Free fermionic and parafermionic quantum spin chains with multispin interactions

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We introduce a family of $Z(N)$ multispin quantum chains with a free fermionic ($N = 2$) or free parafermionic ($N > 2$) eigenspectrum. The models have $(p + 1)$ interacting spins ($p = 1, 2, \dots$), which are Hermitian in the $Z(2)$ (Ising) case and non-Hermitian for $N > 2$. We construct a set of mutually commuting charges that allows us to derive the eigenenergies in terms of the roots of polynomials generated by a recurrence relation of order $(p + 1)$. In the critical limit we identify these polynomials with certain hypergeometric polynomials $_{p+1}F_p$. Also in the critical regime, we calculate the ground-state energy in the bulk limit and verify that they are given in terms of the Lauricella hypergeometric series. The models with special couplings are self-dual and at the self-dual point show a critical behavior with a dynamical critical exponent $z_c = \frac{p+1}{N}$.

Introduction. Models known by the general name of free systems play a crucial role in condensed matter physics and in statistical mechanics of exactly integrable systems. The simplest and prototypical of these models is the quantum Ising model in a transverse field [1,2]. The Hamiltonian defined in a chain of L sites and free boundary conditions is given by

$$\mathcal{H}_I = -\sum_{i=1}^{L-1} \lambda_{2i} \sigma_i^z \sigma_{i+1}^z - \sum_{i=1}^{L} \lambda_{2i-1} \sigma_i^x, \tag{1}$$

where $\sigma_i^{x,z}$ are the standard spin-$\frac{1}{2}$ Pauli matrices attached at the sites ($i = 1, \dots, L$) and the couplings λ_j ($j = 1, \dots, 2L - 1$) play the role of local temperatures. The 2^L eigenvalues of this model can be obtained through a Jordan-Wigner transformation and it has the form

$$E_I = \pm \epsilon_1 \pm \epsilon_2 \pm \cdots \pm \epsilon_L, \tag{2}$$

where the pseudoenergies ϵ_i are functions of $\{\lambda_i\}$. The model is called free since all the 2^L values of E_I are obtained solely from the L pseudoenergies ϵ_i. In this case we have a free fermionic system since each pseudoenergy can appear with both signals as in a standard fermionic system.

A simple direct generalization of (1) with $Z(N)$ symmetry and having a free eigenspectrum was proposed in the late 1980s [3,4],

$$\mathcal{H}_B = -\sum_{i=1}^{L-1} \lambda_{2i} Z_i^\dagger Z_{i+1} - \sum_{i=1}^{L} \lambda_{2i-1} X_i, \tag{3}$$

where X_i and Z_i are obtained from the $Z(N)$ generalizations of the Pauli matrices satisfying the algebra

$$ZX = \omega XZ, \quad X^N = Z^N = 1, \quad Z^\dagger = Z^{N-1}, \quad \omega = e^{\frac{2i\pi}{N}}, \tag{4}$$

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and having the unique irreducible representation [5] in $\mathbb{C}^N$ given by

$$X_{j,k} = \delta_{j,k+1} \pmod{N}, \quad Z_{j,k} = \omega^{j-1} \delta_{j,k}. \tag{5}$$

The spectrum of (3) has the form

$$-E_B = \omega^{s_1} \epsilon_1 + \omega^{s_2} \epsilon_2 + \cdots + \omega^{s_L} \epsilon_L, \tag{6}$$

where $s_i \in \{0, 1, \dots, N-1\}$ and ϵ_i are given in terms of the roots of a given polynomial [3,4]. Clearly, the result (6) reduces to (2) when $N = 2$. Formula (6) was conjectured in Refs. [3,4] and proved in Refs. [6,7]. More recently, it was proved using raising and lowering parafermionic operators [8], generalizing the Clifford algebra method [9] for the free parafermionic system. The work [8] has boosted a number of papers on the spectral problem of (3) [10–15]. In analogy with the Ising chain, the model (3) is free in the sense that the quasienergies are independent of the choice of s_i, despite the model not being Hermitian for $N > 2$.

We recall that the Hamiltonian (3) can be derived from the transfer matrix of the so-called τ_2 model [16,17] under certain restrictions. For the general Hamiltonian arising from the τ_2 model, see, e.g., Refs. [6,10]. To the best of our knowledge, the Hamiltonian (3) or the general case derived from the τ_2 model are so far the only known models with a spectrum of the form (6) when $N > 2$.

The solution of the Ising chain involves the transformation of the Pauli operators in fermions by means of the Jordan-Wigner transformation. As a result, the Hamiltonian acquires a quadratic form in Fermi or Majorana operators. Similarly, using the Fradkin-Kadanoff transformation [18], the Hamiltonian (3) can be written in terms of parafermions [8,19]. However, the resulting Hamiltonian is not quadratic in the parafermions, and yet it has a free parafermionic spectrum.

Recently, a remarkably simple Hamiltonian with three-spin interactions that has a spectrum of the form (2) was introduced

[20]. It is given by

$$\mathcal{H}_F = -\sum_{i=1}^{L-2} \lambda_i \sigma_i^z \sigma_{i+1}^x \sigma_{i+2}^x. \tag{7}$$

The standard Jordan-Wigner transformation applied to (7) yields a Hamiltonian of order four in the Majorana fermions. Given this feature, it is quite surprising that it has a free spectrum.

The common feature of the Hamiltonians (1), (3), and (7) is that their energy density operators satisfy a very simple algebra [8,20] with M generators h_a, namely,

$$h_a h_{a+m} = \omega h_{a+m} h_a \quad \text{for} \quad 1 \leqslant m \leqslant p,$$
$$[h_a, h_b] = 0 \quad \text{for} \quad |a - b| > p, \quad h_a^N = \lambda_a^N, \tag{8}$$

where $p = 1$ for (1) and (3) and $N = p = 2$ for (7). In fact, the Hamiltonians (1), (3), and (7) can be written as

$$-\mathcal{H} = \sum_{i=1}^{M} h_i, \tag{9}$$

where $M = 2L - 1$,

$$h_{2i-1} = \lambda_{2i-1} X_i \quad \text{for} \quad i = 1, \dots, L,$$
$$h_{2i} = \lambda_{2i} Z_i Z_{i+1}^\dagger \quad \text{for} \quad i = 1, \dots, L-1, \tag{10}$$

for (1) and (3) and $M = L - 2$,

$$h_i = \lambda_i \sigma_i^z \sigma_{i+1}^x \sigma_{i+2}^x \quad \text{for} \quad i = 1, \dots, L-2, \tag{11}$$

for (7).

The algebra (8) for $p = 1$ belongs to the realm of generalized Clifford algebras (see, e.g., Refs. [21–28]).

The aim of this Rapid Communication is to argue that Hamiltonians of the form (9) whose energy density operators satisfy (8) for arbitrary positive integer p and N have a free parafermionic spectrum of the form (6). In order to do that, we follow the strategy in Ref. [20]. The idea is that, using (8), one can easily build a set of conserved charges associated with (9) and use them to construct a generating function. This generating function satisfies an important product formula (20), which is similar to that satisfied by the transfer matrix of the τ_2 model [7]. We remark that this procedure is independent of the representation of the algebra (8). Nevertheless, it is easy to find representations of (8) using (4) which leads to interesting quantum spin chains with $(p + 1)$ multispin interactions. For example, $h_i = \lambda_i Z_i Z_{i+1} \cdots Z_{i+p-1} X_{i+p}$ produces a Hamiltonian acting in the vector space $(\mathbb{C}^N)^{\otimes M+p}$,

$$\mathcal{H} = -\sum_{i=1}^{M} h_i = -\sum_{i=1}^{M} \lambda_i Z_i Z_{i+1} \cdots Z_{i+p-1} X_{i+p}. \tag{12}$$

The commuting charges and their generating function. The simple form of the algebra (8) allows one to construct explicitly a set of mutually commuting charges, the Hamiltonian being one of them. As in Ref. [20] for the case $p = 2$, the commuting charge operators $H_M^{(l)}$ are obtained by summing all the products $h_{j_1} h_{j_2} \cdots h_{j_l}$ formed by l generators h_i in the set $\{h_1, \dots, h_M\}$ whose indices difference $|j_i - j_{i\pm1}|$ is larger

than $(p + 1)$, namely,

$$H_M^{(0)} = \mathbb{1}, \quad H_M^{(1)} = -H = \sum_{j=1}^{M} h_j,$$

$$H_M^{(2)} = \sum_{j_1=1}^{M} \sum_{j_2=j_1+p+1}^{M} h_{j_1} h_{j_2}, \quad \dots,$$

$$H_M^{(\bar{M})} = \sum_{j_1=1}^{M} \sum_{j_2=j_1+p+1}^{M} \cdots \sum_{j_{\bar{M}}=j_{\bar{M}-1}+p+1}^{M} h_{j_1} h_{j_2} \cdots h_{j_{\bar{M}}}, \tag{13}$$

where $\bar{M}$ is the largest number of commuting h_j we can obtain from the set $\{h_1, \dots, h_M\}$, and it is given by the integer part of $\frac{M+p}{M+1}$, i.e., $\bar{M} = \lfloor \frac{M+p}{p+1} \rfloor$. We notice that the Hamiltonian (9) is the charge $-H_M^{(1)}$.

The generating function of the charges is defined as

$$G_M(u) = \sum_{l=0}^{\bar{M}} (-u)^l H_M^{(l)}, \tag{14}$$

and we claim that it satisfies the fundamental properties

$$[G_M(u), G_M(v)] = [H_M^{(l)}, G_M(u)] = [H_M^{(l)}, H_M^{(l')}] = 0, \tag{15}$$

for arbitrary u and v and $l, l' = 0, 1, \dots, \bar{M}$.

Actually, the commutativity relations (15) follow from the fact that the generating function $G_M(u)$ and the charges $H_M^{(l)}$ satisfy recurrence relations, namely,

$$H_M^{(l)} = H_{M-1}^{(l)} + h_M H_{M-(p+1)}^{(l-1)}, \tag{16}$$

$$G_M(u) = G_{M-1}(u) - u h_M G_{M-(p+1)}(u), \tag{17}$$

with the initial conditions $H_M^{(0)} = \mathbb{1}$, $H_M^{(l)} = 0$ for $l < 0$ or $M \leqslant 0$, and $G_M(u) = \mathbb{1}$ for $M \leqslant 0$. The proof [29] of (15) follows from the fact that

$$\beta_M^{(l)}(u) = [H_M^{(l)}, G_M(u)] \tag{18}$$

satisfies a recurrence relation with the same form as (17), namely,

$$\beta_M^{(l)}(u) = \beta_{M-1}^{(l)}(u) - u h_M \beta_{M-(p+1)}^{(l)}(u). \tag{19}$$

We remark that (15) has been proved for $N = p = 2$ in Ref. [20] using the fact the generating function admits certain factorizations and it was also proved for $p = 1$, arbitrary N in Ref. [8].

Product formula for the generating function. The free spectrum nature for the Hamiltonian (9) emerges from the following conjecture: The generating function $G_M(u)$ satisfies the product relation,

$$\tau_M(u) \equiv G_M(u) G_M(\omega u) \cdots G_M(\omega^{N-1} u) = P_M^{(p)}(u^N) \mathbb{1}, \tag{20}$$

where $P_M^{(p)}(z)$ is a polynomial of degree $\bar{M}$ in $z = u^N$. Formula (20) has the same form as the one satisfied by the transfer matrix of the τ_2 model [7]. When $N = 2$, (20) is known as the inversion relation [30], since $G_M(-u)$ is proportional to the inverse of $G_M(u)$.

We state the conjecture (20) after extensive analytical and numerical computation. Analytically, we can verify (20) up to $N = \bar{M} = 4$ for $p = 1, 2, 3$. Numerically, we checked it

for small lattice sizes ($\bar{M} \leqslant 8, N \leqslant 6$), with Hamiltonians of dimension up to 6×10^5. Although it is a general conjecture for arbitrary values of N and p, we were able to prove it, after a lengthy calculation, for $N = 2, 3$ and arbitrary values of p [29]. The key point is to show that $\tau_M(u)$ also satisfies the recurrence relation

$$\tau_M(u) = \tau_{M-1}(u) - u^N h_M^N \tau_{M-(p+1)}(u), \quad (21)$$

with $\tau_M(u) = 1$ for $M \leqslant 0$. This expression gives $\tau_1(u) = 1 - u^N h_1^N$ and by iterating (21) we obtain that $\tau_M(u)$ is a polynomial in u^N, as stated in (21). We see from (21) that the coefficients of $P_M^{(p)}(u^N)$ depend on the values $h_i^N = \lambda_i^N$ of the couplings in the Hamiltonians (9) and (12). The proof of (22), similarly as happened with (20), is a consequence of the recurrence Eqs. (17) and (18).

Relation (22) implies the recurrence relation for the polynomial (hereafter we use the variable $z = u^N$ for the polynomial $P_M^{(p)}$),

$$P_M^{(p)}(z) = P_{M-1}^{(p)}(z) - z\lambda_M^N P_{M-(p+1)}^{(p)}(z), \quad (22)$$

with the initial condition $P_l^{(p)}(u^N) = 1$ for $l \leqslant 0$. Comparing (22) with the recursion relation (17), we identify the dependence in $\{\lambda_i^N\}$ of the coefficients $C_M^{(l)}$ in the expansion

$$P_M^{(p)}(z) = \sum_{l=0}^{\bar{M}} (-z)^l C_M^{(l)}. \quad (23)$$

They are obtained by replacing $h_j \to \lambda_j^N$ in (13), i.e.,

$$C_M^{(l)} = \sum_{j_1=1}^{M} \sum_{j_2=j_1+p+1}^{M} \cdots \sum_{j_l=j_{l-1}+p+1}^{M} \lambda_{j_1}^N \lambda_{j_2}^N \cdots \lambda_{j_l}^N, \quad (24)$$

for $l = 0, 1, \ldots, \bar{M}$.

In the special case where $\lambda_j^N = 1$ ($j = 1, \ldots, M$), $C_M^{(l)}$ corresponds to the number of ways to put l particles with an excluded volume of $(p+1)$ lattice units in a lattice of size M, that is, it is given by the binomial coefficient,

$$C_M^{(l)} = \binom{M - p(l-1)}{l} = \frac{(M - p(l-1))!}{(M - p(l-1) - l)!l!}, \quad (25)$$

and we can identify $P_M^{(p)}(z)$ as the generalized hypergeometric polynomial $_{p+1}F_p$ [31],

$$P_M^{(p)}(z) = {}_{p+1}F_p \left(\begin{matrix} -\frac{M+p}{p+1} & -\frac{M+p-1}{p+1} & -\frac{M+p-2}{p+1} & \cdots & -\frac{M}{p+1} \\ -\frac{M+p}{p} & -\frac{M+p-1}{p} & \cdots & -\frac{M+1}{p} \end{matrix} ; \frac{(p+1)^{p+1}}{p^p} z \right)$$

$$= \sum_{l=0}^{\bar{M}} (-1)^l \binom{M - p(1-l)}{l} z^l. \quad (26)$$

It is worth noticing that for the critical quantum Ising chain, where $p = 1$, the polynomial (26) is related to the Chebyshev polynomial of the second type, i.e., $P_M^{(1)}(z) = z^{\frac{M+1}{2}} U_{M+1}(\frac{1}{2z^{\frac{1}{2}}})$.

The solution of the product formula and the spectrum of the Hamiltonian. The eigenspectrum of the Hamiltonian (9) $-H_M^{(1)}$ as well as of all the charges $H_M^{(l)}$ are obtained from the product formula (20). Owing to (15), by applying (20) to a given eigenfunction of $G_M(u)$ with eigenvalue $\Lambda(u)$, we obtain

$$\Lambda(u) \cdots \Lambda(\omega^{N-1}u) = P_M^{(p)}(u^N) = \prod_{i=1}^{\bar{M}} \left(1 - \frac{u^N}{z_i}\right), \quad (27)$$

where z_i are the roots of (23). Relation (27) can be solved in terms of z_i,

$$\Lambda_M(u) = \prod_{i=1}^{\bar{M}} \left(1 - u\frac{\omega^{s_i}}{z_i^{1/N}}\right) = \prod_{i=1}^{\bar{M}} (1 - u\omega^{s_i}\epsilon_i), \quad (28)$$

where $\epsilon_i = z_i^{-1/N}$ and $s_i \in \{0, 1, \ldots, N-1\}$. We verified that all the roots of z_i of (23) are distinct. Therefore, there are $N^{\bar{M}}$ distinct eigenvalues of the generating function. We can expand (28) in terms of its roots,

$$\Lambda_M^{\{s_i\}}(u) = \sum_{l=0}^{\bar{M}} (-1)^l e_l(\omega^{s_1}\epsilon_1, \ldots, \omega^{s_{\bar{M}}}\epsilon_{\bar{M}})u^l, \quad (29)$$

where $e_l(\omega^{s_1}\epsilon_1, \ldots, \omega^{s_{\bar{M}}}\epsilon_{\bar{M}})$ is the lth elementary symmetric polynomial in $\{\omega^{s_1}\epsilon_1, \ldots, \omega^{s_{\bar{M}}}\epsilon_{\bar{M}}\}$ [32].

Since u is arbitrary, applying (14) to a eigenvector with eigenvalue $\Lambda_M^{\{s_i\}}(u)$ yields the eigenvalues $q_{\{s_i\}}^{(l)}$ of the charges $H_M^{(l)}$, i.e.,

$$q_{\{s_i\}}^{(l)} = e_l(\omega^{s_1}\epsilon_1, \ldots, \omega^{s_{\bar{M}}}\epsilon_{\bar{M}}), \quad l = 1, \ldots, \bar{M}. \quad (30)$$

In particular, the Hamiltonian (9) has the the spectrum with the same form as (6),

$$-E^{\{s_i\}} = q_{\{s_i\}}^{(1)} = \omega^{s_1}\epsilon_1 + \omega^{s_2}\epsilon_2 + \cdots + \omega^{s_L}\epsilon_{\bar{M}}. \quad (31)$$

The spectrum of (9) is thus determined by the roots of the polynomial (23) [33]. For general values of the couplings $\{\lambda_i\}$ we expect a quite rich phase diagram (see Ref. [20] for the case $N = p = 2$).

Criticality. Instead of considering the most general models we are going to restrict ourselves to the case where we have only two values for the coupling constants (1 and λ). We consider the model where we alternate the block of p couplings with the values (1 and λ), i.e.,

$$\lambda_{kp+j} = \begin{cases} 1, & k \text{ odd,} \\ \lambda, & k \text{ even,} \end{cases} \quad (32)$$

for $k = 0, 1, 2, \ldots$ and $j = 1, \ldots, p$.

For $p = 1$ we recover (3) with the couplings $\lambda_{2i-1} = 1, \lambda_{2i} = \lambda$, and, for example, for $p = 3$ we have

$$-\mathcal{H}(\lambda) = h_1 + h_2 + h_3 + \lambda(h_4 + h_5 + h_6)$$
$$+ h_7 + h_8 + h_9 + \cdots . \quad (33)$$

This model is gapped (noncritical) at the limiting values $\lambda = 0$ and $\lambda \to \infty$, since the model reduces to a set of disjoint p-interacting systems. On the other hand, by shifting the variables $h_i \to h_{i+p}$, we see that apart from the boundary conditions the eigenenergies of the Hamiltonians with couplings λ and $1/\lambda$ are related, i.e.,

$$\mathcal{H}(\lambda) = \lambda \mathcal{H}\left(\frac{1}{\lambda}\right), \quad (34)$$

and thus the model is self-dual, and in the case we have a single transition from $\lambda = 0$ and $\lambda \to \infty$, it is the phase transition point $\lambda = \lambda_c = 1$. This is the case for $p = 1$ [13] and $N = p = 2$ [20], and for general values of p, as we argue below.

In order to show the criticality at $\lambda = \lambda_c = 1$ we extend the interesting procedure introduced in Ref. [20] for the case $p = 2$, that enables us to find the roots of (26) in the bulk limit $M \to \infty$.

We consider lattice sizes that are multiples of $(p + 1)$, i.e., $M = (p + 1)\bar{M}$. In this case, the recurrence relation (22) for the polynomials (26) gives

$$P_M^{(p)}(z) = P_{M-(p+1)}^{(p)}(z) - \sum_{i=1}^{p+1} \binom{p+1}{i} z^i P_{M-i(p+1)}^{(p)}(z),$$

with the condition $P_{(p+1)l}^{(p)}(z) = 0$ for $l \leqslant 0$.

If we have a root z_k of the $M = (p + 1)\bar{M}$ polynomial, i.e., $P_{(p+1)\bar{M}}^{(p)}(z_k) = 0$, this root should satisfy the $\bar{M} + 1$ coupled difference equations of order $p + 1$,

$$P_{(p+1)(\bar{M}-j)}^{(p)}(z_k) = \sum_{i=0}^{p+1} \binom{p+1}{i} z_k^i P_{(p+1)[\bar{M}-j-(i-1)]}^{(p)}(z_k), \quad (35)$$

with $j = 1, \ldots, \bar{M}$ and initial conditions

$$P_{(p+1)\bar{M}}^{(p)}(z_k) = P_{(p+1)l}^{(p)}(z_k) = 0, \quad l = 1, \ldots, p. \quad (36)$$

The ansatz

$$P_{(p+1)(\bar{M}-j)}^{(p)}(z_k) = \mu^{\bar{M}+p-1-j} z_k^{1-j}, \quad j = 1, \ldots, \bar{M}, \quad (37)$$

gives the characteristic equation

$$(\mu + 1)^{p+1} - z_k^{-1}\mu^p = (\mu - \mu_1) \cdots (\mu - \mu_{p+1}) = 0. \quad (38)$$

The general solution of (35) is then given by a combination of the $(p + 1)$ roots μ_i of the characteristic equation

$$P_{(p+1)(\bar{M}-j)}^{(p)}(z_k) = \sum_{i=1}^{p+1} A_i z_k^{1-j} \mu_i^{\bar{M}+p-1-j}, \quad (39)$$

where A_i ($i = 1, \ldots, p + 1$) are going to be fixed by (36), i.e.,

$$\sum_{i=1}^{p+1} A_i \mu_i^{\bar{M}+p-1} = 0, \quad \sum_{i=1}^{p+1} A_i \mu_i^{p-l-1} = 0, \quad l = 1, \ldots, p,$$
$$(40)$$

where we set $A_{p+1} = 1$.

We could not find a closed form for the roots of the polynomial (38). However, surprisingly, one can find at least one simple root. It is parametrized by the up to now free parameter $\mathfrak{p}_k$,

$$\mu_1 = \frac{\sin\left(\frac{p\mathfrak{p}_k}{p+1}\right)}{\sin\left(\frac{\mathfrak{p}_k}{p+1}\right)} e^{i\mathfrak{p}_k}, \quad z_k^{-1} = \frac{\sin^{p+1}(\mathfrak{p}_k)}{\sin\left(\frac{\mathfrak{p}_k}{p+1}\right) \sin^p\left(\frac{p\mathfrak{p}_k}{p+1}\right)}. \quad (41)$$

Then, using Viète's formulas [34],

$$e_i(\mu_1, \mu_2, \ldots, \mu_{p+1}) = (-1)^i\left[\binom{p+1}{i} - z_k^{-1}\delta_{i,1}\right], \quad (42)$$

for $i = 1, \ldots, p + 1$, we can relate all the other roots to $\mathfrak{p}_k$. By means of (41), the quasiparticle energies in (31) can be written as

$$\epsilon_k = \frac{\sin^{\frac{p+1}{N}}(\mathfrak{p}_k)}{\sin^{\frac{1}{N}}\left(\frac{\mathfrak{p}_k}{p+1}\right) \sin^{\frac{p}{N}}\left(\frac{p\mathfrak{p}_k}{p+1}\right)}, \quad k = 1, \ldots, \bar{M}. \quad (43)$$

Note that the second Eq. (40) fixes the amplitudes A_i in terms of the roots $\mu_i = \mu_i(\mathfrak{p}_k)$. The first equation in (40) then gives a quantization condition for $\mathfrak{p}_k$, which, except for $p = 1$, does not seem to have a closed solution. For $p = 1$, one has

$$\mathfrak{p}_k = \frac{\pi}{\bar{M}+1}k, \quad k = 1, \ldots, \bar{M}, \quad (44)$$

such that the density of roots in the limit $\bar{M} \to \infty$ is $\frac{\Delta\mathfrak{p}_k}{\Delta k} = \frac{\pi}{\bar{M}}$. Based on numerical checks we conjecture that this result generalizes to arbitrary $p > 1$, including the case $p = 2$ considered in Ref. [20].

The ground-state energy per site in the bulk limit. The ground-state energy of the Hamiltonians given by (9) with general p is obtained by taking all the roots ϵ_k with $s_i = 0$ in (31). From the conjecture for the density of roots in the $\bar{M} \to \infty$ limit,

$$\frac{\Delta\mathfrak{p}_k}{\Delta k} = \frac{\pi}{\bar{M}} = \frac{(p+1)\pi}{M}, \quad (45)$$

we have

$$e_\infty^{(p)} \equiv -\frac{E_0}{M} = -\frac{1}{M}\sum_{k=1}^{\bar{M}} \epsilon_k = -\frac{1}{(p+1)\pi}\int_0^\pi \epsilon(\mathfrak{p})d\mathfrak{p}. \quad (46)$$

By means of the change of variables $\sin\left(\frac{\mathfrak{p}_k}{p+1}\right) = \sin\left(\frac{\pi}{p+1}\right)\sqrt{t}$, the integral (46) can be identified as an integral of Euler type in $(p - 1)$ variables which is the integral representation of the Lauricella hypergeometric series $F_D^{(p-1)}$ [35]. The cases $p = 1, 2, 3$ reduce to other known functions. For example, we have

$$e_\infty^{(1)} = -\frac{2^{\frac{2}{N}-1}\Gamma\left(\frac{1}{N}+\frac{1}{2}\right)}{\sqrt{\pi}\left(\frac{1}{N}+1\right)}, \quad (47)$$

$$e_\infty^{(2)} = -\frac{3^{\frac{3}{N}+\frac{1}{2}}\Gamma\left(\frac{3}{N}+1\right)}{2^{\frac{2}{N}+2}\sqrt{\pi}\Gamma\left(\frac{3}{N}+\frac{3}{2}\right)} {}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{1}{N}+\frac{1}{2} \\ \frac{3}{N}+\frac{3}{2}\end{matrix}; \frac{3}{4}\right), \quad (48)$$

$$e_\infty^{(3)} = -\frac{2^{\frac{8}{N}-\frac{3}{2}}\Gamma\left(\frac{4}{N}+1\right)}{3^{\frac{3}{N}}\sqrt{\pi}\Gamma\left(\frac{4}{N}+\frac{3}{2}\right)}$$
$$\times F_1\left(\frac{1}{2}; \frac{1}{2}-\frac{2}{N}, \frac{3}{N}; \frac{4}{N}+\frac{3}{2}; \frac{1}{2}, \frac{2}{3}\right), \quad (49)$$

TABLE I. Ground-state energy per site $-e_\infty^{(p)}$ and dynamical critical exponent z_c for the models (9) and (12), for some values of N and p. The third column shows the extrapolated ($\bar{M} \to \infty$) results obtained from the roots of (26) up to $\bar{M} = 500$. The exact results (47)–(50) are shown in the fourth column. The estimator $z_M^{(p)}$ (see text) and the exact results for the dynamical critical exponent are shown in the last two columns.

N	p	$-e_\infty^{(p)}$: roots	$-e_\infty^{(p)}$: exact	$z_M^{(p)}$	z_c
2	1	0.63661977	0.63661977	1.0000000	1
2	2	0.50000000	0.50000000	1.4999998	3/2
2	3	0.42441318	0.42441318	1.9999994	2
2	4	0.37500000	0.37500000	2.4999996	5/2
3	1	0.56604660	0.56604668	0.6666667	2/3
3	2	0.41349667	0.41349667	0.9999999	1
3	3	0.33333331	0.33333333	1.3333331	4/3
3	4	0.28302332	0.28302334	1.6666663	5/3
4	1	0.53935231	0.53935260	0.5000000	1/2
4	2	0.38137983	0.38137988	0.7499999	3/4
4	3	0.30010545	0.30010544	0.9999998	1
4	4	0.25000000	0.25000000	1.2499997	5/4

where F_1 is the Appel function,

$$e_\infty^{(4)} = -\frac{5^{\frac{5}{N}} \sin\left(\frac{\pi}{5}\right) \Gamma\left(\frac{5}{N}+1\right)}{2^{\frac{8}{N}+1} \sqrt{\pi} \Gamma\left(\frac{5}{N}+\frac{3}{2}\right)}$$
$$\times F_D^{(3)}\left(\frac{1}{2}; \frac{1}{2}+\frac{2}{N}, -\frac{5}{N}, \frac{4}{N}; \frac{5}{N}+\frac{3}{2}; x_1, x_2, x_3\right), \quad (50)$$

where $F_D^{(3)}$ is the Lauricella function with three variables at $x_1 = \frac{1}{2+\frac{2}{\sqrt{5}}}$, $x_2 = \frac{2}{3+\sqrt{5}}$, and $x_3 = \frac{1}{1+\frac{1}{\sqrt{5}}}$. We note that for $N = 2$, the integrals converge to

$$e_\infty^{(p)} = -\frac{\Gamma\left(\frac{1}{2}+\frac{p}{2}\right)}{\Gamma\left(1+\frac{p}{2}\right)} \quad \text{for} \quad N = 2. \quad (51)$$

The numerical solutions for the roots of (26) are simple to obtain numerically. In the third column of Table I we present the values of $e_\infty^{(p)}$ obtained directly from the roots up to $\bar{M} = 500$. The results are the extrapolated ones. The agreement with the exact results obtained from the integral (46), shown in the fourth column of Table I, is remarkable.

The first excited state E_1 is obtained by the addition of all the roots in (31) with $s_i = 0$ except for the largest one ($s_{\bar{M}} = 1$), where $\mathfrak{p}_k = \mathfrak{p}_{\bar{M}} = \pi - \frac{a}{M}$, with a being a harmless

constant. The real part of the energy gap behaves, as $M \to \infty$,

$$\Delta_M^{(p)} = E_1 - E_0 = (1-\omega)\epsilon(\mathfrak{p}_{\bar{M}}) \approx \left(\frac{a}{M}\right)^{z_c}, \quad (52)$$

where $z_c = (p+1)/N$ is the dynamical critical exponent. The case $N = 2$ and $p = 1$ gives the dynamical exponent of the conformally invariant quantum Ising chain ($z_c = 1$), and the case $N = p = 2$ recovers the result $z_c = 3/2$ obtained in Ref. [20]. For the sake of illustration, we also show in the fifth column of Table I the extrapolated results of the estimator $z_M^{(p)} = -\log\left(\Delta_M^{(p)}\right)/\log(M)$ of the dynamical critical exponent for some values of N and p. These results were obtained from the numerical solution of the largest root of the polynomial (26). The agreement with the predicted result $z_c = (p+1)/N$ (last column) is also remarkable.

The result (52) then implies that the Hamiltonians (9) with the couplings (32) and general values of p undergo a continuous phase transition at $\lambda = \lambda_c = 1$ with a dynamical critical exponent $z_c = (p+1)/N$, also shown in the last column of Table I.

Discussion. We have introduced a family of integrable quantum spin chains with multispin interactions and which have a free fermionic or parafermionic spectrum. We believe that many interesting directions of investigation can be pursued. First, it would be important to prove the conjectures we have made, in particular the product formula (20). Next, one should try to construct the so-called parafermionic operators which satisfy a generalized Clifford algebra [8,10–12,20]. This is one step towards the computation of correlation functions for this family of models. It is also very interesting to compute the entanglement entropy, initially for $N = 2$ (in this case, the Hamiltonian is Hermitian) and arbitrary p, and study the role of the polynomials (23) along the lines of Ref. [36]. Another problem is to study the random coupling case. We also think it is worth exploiting the connections between the algebra (8) and the algebras of Onsager and of Temperley-Lieb, as well as trying to consider the models within the quantum inverse scattering method. Going beyond the free boundary case considered here, it is challenging to consider the Hamiltonians with periodic boundary conditions (see Ref. [37] for $p = 1$, $N = 2$); in this case, the product formula (20) is no longer valid, and needs to be generalized.

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[32] The kth elementary symmetric polynomial in n variables $\{x_1, x_2, \ldots, x_n\}$ is defined by $e_k(x_1, x_2, \ldots, x_n) = \sum_{1 \leqslant j_1 < j_2 < \cdots < j_k \leqslant n} x_{j_1} x_{j_2} \cdots x_{j_k}$ for $k = 0, 1, \ldots, n$.

[33] Note that all the levels may be degenerate depending on the representation of the h_i in (9). For the representation (12) we can in fact find [29] several additional charges in involution with $H_M^{(l)}$ ($l = 1, \ldots, M$).

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