Rapid Communications

Particle-hole Pfaffian order in a translationally and rotationally invariant system

Chen Sun,[1,2] Ken K. W. Ma,[1] and D. E. Feldman[1]

[1]*Brown Theoretical Physics Center and Department of Physics, Brown University, Providence, Rhode Island 02912, USA*

[2]*School of Physics and Electronics, Hunan University, Changsha 410082, China*

(Received 6 April 2020; revised 6 September 2020; accepted 14 September 2020; published 24 September 2020)

The PH-Pfaffian (particle-hole Pfaffian) topological order has been proposed as a candidate order for the $\nu = 5/2$ quantum Hall effect. The PH-Pfaffian liquid is known to be the ground state in several coupled wire and coupled stripe constructions. No translationally and rotationally invariant models with the PH-Pfaffian ground state have been identified so far. By employing anyon condensation on top of a topological order, allowed in an isotropic system, we argue that the PH-Pfaffian order is possible in the presence of rotational and translational symmetries.

DOI: 10.1103/PhysRevB.102.121303

I. INTRODUCTION

The topological order on half-integer quantum Hall plateaus has been a subject of much debate [1,2]. There has long been tension between experiment [1,3] and numerics [4–6]. It increasingly appears that multiple topological orders are present in experimentally relevant systems. Indeed, numerical evidence exists for different topological orders on half-integer plateaus in GaAs [5–7] and graphene [8]. Some experiments [9,10] even hint at different topological orders at different magnetic fields on the $5/2$ plateau in GaAs. Such behavior differs profoundly from the intuition that builds on the properties of the simplest and best-understood quantum Hall state at $\nu = 1/3$, where the same Laughlin topological order [11] is believed to be present in a broad range of materials and parameters.

The difficulties with half-integer filling factors reflect a stronger role for composite-fermion (CF) interactions on half-integer plateaus than at most odd-denominator filling factors [12]. Indeed, a great majority of odd-denominator states can be understood as integer quantum Hall states of CFs. Such integer states are present even for noninteracting CFs, and their interaction does not affect qualitative features, such as possible topological orders. In contrast to this picture, noninteracting CFs would not form an incompressible liquid at a half-integer filling [12]. This agrees with the absence of the $1/2$ and $3/2$ plateaus in monolayer GaAs. At the same time, experimental evidence exists for CFs on the quantized $5/2$ plateau [13,14]. This suggests that the $5/2$ plateau forms due to CF interactions. The plateau can be explained by Cooper pairing of CFs [15]. The details of the topological order depend on the pairing channel: Different channels result in eight possible Abelian and eight possible non-Abelian orders [2].

Which one or ones are present in experimentally relevant systems? A preponderance of numerical evidence [1,5,6] points towards Pfaffian [16] and anti-Pfaffian [17,18] liquids in translationally invariant systems. A preponderance of experimental evidence [1,3] suggests the PH-Pfaffian (particle-hole Pfaffian) order [18–22] on the $5/2$ plateau in GaAs. A possible explanation of such discrepancy comes from disorder [20,23–25], inevitable in any sample, but ignored in all numerical studies until a very recent paper [26]. Weak disorder is not believed to affect topological order at $\nu = 1/3$. Strong disorder destroys the $1/3$ plateau. This behavior is the same as in the integer quantum Hall effect. At the same time, disorder can change the pairing channel in a superconductor [27]. This suggests that disorder may change the qualitative physics of the CF superconductor at $\nu = 5/2$. Recent theoretical work [23–25] does predict a complicated phase diagram in the presence of disorder with several topologically ordered phases and a gapless thermal metal. Note that a random potential is not necessary for the stabilization of the PH-Pfaffian liquid. Coupled wire constructions and a coupled stripe construction produce Hamiltonians with the PH-Pfaffian order in the ground state without any randomness [2,28,29]. The common feature of the disorder-based approach with those constructions consists in the absence of translational and rotational symmetry.

A possible lesson might be that the PH-Pfaffian order was impossible in uniform systems. Yet, it was suggested that it might be stabilized by sufficiently strong Landau level mixing (LLM) even in uniform systems [30,31]. If so, a translationally and rotationally invariant model should exhibit PH-Pfaffian order. In this Rapid Communication we argue that the PH-Pfaffian order does emerge in isotropic systems as a result of anyon condensation [32–34] on top of another topological order.

The model system is multicomponent. We argue that for appropriate microscopic interactions, the components may originate from different Landau levels. This makes our model different from constructions in which wave functions of various topological orders are localized in a single Landau level, and thus LLM is ignored. This difference is consistent with recent numerical results [35,36], which suggest that the PH-Pfaffian state loses its gap after projection into the lowest Landau level. It is also consistent with the symmetry-from-no-symmetry principle [20], which postulates that a particle-hole symmetric topological order is only

possible if the particle-hole symmetry is broken by LLM, disorder, or another mechanism or combination of mechanisms.

In what follows, we start with a review of the PH-Pfaffian topological order. We then observe that the edge structure of a PH-Pfaffian liquid can be obtained from a two-component system. One component is made of charged fermions and the other is made of neutral bosons. In the fourth section we argue that the two-component system possesses the PH-Pfaffian order in the bulk. In the final section we propose a scenario as to how such two-component model might be realized in a purely electronic system.

II. PH-PFAFFIAN ORDER

The anyons are labeled by their topological charge $t = 1$, σ, or ψ and the electric charge $ne/4$, where n is odd in the σ-sector and even otherwise. We will use the notation (t, n). The fusion rules are

$$\psi \times \psi = 1;\ \sigma \times \psi = \sigma;\ \sigma \times \sigma = 1 + \psi, \tag{1}$$

where 1 stays for vacuum and ψ is a Majorana fermion. The statistical phase, accumulated by an anyon of type (t_1, n_1) while making a full counterclockwise circle around an anyon of type (t_2, n_2) is

$$\phi = \phi_{\mathrm{nA}}(t_1, t_2, f) + \frac{\pi n_1 n_2}{4}, \tag{2}$$

where the non-Abelian phase ϕ_{nA} depends on the topological charges t_1 and t_2 and on the fusion channel f, Eq. (1). The non-Abelian phase is trivial, $\phi_{\mathrm{nA}} = 0$, if $t_1 = 1, t_2 = 1$, or $t_1 = t_2 = \psi$. For two σ particles, the non-Abelian phase depends on the fusion channel:

$$\phi_{\mathrm{nA}}(\sigma, \sigma, 1) = \pi/4, \quad \phi_{\mathrm{nA}}(\sigma, \sigma, \psi) = -3\pi/4. \tag{3}$$

Finally,

$$\phi_{\mathrm{nA}}(\sigma, \psi, \sigma) = \phi_{\mathrm{nA}}(\psi, \sigma, \sigma) = \pi. \tag{4}$$

The bulk statistics determines the edge Lagrangian density [20]:

$$L = \frac{2}{4\pi}\partial_x\phi_c(\partial_t - v_c\partial_x)\phi_c + i\psi(\partial_t + u\partial_x)\psi, \tag{5}$$

where ψ is a Majorana fermion and the charge mode ϕ_c sets the charge density $e\partial_x\phi_c/2\pi$ on the edge. An edge excitation from the sector (t, n) is created by the operator $t\exp(in\phi_c/2)$, where $t = 1, \sigma, \psi$ acts in the neutral Majorana sector with σ being the twist operator. The electron operator is $\psi\exp(2i\phi_c)$. Both the thermal and electrical conductances are one half of a quantum [19,20]. We include more details about the statistics in the PH-Pfaffian and related orders in the Supplemental Material [37].

III. MODEL: VIEW FROM THE EDGE

Our starting point is a two-component system. One component is a fractional quantum Hall liquid in the anti-Pfaffian state [17,18]. The other component is made of neutral bosons in the Laughlin state at the filling factor $\nu = 1/4$. Rotationally and translationally invariant models with those two orders in their ground states are known.

The anti-Pfaffian order is very similar to the PH-Pfaffian order. The classification of the excitations and their fusion rules are the same. Only a small difference exists in the braiding rules: the Abelian phase has the opposite sign compared to (2):

$$\phi = \phi_{\mathrm{nA}}(t_1, t_2, f) - \frac{\pi n_1 n_2}{4}. \tag{6}$$

The edge theory differs from (5) by the opposite propagation direction of ϕ_c and an additional charge integer mode ϕ_1 with the charge density $e\partial_x\phi_1/2\pi$. The edge Lagrangian density

$$L_{\mathrm{aPf}} = -\frac{2}{4\pi}\partial_x\phi_c(\partial_t + v_c\partial_x)\phi_c + i\psi(\partial_t + u\partial_x)\psi + \frac{1}{4\pi}\partial_x\phi_1(\partial_t - v_1\partial_x)\phi_1 + w\partial_x\phi_1\partial_x\phi_c. \tag{7}$$

Edge excitations are created by the same operators as in the PH-Pfaffian state. There are two electron operators: $\psi\exp(-2i\phi_c)$ and $\exp(i\phi_1)$. The operator $\exp(i\phi_n) = \exp(i[\phi_1 + 2\phi_c])$ creates a neutral fermionic excitation in the Majorana sector ψ. The electrical conductance is half a quantum, as in the PH-Pfaffian state. The thermal conductance is $-1/2$ of a quantum [17,18].

It will be convenient to switch from the variables ϕ_c and ϕ_1 to the neutral mode ϕ_n and the overall charge mode $\phi_\rho = \phi_1 + \phi_c$. The Lagrangian density becomes

$$L_{\mathrm{aPf}} = \frac{2}{4\pi}\partial_x\phi_\rho(\partial_t - v_\rho\partial_x)\phi_\rho + i\psi(\partial_t + u\partial_x)\psi - \frac{1}{4\pi}\partial_x\phi_n(\partial_t + v_n\partial_x)\phi_n + \tilde{w}\partial_x\phi_\rho\partial_x\phi_n. \tag{8}$$

The $\nu = 1/4$ Laughlin state is Abelian [38]. The phase accumulated by a fundamental anyon $\exp(ib)$ on a full counterclockwise circle around an identical anyon is $\pi/2$. The fusion of n fundamental anyons yields a composite anyon $\exp(inb)$. Such anyon accumulates the phase $mn\pi/2$ on a full circle about an anyon of type $\exp(imb)$. As a consequence, $\exp(2ib)$ are fermions. $\exp(4ib)$ is topologically trivial. The edge theory of the Laughlin state assumes the form

$$L_{\mathrm{B}} = \frac{4}{4\pi}\partial_x b(\partial_t - v_b\partial_x)b. \tag{9}$$

The electrical conductance of the neutral bosons is 0. The thermal conductance equals one quantum [39].

We now observe that the sums of the electric and thermal conductances of the bosonic liquid and the anti-Pfaffian liquid equal the electric and thermal conductances of the PH-Pfaffian liquid. This makes us expect that the PH-Pfaffian order should be present in a two-component system made of the anti-Pfaffian and bosonic Laughlin liquids. We start with demonstrating that the edge structure of the PH-Pfaffian liquid can be obtained from such two-component model as illustrated in Fig. 1.

We consider a two-component model with the following Lagrangian density on the edge:

$$L = L_{\mathrm{aPf}} + L_{\mathrm{B}} + u\partial_x\phi_n\partial_x b + U\cos(2\phi_n - 4b). \tag{10}$$

The cosine term is allowed in the action since it is topologically trivial and conserves the electric charge. For simplicity, we assume [40] that $\tilde{w} = 0$ in L_{aPf}. The results do not change

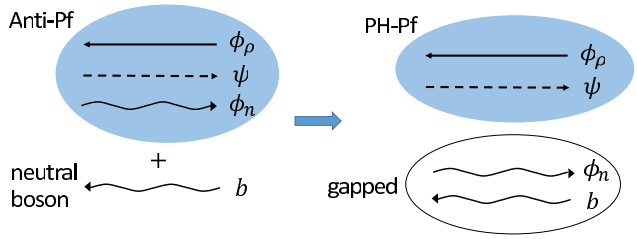

FIG. 1. Emergence of the PH-Pfaffian edge structure. Left panel: edge modes of two noninteracting layers with the anti-Pfaffian and Laughlin orders. Right panel: the counterpropagating modes ϕ_n and b are gapped out, and the remaining two modes exhibit the edge structure of a PH-Pfaffian liquid.

for a finite small $\tilde{w}$. The two counterpropagating modes b and ϕ_n are gapped out if the cosine term is relevant in the renormalization group sense. After introducing a new field $\phi_b = -2b$, the contribution to the Lagrangian density that depends on ϕ_n and ϕ_b becomes

$$L_{\mathrm{n,b}} = \frac{1}{4\pi}[\partial_x\phi_b(\partial_t - v_b\partial_x)\phi_b - \partial_x\phi_n(\partial_t + v_n\partial_x)\phi_n] - \frac{u}{2}\partial_x\phi_n\partial_x\phi_b + U\cos[2(\phi_n + \phi_b)]. \quad (11)$$

The stability of the edge requires $|\pi u| \leqslant \sqrt{v_b v_n}$.

The Lagrangian density $L_{\mathrm{n,b}}$ can be diagonalized by the transformation [38]:

$$\phi_b = \cosh\theta\,\tilde{\phi}_b + \sinh\theta\,\tilde{\phi}_n, \quad (12)$$

$$\phi_n = \sinh\theta\,\tilde{\phi}_b + \cosh\theta\,\tilde{\phi}_n, \quad (13)$$

$$\tanh 2\theta = -\frac{2\pi u}{v_b + v_n}. \quad (14)$$

In the new basis, the cosine term becomes

$$L_{\mathrm{tun}} = U\cos[2(\cosh\theta + \sinh\theta)(\tilde{\phi}_n + \tilde{\phi}_b)]. \quad (15)$$

Its scaling dimension can be deduced [41]:

$$\Delta = 4(\cosh 2\theta + \sinh 2\theta) = 4\sqrt{\frac{v_b + v_n - 2\pi u}{v_b + v_n + 2\pi u}}. \quad (16)$$

When $\Delta < 2$, L_{tun} is relevant and gaps out ϕ_b and ϕ_n. This happens for

$$\frac{3(v_b + v_n)}{10\pi} < u < \frac{\sqrt{v_n v_b}}{\pi}. \quad (17)$$

The remaining two gapless modes ϕ_ρ and ψ are described by the action identical to the PH-Pfaffian action (5).

IV. MODEL: VIEW FROM THE BULK

The action (10) is the key to the bulk model. Indeed, $\cos(2\phi_n - 4b)$ can be represented in the form $\hat{B}\hat{B} + \hat{B}^\dagger\hat{B}^\dagger$, where $\hat{B}$ creates an excitation $B = (\psi, 0)\exp(2ib)$. Such excitation is a product of two fermions and hence a boson. The edge action thus suggests to consider the condensation of bosons B. The condensation results in the confinement of many anyon types [32–34]. As we will see, the statistics of the remaining deconfined excitations is PH-Pfaffian. We argue that B is condensable in two ways: using algebraic theory of anyons and Refs. [42–48] in the Supplemental Material [37] and using the above edge construction in the end of this section.

Deconfined excitations braid trivially with B. Hence, the only nontrivial deconfined excitation of the Bose liquid is $\exp(2ib)$. The deconfined excitations of the anti-Pfaffian liquid are $(\psi, 2n)$ and $(1, 2n)$. The attachment of any number of bosons B does not change the superselection sector of an excitation. Thus, $\exp(2ib)$ and ψ can be identified.

What about deconfined anyons that combine topological excitations of the Bose and anti-Pfaffian subsystems? First, we can combine any number of deconfined excitations in the Bose and anti-Pfaffian sectors. This yields anyons of the types $(t, 2n)\exp(2mib)$, where $t = 1, \psi$. By attaching $(n - m)$ B particles, any such anyon can be reduced to the standard type $(t', 2n)\exp(2nib)$, where $t' = 1, \psi$ is not necessarily the same as t. In addition to products of deconfined excitations of the two subsystems, deconfined excitations exist in the σ sector: $(\sigma, 2n + 1)\exp([2m + 1]ib)$. Without loss of generality we can set $n = m$ since attaching $(n - m)$ bosons B changes $(\sigma, 2n + 1)\exp([2m + 1]ib)$ into $(\sigma, 2n + 1)\exp([2n + 1]ib)$. Thus, all superselection sectors can be labeled as $(t, n)\exp(inb)$.

Neither of those sectors splits. Indeed, only non-Abelian anyons can split and only if the fusion of such an anyon with its antiparticle contains orthogonal copies of the vacuum of the condensed phase [32]. One sees that this does not happen in our problem.

We will now observe that all deconfined anyons can be identified with excitations of a PH-Pfaffian liquid. We identify $(t, n)\exp(inb)$ with the (t, n) anyon of the PH-Pfaffian order. All fusion rules are satisfied after such identification. The non-Abelian part of the braiding phase (2) is also correct. The Abelian part of the mutual braiding phase of the anyons $(t_1, n_1)\exp(in_1 b)$ and $(t_2, n_2)\exp(in_2 b)$ is now the sum of the anti-Pfaffian contribution $-n_1 n_2\pi/4$ and the Laughlin contribution $n_1 n_2\pi/2$. This gives the correct PH-Pfaffian value.

The above discussion assumes that B is condensable. While this is plausible, can this be placed on a more rigorous footing? In addition to the discussion in the Supplemental material [37], we support this point with a coupled stripe construction (Fig. 2) in the spirit of Ref. [2]. The bulk anti-Pfaffian and Laughlin orders can be obtained in a system of narrow stripes with anti-Pfaffian and Laughlin edge modes, in which counterpropagating modes of neighboring stripes gap each other. The charge modes are gapped out by the operators that tunnel charge $2e$. Neutral modes are gapped out by operators that tunnel electrons and trivial bosons $\exp(4ib)$. We next add cosine terms $-A\cos(2b_L - \phi_{n,R})\cos(2b_R - \phi_{n,L})$ on each stripe, where the indices L and R show the right- and left-moving modes on the edges of the stripe. Such contribution creates trivial topological charge in each stripe. We also add interstripe tunneling between stripes i and $i + 1$ of the form $-A'\cos(4b_{i+1,L} - 2\phi_{n,i+1,R} - 4b_{i,R} + 2\phi_{n,i,L})$. We assume that the amplitudes A and A' are much greater than the amplitudes of any other tunneling terms. One sees that the system remains gapped in the bulk and the boson operator $\exp(2ib - i\phi_n)$ acquires a nonzero expectation value. This

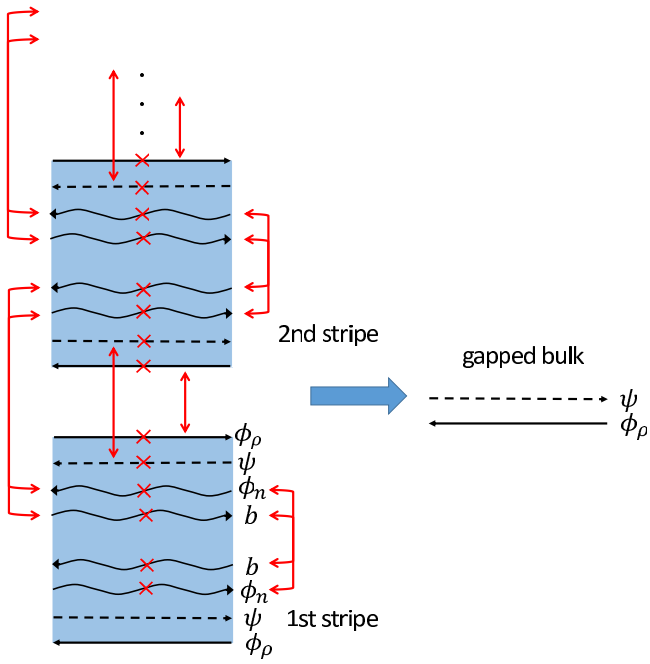

FIG. 2. A coupled-stripe construction of the PH-Pfaffian order. The red lines with arrows represent interactions which gap out the connected modes.

suggests that B can condense in an anisotropic system. Since condensability is a topological property, the condensation of B should also be possible in a rotationally and translationally invariant system.

V. CONCLUSIONS

The above model demonstrates that the PH-Pfaffian topological order can be obtained in a uniform system. All other known models [2,23–25,28,29] with that order break the translational and rotational symmetry either because of impurities or because the models consist of coupled wires or stripes. Note that crystal structure implies that quantum Hall effect systems are never exactly isotropic even in the absence of disorder, but this makes little difference at relevant electron densities.

Since our model combines fermions in the anti-Pfaffian state with neutral bosons, its most natural realization would come from cold atoms. The model may seem disconnected from the physics of the $5/2$ plateau in semiconductors, where only fermions are present. We propose a scenario that makes a connection with a purely fermionic system. We assume that electrons are present in four spin-resolved Landau levels. Electrons in one level exhibit the anti-Pfaffian order and form one of the two subsystems we need. The electrical conductance of the anti-Pfaffian subsystem is one half of a quantum. The other three partially and fully filled Landau levels contribute two quanta to the electrical conductance, as necessary for the total conductance of $5/2$. One Landau level is fully filled. The sum of the filling factors of the other two is 1. Thus, the number of the holes in one of those Landau levels equals the number of the electrons in the other. We assume that all holes from one level combine with the electrons from the other level to form neutral bosons. The bosons form the Laughlin $\nu = 1/4$ state, provided that their two-body interaction favors the relative angular momentum $+4$, where the plus emphasizes that only one sign of the angular momentum along the z axis is favored. An appropriate choice of the interaction between the bosons and the fermions in the anti-Pfaffian state yields the desired model system [37].

The boson interaction breaks the time-reversal symmetry. This property is not shared by the Coulomb interaction in realistic samples. The time-reversal symmetry is broken instead by the external magnetic field to which neutral bosons do not minimally couple. Even if the interaction with the magnetic field is the only contribution to the microscopic Hamiltonian that breaks the time-reversal symmetry, it is possible that additional symmetry-breaking interactions are generated in the effective low-energy Hamiltonian. Of course, it may well be that this does not happen for realistic Coulomb interactions. The point of our model is to show that the PH-Pfaffian order is possible without breaking the translational symmetry. More research is needed to understand if the PH-Pfaffian order could be stabilized in realistic semiconductor heterostructures in the absence of random impurities.

ACKNOWLEDGMENTS

We acknowledge a useful discussion with M. V. Milovanović. This work was supported by the J. Michael Kosterlitz Postdoctoral Fellowship at Brown University (C.S.), the Galkin Fellowship (K.K.W.M.), and by NSF Grant No. DMR-1902356.

[1] M. Heiblum and D. E. Feldman, in *Fractional Quantum Hall Effects: New Developments*, edited by B. I. Halperin and J. K. Jain (World Scientific, Singapore, 2020); Int. J. Mod. Phys. B **35**, 2030009 (2020).
[2] K. K. W. Ma and D. E. Feldman, Phys. Rev. B **100**, 035302 (2019).
[3] M. Banerjee, M. Heiblum, V. Umansky, D. E. Feldman, Y. Oreg, and A. Stern, Nature (London) **559**, 205 (2018).
[4] R. H. Morf, Phys. Rev. Lett. **80**, 1505 (1998).
[5] E. H. Rezayi, Phys. Rev. Lett. **119**, 026801 (2017).
[6] K. Pakrouski, M. R. Peterson, T. Jolicoeur, V. W. Scarola, C. Nayak, and M. Troyer, Phys. Rev. X **5**, 021004 (2015).
[7] W. Luo and T. Chakraborty, Phys. Rev. B **96**, 081108(R) (2017).
[8] Y.-H. Wu, T. Shi, and J. K. Jain, Nano Lett. **17**, 4643 (2017).
[9] W. Pan, A. Serafin, J. S. Xia, L. Yin, N. S. Sullivan, K. W. Baldwin, K. W. West, L. N. Pfeiffer, and D. C. Tsui, Phys. Rev. B **89**, 241302(R) (2014).
[10] N. Samkharadze, D. Ro, L. N. Pfeiffer, K. W. West, and G. A. Csáthy, Phys. Rev. B **96**, 085105 (2017).
[11] R. B. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983).
[12] J. K. Jain, *Composite Fermions* (Cambridge University Press, Cambridge, 2007).
[13] R. L. Willett, K. W. West, and L. N. Pfeiffer, Phys. Rev. Lett. **88**, 066801 (2002).

[14] M. S. Hossain, M. K. Ma, M. A. Mueed, L. N. Pfeiffer, K. W. West, K. W. Baldwin, and M. Shayegan, Phys. Rev. Lett. **120**, 256601 (2018).
[15] N. Read and D. Green, Phys. Rev. B **61**, 10267 (2000).
[16] G. Moore and N. Read, Nucl. Phys. B **360**, 362 (1991).
[17] M. Levin, B. I. Halperin, and B. Rosenow, Phys. Rev. Lett. **99**, 236806 (2007).
[18] S.-S. Lee, S. Ryu, C. Nayak, and M. P. A. Fisher, Phys. Rev. Lett. **99**, 236807 (2007).
[19] D. T. Son, Phys. Rev. X **5**, 031027 (2015).
[20] P. T. Zucker and D. E. Feldman, Phys. Rev. Lett. **117**, 096802 (2016).
[21] L. Fidkowski, X. Chen, and A. Vishwanath, Phys. Rev. X **3**, 041016 (2013).
[22] P. Bonderson, C. Nayak, and X.-L. Qi, J. Stat. Mech. (2013) P09016.
[23] D. F. Mross, Y. Oreg, A. Stern, G. Margalit, and M. Heiblum, Phys. Rev. Lett. **121**, 026801 (2018).
[24] C. Wang, A. Vishwanath, and B. I. Halperin, Phys. Rev. B **98**, 045112 (2018).
[25] B. Lian and J. Wang, Phys. Rev. B **97**, 165124 (2018).
[26] W. Zhu and D. N. Sheng, Phys. Rev. Lett. **123**, 056804 (2019).
[27] V. P. Mineev and K. V. Samokhin, *Introduction to Unconventional Superconductivity* (Gordon and Breach Science, New York, 1999).
[28] C. L. Kane, A. Stern, and B. I. Halperin, Phys. Rev. X **7**, 031009 (2017).
[29] Y. Fuji and A. Furusaki, Phys. Rev. B **99**, 035130 (2019).
[30] L. Antonić, J. Vučičević, and M. V. Milovanović, Phys. Rev. B **98**, 115107 (2018).
[31] S. Durdević and M. V. Milovanović, Phys. Rev. B **100**, 195303 (2019).
[32] F. J. Burnell, Annu. Rev. Condens. Matter Phys. **9**, 307 (2017).
[33] I. S. Eliëns, J. C. Romers, and F. A. Bais, Phys. Rev. B **90**, 195130 (2014).
[34] T. Neupert, H. He, C. von Keyserlingk, G. Sierra, and B. A. Bernevig, Phys. Rev. B **93**, 115103 (2016).
[35] A. C. Balram, M. Barkeshli, and M. S. Rudner, Phys. Rev. B **98**, 035127 (2018).
[36] R. V. Mishmash, D. F. Mross, J. Alicea, and O. I. Motrunich, Phys. Rev. B **98**, 081107(R) (2018).
[37] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevB.102.121303 for a detailed discussion of statistics in the PH-Pfaffian, anti-Pfaffian, and Laughlin orders, a discussion of condensability of boson B, and a toy Hamiltonian with a condensate of B in its ground state.
[38] X.-G. Wen, *Quantum Field Theory of Many-Body Systems: From the Origin of Sound to an Origin of Light and Electrons* (Oxford University Press, New York, 2004).
[39] C. L. Kane and M. P. A. Fisher, Phys. Rev. B **55**, 15832 (1997).
[40] $\tilde{w} = 0$ in the low-energy fixed point, describing a realistic disordered edge [17,18].
[41] J. von Delft and H. Schoeller, Ann. Phys. **7**, 225 (1998).
[42] A. Kitaev, Ann. Phys. **321**, 2 (2006).
[43] P. H. Bonderson, Ph.D. thesis, Caltech, 2007.
[44] D. E. Feldman and A. Kitaev, Phys. Rev. Lett. **97**, 186803 (2006).
[45] E. Rowell, R. Stong, and Z. Wang, Commun. Math. Phys. **292**, 343 (2009).
[46] T. H. Hansson, M. Hermanns, S. H. Simon, and S. F. Viefers, Rev. Mod. Phys. **89**, 025005 (2017).
[47] M. Greiter, *Mapping of Parent Hamiltonians: From Abelian and Non-Abelian Quantum Hall States to Exact Models of Critical Spin Chains* (Springer, New York, 2011).
[48] *BCS: 50 Years*, edited by L. N. Cooper and D. E. Feldman (World Scientific, Singapore, 2011).