Rapid Communications

Critical behavior at the integer quantum Hall transition in a network model on the kagome lattice

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We study a network model on the kagome lattice (NMKL). This model generalizes the Chalker-Coddington network model for the integer quantum Hall transition. Unlike random network models we studied earlier, the geometry of the kagome lattice is regular. Therefore, we expect that the critical behavior of the NMKL should be the same as that of the Chalker-Coddington model. We numerically compute the localization length index ν in the NKML. Our result $\nu = 2.658 \pm 0.046$ is close to Chalker-Coddington model values obtained in a number of recent papers. We also map the NMKL to the Dirac fermions in random potentials and in a fixed periodic curvature background. The background turns out irrelevant at long scales. Our numerical and analytical results confirm our expectation of the universality of critical behavior on regular network models.

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Introduction. The integer quantum Hall (IQH) transition [1] is a quantum phase transition accompanied by universal critical phenomena. A central characteristic of the transition is the exponent ν describing the divergence of the localization length of single-particle wave functions with energies E close to critical energies E_c:

$$\xi \sim |E - E_c|^{-\nu}. \tag{1}$$

Multiple experiments [2–10] demonstrated scaling near the integer QH transition in various systems. Most experiments can access only the product νz, where z is a dynamical critical exponent. With a common assumption that $z = 1$, all experiments are consistent with the value $\nu_{\rm expt} \approx 2.38$. The attempt [9] to establish the value $z = 1$ from scaling with the sample size remains somewhat controversial due to a possible presence of nonuniversal, sample-dependent parameters (see Ref. [11]).

The QH plateaus separated by the transition are successfully described by models of noninteracting electrons in the presence of disorder. In this approximation the transition is an Anderson transition [12]. Even with this simplification the problem of the IQH transition is notoriously difficult. In early analytic investigations, the IQH effect was described by a nonlinear σ model [13–17], and a two-parameter renormalization group flow was suggested in [18,19]. Later, various versions of the Wess-Zumino model were proposed as the critical field theories underlying the IQH critical point [20–22]. More recently, analytical results for the multifractal critical exponents were obtained using conformal field theory [23–25]. The latter paper [25] notably predicts logarithmic (as opposed to power-law) scaling effectively meaning $\nu = \infty$.

There are many numerical simulations of noninteracting models of the IQH transition. One of the better studied models is the Chalker-Coddington network model on a square lattice [26,27]. Recent accurate simulations of the Chalker-Coddington model [28–33] give the value ν in the range 2.56–2.62, which is definitely different from the experimental value. Similar values have been obtained in numerical simulations of other noninteracting models of the IQH transition [34–38].

The likely source for the discrepancy between the experimental and numerical values of ν are the electron-electron interactions [39–42]. Recently, we have proposed another possible reason for the discrepancy, and studied a version of the network model on random graphs [43,44]. Our results suggest that the additional geometric randomness is relevant: it changes the localization length exponent to $\nu \approx 2.37$ and places the random network model in a different universality class than the regular Chalker-Coddington model.

Random graphs that we have studied are dual to random quadrangulations. A polygon with n sides in a random graph is dual to a vertex where n quadrangles meet. If all quadrangles are viewed as squares, the deficit angle at the vertex is $R_n = (4 - n)\pi/2$, and this can be interpreted as discrete curvature of a conical singularity at the vertex. Random networks contain randomly placed curvatures, and averaging over geometric randomness can be interpreted as integration over configurations of quenched random metric. Any two-dimensional (2D) model with interacting matter fields defined by an evolution operator (R matrix) can be studied on random surfaces [45], similar to critical 2D minimal models coupled to quantum gravity on triangulated random surfaces [46–49].

Our previous results [43,44] raise the issue of universality of critical behavior of network models. We expect that if the network is not random but contains periodically placed fixed curvatures, then the exponent ν should be the same as in the Chalker-Coddington model. One such network can be defined on the kagome lattice which contains triangles with $R_3 = \pi/2$ and hexagons with $R_6 = -\pi$ in a periodic arrangement.

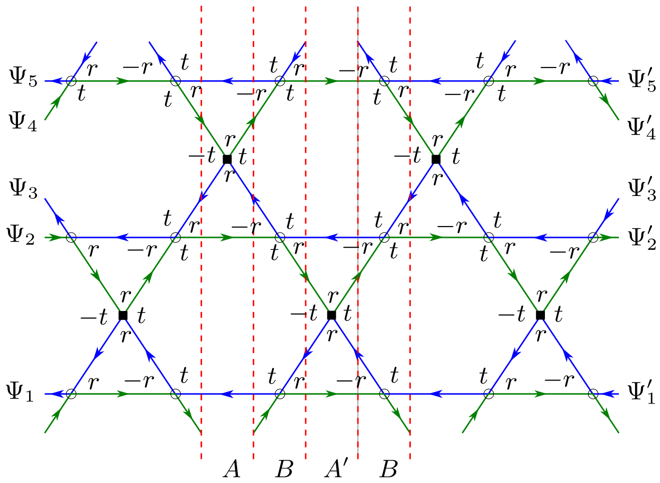

FIG. 1. The network model on the kagome lattice. Left running channels are shown in blue; right running channels in green. The column transfer matrices A, B, and A' are framed by red dashed lines. There are two different types of vertices, a and b, shown as squares (a) and circles (b).

In this Rapid Communication we study the network model on the kagome lattice (NMKL) analytically and numerically, determine its critical behavior, and confirm our expectation of the universality of the exponent ν.

The model. The NMKL is shown in Fig. 1. A state of the network $|\psi\rangle = \sum_l \psi_l |l\rangle$ is a vector in $\mathcal{H} = \mathbb{C}^{N_L}$, where N_L is the number of links, and $|l\rangle$ are basis vectors associated with each link l.

States of the network evolve in discrete time, each time step described by a unitary matrix $\mathcal{U}$ acting on $\mathcal{H}$ whose matrix elements $\mathcal{U}_{ll'}$ are nonzero only if l' and l are incoming and outgoing links at the same node. In this case $\mathcal{U}_{ll'} = e^{i\phi_l}\mathcal{S}_{ll'}$, where ϕ_l are random phases uniformly distributed on $[0, 2\pi)$, and the scattering matrix $\mathcal{S}$ depends on the type of node (square or circle in Fig. 1):

$$\mathcal{S}_a = \begin{pmatrix} -t & r \\ r & t \end{pmatrix}, \quad \mathcal{S}_b = \begin{pmatrix} -r & t \\ t & r \end{pmatrix}. \tag{2}$$

This choice assigns probabilities t^2 for all right turns and $r^2 = 1 - t^2$ for all left turns.

The clean model where all $\phi_l = 0$ is periodic and is easily solved in the momentum space (see [50], Sec. I). We find that at the critical point of the clean model $t_0 = \sqrt{3}/2$ $(r_0 = 1/2)$, the spectrum of the quasienergy $\varepsilon = -i \ln \mathcal{U}$ contains a gapless Dirac cone, so the long-distance description is the 2D Dirac fermion. In analogy with the analysis of Refs. [43,51], the addition of weak randomness in the phases ϕ_l leads to the field theory of a Dirac fermion with random mass, coupled to random scalar and vector potentials, and a fixed periodic curvature background. The periodic nature of the curvature background on the lattice scale makes it irrelevant in the long-distance limit, and leads to the same model of Dirac fermions as in Ref. [51] for the Chalker-Coddington model. This enforces our expectation that the critical behavior of the NMKL is the same as that of the Chalker-Coddington model.

Numerical procedure. To compute critical exponents of the NMKL we use the transfer-matrix method [52,53]. For finite networks of length L, with M channels in each direction, and periodic boundary conditions in the transverse direction, we compute the product $T_L = \prod_{j=1}^{L} AU_{1j}BU_{2j}A'U_{3j}BU_{4j}$ of transfer matrices for L layers. Each layer is split into four sublayers, as indicated in Fig. 1. The $2M \times 2M$ transfer matrices for the sublayers, A, A', and B, contain 2×2 matrices a and b for the scattering nodes, and 2×2 identity matrices 1_2:

$$A = \mathrm{diag}(1_2, a, \dots, 1_2, a),$$
$$A' = \mathrm{diag}(a, 1_2, \dots, a, 1_2),$$
$$B = \begin{pmatrix} 1/t & 0 & \cdots & r/t \\ 0 & b & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ r/t & 0 & \cdots & 1/t \end{pmatrix},$$
$$a = \begin{pmatrix} 1/r & t/r \\ t/r & 1/r \end{pmatrix}, \quad b = \begin{pmatrix} 1/t & r/t \\ r/t & 1/t \end{pmatrix}. \tag{3}$$

In addition, the random phases ϕ_l are combined into diagonal matrices $U_{ll'} = \exp{(i\phi_l)}\,\delta_{ll'}$.

The transmission and reflection amplitudes t and r at each node are shown in Fig. 1. We parametrize them as

$$r = (1 + 3e^{2x})^{-1/2}, \quad t = \left(1 + \tfrac{1}{3}e^{-2x}\right)^{-1/2}. \tag{4}$$

Here $x = 0$ corresponds to the critical point of the clean model without randomness. This parametrization resembles that traditionally used for the Chalker-Coddington model. However, in the latter case there is a symmetry with respect to rotations by 90° that results in the invariance of the spectrum of T_L upon exchange $x \leftrightarrow -x$ or, equivalently, $t \leftrightarrow r$ even in the presence of random phases. In the NMKL there is no such symmetry, and the critical point in the random model is not expected to be at $x = 0$. The determination of the value of x_c will be carried out simultaneously with that of the critical exponents by fitting the expected scaling function to the numerically obtained data.

T_L is a product of random matrices. According to Oseledec's theorem [54] the Lyapunov exponents defined as the eigenvalues of $\log[T_L T_L^\dagger]/2L$, tend to nonrandom values as $L \to \infty$. The smallest positive Lyapunov exponent γ is inversely proportional to the localization length in the quasi-one-dimensional system with width M. The product $\Gamma = \gamma M$ (the "dimensionless" Lyapunov exponent) becomes a universal quantity in the limit $M \to \infty$ at the critical point. In practice, a finite-size-scaling analysis relates Γ to critical exponents of the NMKL. In addition, Tutubalin's theorem [55] states that for finite systems with $L \gg 1$, the Lyapunov exponents have Gaussian distributions with variance $\sim(M/L)^{1/2}$. If we consider an ensemble of N random networks, the variance decreases to $\sim(M/LN)^{1/2}$. Therefore, our strategy is to consider large numbers of long systems to create ensembles of γ that have distributions close to Gaussian.

In this work we used networks of length $L = 5 \times 10^6$ and created ensembles of Lyapunov exponents γ labeled by $a = 1, \dots, N_{\text{ens}}$, where $N_{\text{ens}} = 200$ is the number of pairs $(x, M)_a$ that we used. The widths M take ten values $M = 20, 40, \dots, 200$, and the 20 values of x in the range $[0.24, 0.3]$ were chosen adaptively to get more data points in the vicinity of the (*a priori* unknown) critical point x_c [which we estimate to be $x_c = 0.268(1)$]. The numbers N_a of Lyapunov exponents in each ensemble are given in Table I in [50], Sec. II; most of them are $N_a = 624$. The total number of Lyapunov exponents in all ensembles is $N_{\text{Lyapunov exponent}} = 130\,896$.

Computing large products T_L directly is not possible, as many entries of the products grow exponentially with L. This problem is often overcome using the QR decomposition [52,53,56], where matrices T in the product are decomposed as $T = QR$ with unitary matrix Q and upper right triangular matrix R. An alternative is to use the LU decomposition $T = PLU$ using a lower triangular matrix L with unit diagonal, a permutation matrix P and an upper triangular matrix U (see Ref. [57] for details). Simulations with the LU decomposition are about two times faster than those with the QR decomposition.

We have generated pairs of large ensembles of Lyapunov exponents γ for multiple pairs (x, M) using both the QR and the LU decompositions and created a histogram for each ensemble. The histograms are very well described by normal distributions as confirmed by Gaussian fits. The centers of the Gaussian peaks in a pair corresponding to the ensembles generated by the QR and the LU decompositions differ by orders of magnitude less than the peaks widths. The widths of peaks in each such pair agrees with the same precision as the centers of the peaks do.

The fitting procedure. Near the critical point in a system of finite width M, the Lyapunov exponent Γ is expected [28,52,53] to exhibit the following scaling behavior:

$$\Gamma = F_\Gamma[M^{1/\nu} u_0(x), f(M)\, u_1(x)], \tag{5}$$

where F_Γ is a scaling function of the relevant field $u_0(x)$ and the leading irrelevant field $u_1(x)$. In the limit $M \to \infty$ the contribution of the irrelevant field should vanish, so $f(M)$ should decrease with M. If the field u_1 is truly irrelevant, we have $f(M) = M^y$ with a negative exponent $y < 0$. Recently, it was suggested that one might need to include two irrelevant fields [33], or that the field u_1 can be marginally irrelevant. The latter case would correspond to $f(M) = (\ln M)^b$ with some negative $b < 0$ [30,33]. We comment on results by fits with two irrelevant fields below.

On the left-hand side of (5) we use the numerical values of γ extracted from T_L for various combinations x and M. The scaling function Γ is expanded in its arguments, and we assume that the scaling fields u_i are polynomials in x. Since we do not have symmetry under $x \leftrightarrow -x$, and the critical point $x_c \neq 0$, we do not restrict $u_0(x), u_1(x)$ to be even or odd. Then we get

$$\begin{aligned} F_\Gamma[u_0 M^{1/\nu}, u_1 M^y] = {} & \Gamma_{00} + \Gamma_{01} u_1 M^y + \Gamma_{20} u_0^2 M^{2/\nu} \\ & + \Gamma_{02} u_1^2 M^{2y} + \Gamma_{21} u_0^2 u_1 M^{2/\nu} M^y \\ & + \Gamma_{03} u_1^3 M^{3y} + \cdots, \end{aligned} \tag{6}$$

with fields $u_{0,1} = p_{0,1}(x - x_c)$, and polynomials

$$p_0(x) = x + \sum_{k=2}^{m} a_k x^k, \quad p_1(x) = 1 + \sum_{k=1}^{n} b_k x^k. \tag{7}$$

Since the overall scaling of the fields is arbitrary, the leading coefficients in (7) are set to 1.

The critical exponents ν and y, and the critical amplitude ratio $\Gamma_c \equiv \Gamma_{00}$ are the most interesting universal characteristics of the IQH transition. The latter is related to one of the multifractal exponents by $\Gamma_c = \pi(\alpha_0 - 2)$ [58,59]. These quantities, together with a finite number of expansion coefficients in Eqs. (6) and (7), form sets $\Lambda = \{\nu, y, \Gamma_{ij}, a_k, b_l\}$ of the fitting parameters. The fits should use as few fitting parameters as possible while reproducing the data as well as possible, and we use several criteria to assess the quality of our fits. Details of our best-fitting procedures are presented in [50], Sec. II.

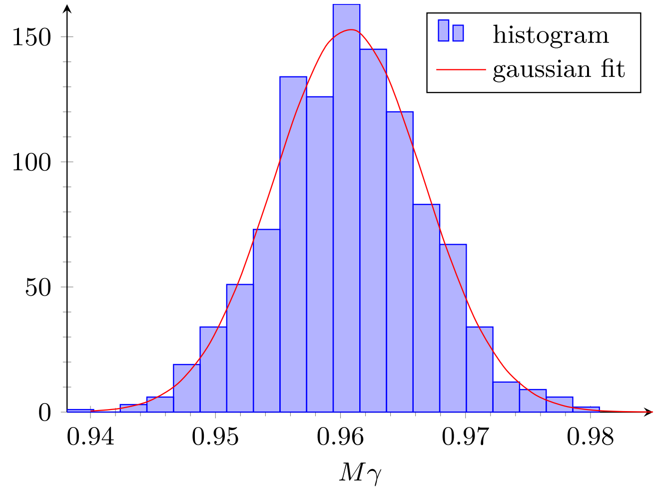

FIG. 2. Histogram for $M = 180$ and $x = 0.255$. The ensemble consists of 1088 Lyapunov exponents.

Results. In Fig. 2 we present an example of a histogram for the distribution of Γ for $M = 180$ and $x = 0.255$. The distribution is fitted to a Gaussian, and the Gaussian fit is very accurate in full accord with Tutubalin's central limit theorem [55]. As discussed in [50], each distribution for a given $(x, M)_a$ defines one data point and its error bars, as well as weights for the fitting procedures.

In Fig. 3 we plot the numerical data points for Lyapunov exponents, together with the scaling function F_Γ that results from one of our two best fits. The two fits give the following

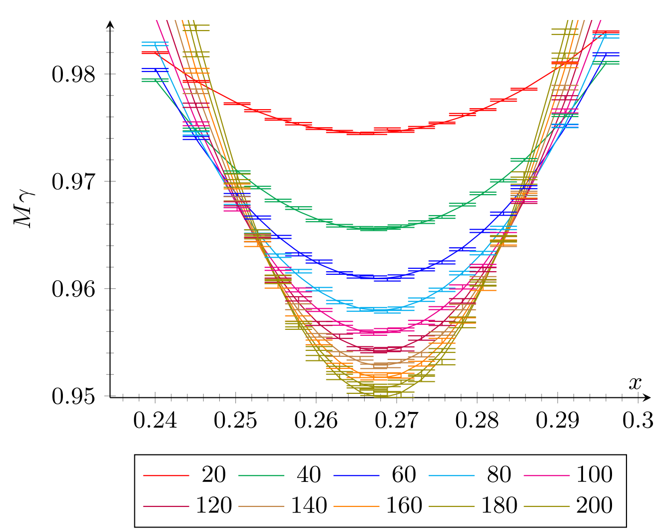

FIG. 3. Results of the best fit for the Lyapunov exponents. The system widths M are color-coded as indicated below the figure.

values of the critical parameters (and the 95% confidence bounds):

$$\nu = 2.658 \quad (2.612, 2.704), \tag{8}$$

$$y = -0.1511 \quad (-0.4307, 0.1284), \tag{9}$$

$$\Gamma_c = 0.9166 \quad (0.884, 0.9493), \tag{10}$$

and

$$\nu = 2.659 \quad (2.614, 2.704), \tag{11}$$

$$y = -0.07007 \quad (-0.1625, 0.02232), \tag{12}$$

$$\Gamma_c = 1.02 \quad (0.5593, 1.481). \tag{13}$$

Other fitting parameters Λ are presented in [50].

Discussion and outlook. We have studied the integer quantum Hall transition in the network model on the kagome lattice (NMKL). We have argued that the model should exhibit critical properties that are the same as the Chalker-Coddington model on a square lattice. We simulated the NKML numerically using the transfer matrix approach, and obtained a number of critical properties, including the value

$$\nu = 2.658 \pm 0.046 \tag{14}$$

for the localization length exponent. Our result is close to the Chalker-Coddington model value obtained in a number of recent papers [28–38], $\nu \sim 2.56$–2.62, though somewhat higher and with larger error bars. Let us discuss these deviations.

The error bars in Eq. (14) are larger by about a factor of 5 compared to the cited studies of the Chalker-Coddington model. The reason for the lower accuracy is absence of symmetry between the transmission and reflection amplitudes t and r that requires us to treat the critical point x_c on par with other fitting parameters, thereby increasing their total number. This is not necessary for the Chalker-Coddington model where the critical point is fixed by symmetry.

The average value $\nu = 2.658$ is higher than the values reported for the Chalker-Coddington model, but within the error bars the exponents for the network models on the kagome and the square lattices agree. We also performed fits for the NMKL including two irrelevant fields, and found that those fits resulted in values for ν that are lower than those with one irrelevant field. Unfortunately, the fits with two irrelevant fields inevitably produced larger error bars, since they involved larger numbers of fit parameters.

The main message is that our result (14) implies that the universality class of the transition in NMKL is the same as in the Chalker-Coddington model, in spite of the presence of nonzero, periodically distributed curvature. Such regular, nonfluctuating curvature background turns out to be irrelevant and does not change the critical behavior. In contrast, in our previous papers [43,44] we considered network models with geometric disorder in the form of random, fluctuating curvatures, indicating the presence of an effective 2D gravitational field in the system. Inclusion of a random curvature made the structurally disordered network models different from the flat Chalker-Coddington model, and resulted in the critical properties that were different for the two models, signifying the relevance of geometric disorder.

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