (for $t < N - 2$). This is reminiscent of the spectral problem for arithmetic surfaces of constant negative curvature, where correlations do decay exponentially, but the system spectrum exhibits Poissonian spectral statistics due to the existence of an infinite number of Hecke operators commuting with the system Hamiltonian. In the same spirit we expect that for the dual-unitary KIC model at $h = \pi/3$ there exists an additional number of symmetries splitting the system's spectrum into uncorrelated subspectra. Clarification of their exact nature is important, but beyond the scope of the present contribution.

VI. CONCLUSIONS

We derived an analytic formula, relating correlators $C(n, t)$ between operators with two-point support for $n = t$ (light-cone edge) to the expectation values of a transfer operator $\mathbf{T}$ with small dimensions. The result holds for a sufficiently long generic KC and does not require fine-tuned system parameters. For dual-unitary KCs this allows for a full characterization of the correlator behavior in terms of the $\mathbf{T}$ spectrum, as $C(n, t) = 0$ for $n \neq t$ in this case. For the FTC we go much further and obtain an explicit analytical expression for correlations between operators supported on pairs of adjacent sites. The results are illustrated on the example of the KIC.

The above results allow for several generalizations. First, models with a larger range of interactions can be treated in a similar manner. For systems with r-point interactions, $H_I = \sum_{i=1}^{N} f_1(s_{1+i}, \ldots, s_{r+i})$, the correlations at the light-cone edge $n = rt$ can be expressed through transfer operators of the dimension $L^r \times L^r$. Second, in the present work we restricted our considerations to correlators between operators with two-point support. An analogous result holds for correlations between operators with a larger support, i.e., $\Sigma_k^{(l)} = \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes \mathbf{q}_{k+1} \otimes \cdots \otimes \mathbf{q}_{k+l} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}$. In general, the correlators $\langle \Sigma_0^{(l)} \Sigma_n^{(l)} \rangle$ can be expressed through expectation values of transfer operators $\mathbf{T}_l$ with the dimensions $L^l \times L^l$. By using this, the correlators $C(n, t)$ in Eq. (10) can be evaluated above the light-cone edge $t = n + l$, $l > 0$, as well. To this end one fixes all $\mathbf{q}_i$ in $\Sigma_0^{(l)}$, $\Sigma_n^{(l)}$ to $\mathbb{1}$, except $\mathbf{q}_l, \mathbf{q}_{n+l}$. The price to pay is in the dimension of the transfer operators: the dimension of $\mathbf{T}_l$ increases exponentially with l. It remains an interesting open question whether the spectrum of $\mathbf{T}_l$ can be calculated exactly for FTC in the case $l > 2$.

Finally, the semiclassical limit $L \to \infty$ of FTC deserves a separate study. The classical model emerging in this limit is nothing more than a (perturbed) coupled cat map lattice considered in [16,17]. Depending on the functions $\lambda_i(s)$ and $\lambda_i'(s)$, this model exhibits different dynamical behaviors in the classical limit, ranging from full chaos to full integrability. However, for a finite dimension L of the local Hilbert space the transfer operator $\mathbf{T}$, in general, contains no eigenvalues on the unit circle except the trivial one, associated with the unit operator. Thus, independently of the underlying classical dynamics, FTC systems generically show exponential decay of two-point correlators for a fixed L in the thermodynamic limit $N \to \infty$. On the other hand, if the semiclassical limit $L \to \infty$ is taken first (or simultaneously with the thermodynamic limit), the gap in the transfer operator spectrum might close, such that no exponential decay is observed for any finite N. This shows that the emerging theory is very sensitive to the order of the thermodynamic and semiclassical limits.

ACKNOWLEDGMENTS

We thank T. Prosen for useful discussion. B.G. acknowledges support from the Israel Science Foundation through Grant No. 2089/19. D.W. and T.G. thank the Deutsche Forschungsgemeinschaft for support received through Project No. 402552305.

APPENDIX A: RELATION TO CIRCUIT LATTICES

For the sake of comparison it is instructive to observe a connection between quantum kicked chains considered in this work and circuit lattices. Such a connection can be established when both the chain length N and the propagation times t are even. It is straightforward to see that the quantum evolution operator U^{2t} for even times can be cast into the form

$$U^{2t} = U_I^e U_{\rm circ}^t \left(U_I^e\right)^\dagger. \tag{A1}$$

Here, the operator U_I^e corresponds to the even "half of the interaction":

$$\langle \boldsymbol{s}|U_I^e[f_1]|\boldsymbol{s}'\rangle = \delta(\boldsymbol{s},\boldsymbol{s}')e^{i\sum_{n=1}^{N/2} f_1(s_{2n},s_{2n+1})}, \tag{A2}$$

and the evolution $U_{\rm circ}$ has the form

$$U_{\rm circ} = \mathbb{T} U_I^e U_K U_I^e \mathbb{T}^\dagger U_I^e U_K U_I^e, \tag{A3}$$

where $\mathbb{T}$ is the circular shift operator on a lattice of N sites. Note that $U_{\rm circ}$ has a special structure, characteristic of circuit lattice evolution (see, e.g., [22]). The role of the unitary gate operator is fulfilled here by

$$U_{\rm gate} = u_1^e\,(u_2 \otimes u_2)\,u_1^e, \tag{A4}$$

where the diagonal matrix

$$\langle s_1 s_2|u_1^e|s_1' s_2'\rangle = \delta(s_1, s'_1)\delta(s_2, s'_2)e^{if_1(s_1,s_2)}$$

is a restriction of U_I^e to two adjacent lattice sites.

By Eq. (A3) we find for the two-point correlator

$$\mathrm{Tr}\,(U^t Q_1 U^{-t} Q_2) = \mathrm{Tr}\left(U_{\rm circ}^t \tilde{Q}_1 U_{\rm circ}^{-t}\tilde{Q}_1\right), \tag{A5}$$

where $\tilde{Q}_i = (U_I^e)^\dagger Q_i U_I^e$. Since U_I^e couples two neighboring sites, any strictly local operator with one-point support in the kicked model corresponds to a two-site operator of the respective circuit model.

APPENDIX B: GRAPHICAL METHOD FOR EVALUATION OF CORRELATORS

Correlation function between a number of local observables in the Floquet KC (1) can be written in the form of a partition function,

$$Z = \frac{1}{L^{Nt}} \sum_{\{s_{m,k}|(m,k)\in\mathcal{L}_1\}} e^{-i\mathcal{F}(\{s_{m,k}\})} \times \prod_{(m,k)\in\mathcal{L}_2} \delta(s_{m,k}, s_{m,1-k+2t}) D(s_{z_1},\ldots,s_{z_n}), \tag{B1}$$

where the last factor, D, depends on a finite number of lattice sites, $\mathcal{L}_0 = \{z_1,\ldots,z_l\}$, corresponding to the location of the observables. The above sum, in general, runs over a subset $\mathcal{L}_1$ of sites from the $2t \times N$ lattice $\mathcal{L}_{N\times 2t} = \{(m,k)|k = 1,\ldots,2t, m = 1,\ldots,N\}$ while the product in Eq. (B1) is, furthermore, restricted to a subset $\mathcal{L}_2 \subseteq \mathcal{L}_1$. In what follows we distinguish between three type of points $(m,k) \notin \mathcal{L}_0$ of the spatial-temporal lattice $\mathcal{L}_{N\times 2t}$ and introduce the corresponding symbolic notation for lattice sites:

Type 1. $(m,k) \notin \mathcal{L}_1$; i.e., there is no summation over the variables $s_{m,k}, s_{m,1-k+2t}$ in the partition function. The sites of this type are depicted by empty circles {○}.

Type 2. $(m,t) \in \mathcal{L}_2$; i.e., there is summation over the variables $s_{m,k}, s_{m,1-k+2t}$ coupled by the term $\delta(s_{m,k}, s_{m,1-k+2t})$. The sites of this type are depicted by full red circles {●}.

Type 3. $(m,k) \in \mathcal{L}_1\backslash\mathcal{L}_2$; i.e., there is summation over uncoupled variables $s_{m,k}, s_{m,1-k+2t}$. The sites of this type are depicted by full black circles {●}.

Having this notation at hand, we can uniquely encode a partition function of the type (B1) by filling nodes (m,k) of the lattice $\mathcal{L}_1\backslash\mathcal{L}_0$ with symbols drawn from the alphabet {○, ●, ●} (see Figs. 1 and 2).

Thanks to the unitarity of the operator u_2 a simple graphical method for calculation of partition functions like Eq. (B1) can be developed. To this end we establish "contraction rules" for sites of $\mathcal{L}_{N\times 2t}\backslash\mathcal{L}_0$. Let (m,k) be a site of type 2 such that three of its neighbors are of type 2, and the fourth one is of type 3. It can be easily shown that after summation over $s_{m,k}, s_{m,1-k+2t}$ variables the fourth site becomes of type 2 as well, while (m,k) becomes of type 1 (see Fig. 5). Indeed, whenever $(m-1,k),(m+1,k),(m,k),(m,k-1) \in \mathcal{L}_2$ we have for sum over $s_{m,k}, s_{m,1-k+2t}$ variables in Eq. (B1)

$$\frac{1}{L}\sum_{s_{m,k}}\sum_{s_{m,1-k+2t}} e^{-i(f_2(s_{m,k},s_{m,k+1})-f_2(s_{m,1-k+2t},s_{m,-k+2t}))}\delta(s_{m,k}, s_{m,1-k+2t}) = \delta(s_{m,k+1}, s_{m,-k+2t}). \tag{B2}$$

In an analogous way one can obtain all other contraction rules illustrated in Fig. 5. Note that the above contraction rules are akin to the operator "fusion rules" introduced in [22].

Obviously, each contraction leads to the removal of two summation variables from the sum (B1) without changing its form. As a result, by consecutive applications of the contraction rules the initial partition function can be reduced to the state where the vast majority of the summation variables are excluded from the sum (B1). The remaining sum can be then represented with the help of a transfer operator of a small dimension, independent of N.

APPENDIX C: CORRELATIONS IN THE FTC MODEL

For the FTC model, Eq. (15) can be utilized to obtain an explicit formula for correlators. To this end we expand the right-hand side of Eq. (15) in the basis of eigenstates (21) of $\mathbf{T}$. This yields

$$C_t = \sum_{m=0}^{\lfloor L/2\rfloor} (\mathbf{t}_m)^{t-2}\left(2-\delta_{m,0}-\delta_{m,\frac{L}{2}}\right)\mathrm{Re}\left[A_m(\mathbf{q}_4)\left\langle\bar{\Phi}_{\mathbf{q}_1\mathbf{q}_2}\middle|\Phi_{\mathbf{q}_3\mathbf{e}_m}\right\rangle\right] \tag{C1}$$

for even t and

$$C_t = \sum_{m=0}^{\lfloor L/2\rfloor} (\mathbf{t}_m)^{t-2}\left(2-\delta_{m,0}-\delta_{m,\frac{L}{2}}\right) \times \mathrm{Re}\left[e^{-i\phi}A_m^*(\mathbf{q}_4)\left\langle\bar{\Phi}_{\mathbf{q}_1\mathbf{q}_2}\middle|\Phi_{\mathbf{q}_3\mathbf{e}_m}\right\rangle\right], \tag{C2}$$

for odd t, where

$$A_m(\mathbf{q}) = \frac{1}{L}\sum_{s=1}^{L} e^{i2\pi sm/L}\langle s|\mathbf{q}|s\rangle.$$

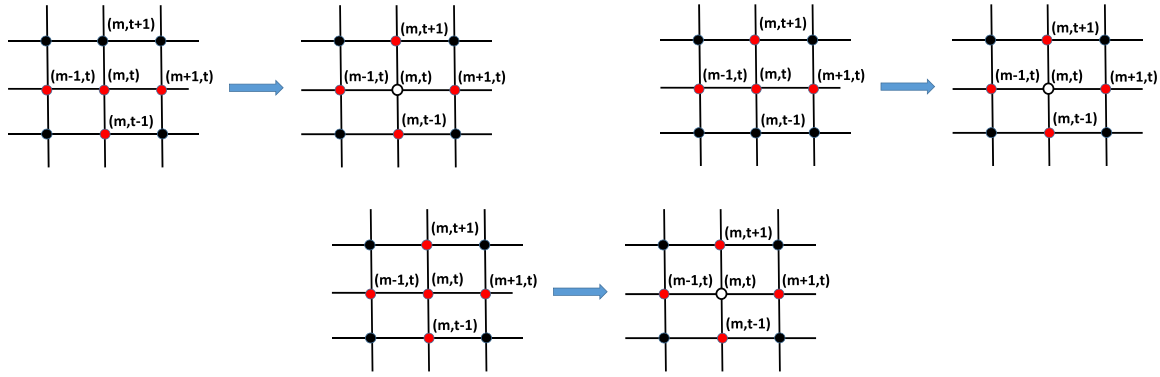

FIG. 5. The figure illustrates contraction rules for lattice sites (m,t) belonging to the set $\mathcal{L}_2$. The four figures above correspond to the case where three out of four neighbors of (m,t) belong to the set $\mathcal{L}_2$. The bottom figure illustrates the case where all four neighbors belong to $\mathcal{L}_2$.

The scalar products $\langle\bar{\Phi}_{\mathbf{q}_1\mathbf{q}_2}|\Phi_{\mathbf{q}_3\mathbf{e}_m}\rangle$ can be easily evaluated by using Eqs. (17) and (18):

$$\left\langle\bar{\Phi}_{\mathbf{q}_1\mathbf{q}_2}\middle|\Phi_{\mathbf{q}_3\mathbf{e}_m}\right\rangle = A_m(\mathbf{q}_1^c)B_m^{(1)}(\mathbf{q}_3)B_m^{(2)}(\mathbf{q}_2^c),$$

$$B_m^{(j)}(\mathbf{q}) = \frac{1}{L}\sum_{s=1}^{L} e^{i(\mu_j(s)-\mu_j(s^{(m)}))}\langle s|\mathbf{q}|s^{(m)}\rangle, \quad \text{(C3)}$$

with $j=1,2$, $s^{(m)} = 1+(s+m-1) \bmod L$, and $\mu_1(s) = -\lambda_1(s)-\lambda_1'(s)-\lambda_2(s)$, $\mu_2(s) = \lambda_1(s)+\lambda_1'(s)+\lambda_2'(s)$, respectively. Note that the constants $A_m(\mathbf{q})$ and $B_m^{(j)}(\mathbf{q})$ can be also written in a more compact form as

$$A_m(\mathbf{q}) = \frac{1}{L}\,\mathrm{Tr}\left(\Gamma_0^m\mathbf{q}\right),$$

$$B_m^{(j)}(\mathbf{q}) = \frac{1}{L}\,\mathrm{Tr}\left(\Gamma_j\mathbf{q}\Gamma_j^\dagger\,\mathbb{T}^m\right), \quad \text{(C4)}$$

where $\mathbb{T}$ is the circular shift operator, $\mathbb{T}|s\rangle = |s^{(m)}\rangle$, and Γ_j, $j=1,2,3$, are the diagonal matrices:

$$\Gamma_0 = \mathrm{diag}\{e^{i2\pi s/L}\}_{s=1}^L, \quad \text{(C5)}$$

$$\Gamma_j = \mathrm{diag}\{e^{i\mu_j(s)}\}_{s=1}^L, \quad j=1,2. \quad \text{(C6)}$$

APPENDIX D: APPLICATION TO THE KIC MODEL

The KIC model provides a minimal realization of model (1) with $L=2$. The KIC evolution is governed by the Hamiltonians

$$H_l = \sum_{n=1}^{N} J\hat{\sigma}_n^z\hat{\sigma}_{n+1}^z + h\hat{\sigma}_n^z, \quad H_K = b\sum_{n=1}^{N}\hat{\sigma}_n^x, \quad \text{(D1)}$$

$$\hat{\sigma}_n^\alpha = \underbrace{\mathbb{1}\otimes\cdots\otimes\mathbb{1}}_{n-1}\otimes\sigma^\alpha\otimes\underbrace{\mathbb{1}\otimes\cdots\otimes\mathbb{1}}_{N-n},$$

where σ_n^α, $\alpha = x,y,z$, are Pauli matrices. For the sake of simplicity of exposition we restrict our considerations to $b=\pi/4$ and arbitrary J and h. Note that the dual-unitary case corresponds to $J=b=\pi/4$. The resulting evolution operators U_K and U_l take the form (4) with the functions

$$f_1 = -Jmn - \frac{h}{2}(m+n), \quad f_2 = \frac{\pi}{4}(mn-1),$$

$m,n=\pm1$, defining the two unitary matrices u_1 and u_2:

$$u_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i(J+h)} & e^{iJ} \\ e^{iJ} & e^{-i(J-h)}\end{pmatrix},$$

$$u_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1\end{pmatrix}. \quad \text{(D2)}$$

After inserting f_1 and f_2 into Eq. (16) we obtain

$$\mathbf{T} = \frac{1}{2}\begin{pmatrix} \cos^2 h_+ & \sin^2 h & \sin^2 h_+ & \cos^2 h \\ \sin^2 h_+ & \cos^2 h & \cos^2 h_+ & \sin^2 h \\ \sin^2 h & \cos^2 h_- & \cos^2 h & \sin^2 h_- \\ \cos^2 h & \sin^2 h_- & \sin^2 h & \cos^2 h_- \end{pmatrix}, \quad \text{(D3)}$$

where $h_+ = h+J-\pi/4$, $h_- = h-J+\pi/4$. The four eigenvalues of $\mathbf{T}$ are

$$\mu_1 = 1, \quad \mu_2 = \cos 2h\sin^2 2J, \quad \mu_3 = 0, \quad \mu_4 = 0.$$

As a result, the nth power of $\mathbf{T}$ is given for $n>1$ by

$$\mathbf{T}^n = \mu_2^n\,\Phi_2\otimes\bar{\Phi}_2 + \Phi_1\otimes\Phi_1 \quad \text{(D4)}$$

with $\Phi_1 = \frac{1}{2}(1,1,1,1)^T$ being the eigenvector of $\mathbf{T}$ for the leading eigenvalue μ_1 and

$$\Phi_2 = \frac{1}{c+d}(c,-c,-d,d)^T,$$

$$\bar{\Phi}_2 = \frac{1}{c+d}(c,-d,-c,d), \quad \text{(D5)}$$

$c = \cos 2h_+ + \cos 2h$, $d = \cos 2h_- + \cos 2h$, are the left and right eigenvectors corresponding to μ_2.

1. Local operator correlators

To evaluate correlators note that the operators $u_2\mathbf{q}_1u_2^\dagger$, $\mathbf{q}_4$ contribute only diagonal elements into Eqs. (17) and (18). In the case of the KIC model this means that only the spin combinations $\Sigma_{n_1} = \hat{\sigma}_{n_1}^\alpha\hat{\sigma}_{n_1+1}^\beta$ and $\Sigma_{n_2} = \hat{\sigma}_{n_2}^\gamma\hat{\sigma}_{n_2+1}^\delta$ for $\alpha=y$, $\delta=z$ might have $C_t\neq0$. By using the representation (D4) we have for the correlator (15)

$$C_t \equiv C(t,t) = \mu_2^{t-2}\langle\bar{\Phi}_{\sigma^y\sigma^\beta}|\Phi_2\rangle\langle\bar{\Phi}_2|\Phi_{\sigma^y\sigma^z}\rangle + \langle\bar{\Phi}_{\sigma^y\sigma^\beta}|\Phi_1\rangle\langle\Phi_1|\Phi_{\sigma^y\sigma^z}\rangle, \quad \text{(D6)}$$

where the vectors $\bar{\Phi}_{\sigma^y\sigma^\beta}$ and $\Phi_{\sigma^\gamma\sigma^z}$ are calculated by Eqs. (17) and (18). Explicitly, they are given by

$$\Phi_{\sigma^y\sigma^z} = \bar{\Phi}_{\sigma^y\sigma^z} = \frac{\sin 2J}{2}\begin{pmatrix} -\sin(2h-2J) \\ \sin(2h-2J) \\ -\sin(2h+2J) \\ \sin(2h+2J)\end{pmatrix},$$

$$\Phi_{\sigma^x\sigma^z} = \bar{\Phi}_{\sigma^y\sigma^x} = \frac{\sin 2J}{2}\begin{pmatrix} -\cos(2h-2J) \\ -\cos(2h+2J) \\ \cos(2h-2J) \\ \cos(2h+2J)\end{pmatrix}. \quad \text{(D7)}$$

After inserting Eqs. (D5) and (D7) into Eq. (D6) we obtain

$$C_t = \mathcal{C}^{\gamma\delta}_{\alpha\beta}(\cos 2h \sin^2 2J)^t, \quad \text{(D8)}$$

where prefactors $\mathcal{C}^{\gamma\delta}_{\alpha\beta}$ are given by

$$\mathcal{C}^{yz}_{yz} = 1, \quad \mathcal{C}^{xz}_{yx} = \tan^2 2h, \quad \mathcal{C}^{xz}_{yz} = \mathcal{C}^{yz}_{yx} = -\tan 2h, \quad \text{(D9)}$$

while $C^{\gamma\delta}_{\alpha\beta}$ are zero for all other spin combinations.

2. Strictly local operator correlators

By using the representation (D4) we have for the correlator (27)

$$C^{\alpha\beta}(t-1,t) = \mu_2^{t-2}\langle\bar{\Phi}_{\mathbb{1}\sigma^\alpha}|\Phi_2\rangle\langle\bar{\Phi}_2|\Phi_{\sigma^\beta\mathbb{1}}\rangle + \langle\bar{\Phi}_{\mathbb{1}\sigma^\alpha}|\Phi_1\rangle\langle\Phi_1|\Phi_{\sigma^\beta\mathbb{1}}\rangle, \quad \text{(D10)}$$

where the vectors $\bar{\Phi}_{\mathbb{1}\sigma^\alpha}$ and $\Phi_{\sigma^\beta\mathbb{1}}$ can be calculated by Eqs. (17) and (18). Explicitly, they are given by

$$\Phi_{\sigma^y\mathbb{1}} = \frac{\cos 2J}{2}\begin{pmatrix} \cos(2h-2J) \\ -\cos(2h-2J) \\ \cos(2h+2J) \\ -\cos(2h+2J)\end{pmatrix}, \quad \Phi_{\sigma^z\mathbb{1}} = 0,$$

$$\Phi_{\sigma^x\mathbb{1}} = \frac{\cos 2J}{2}\begin{pmatrix} -\sin(2h-2J) \\ -\sin(2h+2J) \\ \sin(2h-2J) \\ \sin(2h+2J)\end{pmatrix}, \quad \text{(D11)}$$

and $\Phi_{\sigma^y\mathbb{1}} = \bar{\Phi}_{\mathbb{1}\sigma^z}$, $\Phi_{\sigma^z\mathbb{1}} = \bar{\Phi}_{\mathbb{1}\sigma^y}$, and $\Phi_{\sigma^x\mathbb{1}} = \bar{\Phi}_{\mathbb{1}\sigma^x}$. After substitution of Eqs. (D11) into Eq. (D10) one has

$$C^{\alpha\beta}(t-1,t) = \mathcal{C}^{\alpha\beta}(\cos 2h \sin^2 2J)^t \cot^2 2J \quad \text{(D12)}$$

with the coefficients given by

$$\mathcal{C}^{xx} = 1, \quad \mathcal{C}^{xy} = \mathcal{C}^{zx} = \tan 2h, \quad \mathcal{C}^{zy} = \tan^2 2h,$$

and zeros for all other α, β combinations.

3. Dual-unitary case

The dual-unitary KIC is governed by the Hamiltonians (D1) with $J = b = \pi/4$. The strength h of the magnetic field in the z direction is arbitrary and might, in general, depend on the chain site. The resulting two unitary matrices u_1 and u_2 are given by

$$u_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i(\frac{\pi}{4}+h)} & e^{i\frac{\pi}{4}} \\ e^{i\frac{\pi}{4}} & e^{-i(\frac{\pi}{4}-h)}\end{pmatrix},$$

$$u_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1\end{pmatrix}, \quad \text{(D13)}$$

at the ith site of the chain. Note that u_1 and u_2 can be expressed through the DFT matrix F as

$$u_1 = \begin{pmatrix} e^{-\frac{ih}{2}} & 0 \\ 0 & e^{\frac{i(\pi+h)}{2}}\end{pmatrix}F\begin{pmatrix} e^{-\frac{i(\pi+2h)}{4}} & 0 \\ 0 & e^{\frac{i(\pi+2h)}{4}}\end{pmatrix},$$

$$u_2 = \begin{pmatrix} 1 & 0 \\ 0 & e^{-\frac{i\pi}{2}}\end{pmatrix}F\begin{pmatrix} 1 & 0 \\ 0 & e^{-\frac{i\pi}{2}}\end{pmatrix}, \quad F = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1\end{pmatrix}.$$

This implies that KIC is just a particular case of the FTC model for $L = 2$ with the parameters

$$\Lambda_1 = \mathrm{diag}\{e^{-ih/2}, e^{i(\pi+h)/2}\},$$
$$\Lambda_1' = \mathrm{diag}\{e^{-i(\pi+2h)/4}, e^{i(\pi+2h)/4}\},$$
$$\Lambda_2 = \Lambda_2' = \mathrm{diag}\{1, e^{-i\pi/2}\}.$$

Inserting into Eq. (16) the corresponding functions f_1 and f_2 yields

$$\mathbf{T}_i = \frac{1}{2}\begin{pmatrix} a_i & b_i & b_i & a_i \\ b_i & a_i & a_i & b_i \\ b_i & a_i & a_i & b_i \\ a_i & b_i & b_i & a_i\end{pmatrix}, \quad \text{(D14)}$$

where $a_i = \cos^2 h_i$, $b_i = \sin^2 h_i$, and h_i is the value of the local magnetic field at the ith site of the chain. The four eigenvalues of this matrix are $\{1, \cos 2h_i, 0, 0\}$ in agreement with the results of [22].

For the operators $\Sigma_{n_1} = \hat{\sigma}^y_{n_1}\hat{\sigma}^\beta_{n_1+1}$ and $\Sigma_{n_2} = \hat{\sigma}^\gamma_{n_2}\hat{\sigma}^z_{n_2+1}$, the corresponding vectors (17) and (18) are given by

$$\bar{\Phi}_{\sigma_y\sigma_z} = \Phi_0 \cos 2h_{n_1+1}, \quad \bar{\Phi}_{\sigma_y\sigma_x} = -\Phi_0 \sin 2h_{n_1+1},$$
$$\Phi_{\sigma_y\sigma_z} = \Phi_0 \cos 2h_{n_2}, \quad \Phi_{\sigma_x\sigma_z} = -\Phi_0 \sin 2h_{n_2}, \quad \text{(D15)}$$

with $\Phi_0 = \frac{1}{2}(1,-1,-1,1)^T$ being the eigenvector of $\mathbf{T}_i$ for the eigenvalue $\cos 2h_i$. All other combinations of x, y, z give rise to zero vectors. Importantly, the vector Φ_0 is independent of h_i. As a result, after inserting Eqs. (D15) into Eq. (15) with $\mathbf{T}^{n-2} \equiv \prod_{i=n_1+1}^{n_2-1}\mathbf{T}_i$ we obtain

$$C(n,t) = \delta(n,t)\mathcal{C}^{\delta\gamma}_{\alpha\beta}\prod_{i=n_1+1}^{n_2}\cos 2h_i, \quad \text{(D16)}$$

$n = n_2 - n_1 < N/2$, where the prefactors $\mathcal{C}^{\delta\gamma}_{\alpha\beta}$ depend on the operators $\Sigma_{n_1} = \hat{\sigma}^\alpha_{n_1}\hat{\sigma}^\beta_{n_1+1}$ and $\Sigma_{n_2} = \hat{\sigma}^\gamma_{n_2}\hat{\sigma}^\delta_{n_2+1}$. Specifically, $\mathcal{C}^{yz}_{yz} = 1$, $\mathcal{C}^{xz}_{yx} = \tan 2h_{n_1+1}\tan 2h_{n_2}$, $\mathcal{C}^{yz}_{yx} = -\tan 2h_{n_1+1}$, $\mathcal{C}^{xz}_{yz} = -\tan 2h_{n_2}$, and $C^{\gamma\delta}_{\alpha\beta}$ are zero for all other spin combinations.

[1] I. Bloch, J. Dalibard, and W. Zwerger, Many-body physics with ultracold gases, Rev. Mod. Phys. **80**, 885 (2008).

[2] G. D. Mahan, *Condensed Matter in a Nutshell* (Princeton University Press, Princeton, NJ, 2011), Vol. 8.

[3] M. Schreiber, S. S. Hodgman, P. Bordia, H. P. Lüschen, M. H. Fischer, R. Vosk, E. Altman, U. Schneider, and I. Bloch, Observation of many-body localization of interacting fermions in a quasirandom optical lattice, Science **349**, 842 (2015).

[4] J. Simon, W. S. Bakr, R. Ma, M. E. Tai, P. M. Preiss, and M. Greiner, Quantum simulation of antiferromagnetic spin chains in an optical lattice, Nature **472**, 307 (2011).

[5] T. Engl, J. Dujardin, A. Argüelles, P. Schlagheck, K. Richter, and J. D. Urbina, Coherent Backscattering in Fock Space: A Signature of Quantum Many-Body Interference in Interacting Bosonic Systems, Phys. Rev. Lett. **112**, 140403 (2014).

[6] R. Dubertrand and S. Müller, Spectral statistics of chaotic many-body systems, New J. Phys. **18**, 033009 (2016).

[7] P. Ponte, Z. Papić, F. Huveneers, and D. A. Abanin, Many-Body Localization in Periodically Driven Systems, Phys. Rev. Lett. **114**, 140401 (2015).

[8] Y. Y. Atas and E. Bogomolny, Spectral density of a one-dimensional quantum Ising model: Gaussian and multi-Gaussian approximations, J. Phys. A: Math. Theor. **47**, 335201 (2014).

[9] J. P. Keating, N. Linden, and H. J. Wells, Spectra and eigenstates of spin chain Hamiltonians, Commun. Math. Phys. **338**, 81 (2015).

[10] S. Czischek, M. Gärttner, M. Oberthaler, M. Kastner, and T. Gasenzer, Quenches near criticality of the quantum Ising chain—power and limitations of the discrete truncated Wigner approximation, Quantum Sci. Technol. **4**, 014006 (2018).

[11] A. Altland and B. D. Simons, *Condensed Matter Field Theory*, 2nd ed. (Cambridge University Press, Cambridge, UK, 2010).

[12] J. P. Sethna, *Statistical Mechanics: Entropy, Order Parameters and Complexity*, 1st ed. (Oxford University Press, Oxford, UK, 2006).

[13] M. Akila, D. Waltner, B. Gutkin, and T. Guhr, Particle-time duality in the kicked Ising spin chain, J. Phys. A **49**, 375101 (2016).

[14] M. Akila, D. Waltner, B. Gutkin, P. Braun, and T. Guhr, Semiclassical Identification of Periodic Orbits in a Quantum Many-Body System, Phys. Rev. Lett. **118**, 164101 (2017).

[15] M. Akila, B. Gutkin, P. Braun, D. Waltner, and T. Guhr, Semiclassical prediction of large spectral fluctuations in interacting kicked spin chains, Ann. Phys. **389**, 250 (2018).

[16] B. Gutkin and V. Osipov, Classical foundations of many-particle quantum chaos, Nonlinearity **29**, 325 (2016).

[17] B. Gutkin, L. Han, R. Jafari, A. K. Saremi, and P. Cvitanović, Linear encoding of the spatiotemporal cat map, arXiv:1912.02940.

[18] B. Bertini, P. Kos, and T. Prosen, Exact Spectral Form Factor in a Minimal Model of Many-Body Quantum Chaos, Phys. Rev. Lett. **121**, 264101 (2018).

[19] B. Bertini, P. Kos, and T. Prosen, Entanglement Spreading in a Minimal Model of Maximal Many-Body Quantum Chaos, Phys. Rev. X **9**, 021033 (2019).

[20] P. Braun, D. Waltner, M. Akila, B. Gutkin, and T. Guhr, Transition from quantum chaos to localization in spin chains, Phys. Rev. E **101**, 052201 (2020).

[21] R. Pal and A. Lakshminarayan, Entangling power of time-evolution operators in integrable and nonintegrable many-body systems, Phys. Rev. B **98**, 174304 (2018).

[22] B. Bertini, P. Kos, and T. Prosen, Exact Correlation Functions for Dual-Unitary Lattice Models in $1+1$ Dimensions, Phys. Rev. Lett. **123**, 210601 (2019).

[23] S. Gopalakrishnan and A. Lamacraft, Unitary circuits of finite depth and infinite width from quantum channels, Phys. Rev. B **100**, 064309 (2019).

[24] B. Bertini, P. Kos, and T. Prosen, Operator entanglement in local quantum circuits I: Maximally chaotic dual-unitary circuits, SciPost Phys. **8**, 067 (2020).

[25] J. Avan, V. Caudrelier, A. Doikou, and A. Kundu, Lagrangian and Hamiltonian structures in an integrable hierarchy and space-time duality, Nucl. Phys. B **902**, 415 (2016).

[26] L. Piroli, B. Bertini, J. I. Cirac, and T. Prosen, Exact dynamics in dual-unitary quantum circuits, Phys. Rev. B **101**, 094304 (2020).

[27] Z. Krajnik and T. Prosen, Kardar-Parisi-Zhang physics in integrable rotationally symmetric dynamics on discrete space-time lattice, J. Stat. Phys. **179**, 110 (2020).

[28] T. Zhou and A. Nahum, The Entanglement Membrane in Chaotic Many-Body Systems, Phys. Rev. X **10**, 031066 (2020).

[29] D. Goyeneche, D. Alsina, J. I. Latorre, A. Riera, and K. Życzkowski, Absolutely maximally entangled states, combinatorial designs, and multiunitary matrices, Phys. Rev. A **92**, 032316 (2015).

[30] S. A. Rather, S. Aravinda, and A. Lakshminarayan, Creating Ensembles of Dual Unitary and Maximally Entangling Quantum Evolutions, Phys. Rev. Lett. **125**, 070501 (2020).

[31] P. W. Claeys and A. Lamacraft, Maximum velocity quantum circuits, Phys. Rev. Research **2**, 033032 (2020).

[32] W. Tadej and K. Życzkowski, A concise guide to complex Hadamard matrices, Open Syst. Inf. Dyn. **13**, 133 (2006).

[33] C. N. Yang and T. D. Lee, Statistical theory of equations of state and phase transitions. I. Theory of condensation, Phys. Rev. **87**, 404 (1952).

[34] T. D. Lee and C. N. Yang, Statistical theory of equations of state and phase transitions. II. Lattice gas and Ising model, Phys. Rev. **87**, 410 (1952).

[35] V. Matveev and R. Shrock, On properties of the Ising model for complex energy/temperature and magnetic field, J. Phys. A: Math. Theor. **41**, 135002 (2008).

[36] V. Oganesyan and D. A. Huse, Localization of interacting fermions at high temperature, Phys. Rev. B **75**, 155111 (2007).

[37] Y. Y. Atas, E. Bogomolny, O. Giraud, and G. Roux, Distribution of the Ratio of Consecutive Level Spacings in Random Matrix Ensembles, Phys. Rev. Lett. **110**, 084101 (2013).