

## Symmetry-protected topological phases beyond groups: The $q$ -deformed Affleck-Kennedy-Lieb-Tasaki model

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We argue that the  $q$ -deformed spin-1 Affleck-Kennedy-Lieb-Tasaki (AKLT) Hamiltonian should be regarded as a representative of a symmetry-protected topological phase. Even though it fails to exhibit any of the standard symmetries known to protect the Haldane phase, it still displays all characteristics of this phase: fractionalized spin- $\frac{1}{2}$  boundary spins, nontrivial string order, and—when using an appropriate definition—a twofold degeneracy in the entanglement spectrum. We trace these properties back to the existence of an  $SO_q(3)$  quantum group symmetry and speculate about potential links to discrete duality symmetries. We expect our findings and methods to be relevant for the identification, characterization, and classification of other symmetry-protected topological phases with nonstandard symmetries.

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### I. INTRODUCTION

The Haldane phase of antiferromagnetic spin-1  $SU(2)$  quantum spin chains is one of the prototypes of symmetry-protected topological (SPT) phases [1–3]. It exhibits a unique ground state, an excitation gap, a twofold degeneracy in the bipartite entanglement spectrum and, at least when  $SO(3)$  spin-rotation symmetry is preserved, fractional boundary spins as well as nontrivial string order. While these facts have only been verified numerically for the spin-1 Heisenberg model, the associated Affleck-Kennedy-Lieb-Tasaki (AKLT) model [4,5] provides an alternative representative of the same phase in which all these properties can be established with full analytical rigor and where the existence of nontrivial string order can be linked to the breaking of a hidden  $\mathbb{Z}_2 \times \mathbb{Z}_2$  symmetry [4–8]. This is due to the fact that the ground state of the AKLT Hamiltonian has a simple representation as a matrix product state (MPS) [4,5,9,10]. (See also Refs. [11–13] for a general discussion of MPS and finitely correlated states.)

In view of the simple construction and intriguing properties of the AKLT model it is no surprise that variations of this construction have been applied to other symmetries, notably quantum group deformations of the  $SU(2)$  spin-rotation symmetry [14–19]. We will call the resulting spin-1 model which is based on a  $U_q[su(2)]$  quantum group symmetry the  $q$ AKLT model. From the construction of the  $q$ AKLT Hamiltonian as the parent Hamiltonian of an MPS it is evident that the model exhibits fractionalized boundary spins. Moreover, it has been known for a very long time that the model has nontrivial string order which is linked to the breaking of a generalized duality-type  $\mathbb{Z}_2 \times \mathbb{Z}_2$  symmetry [18]. Its entanglement properties [20] and correlation functions [16] have also been studied in great detail.

However, while these properties are all well known, at least among experts, a link between the  $q$ AKLT model and SPT

phases has yet to be established [21]. As we will discuss, this is (presumably) related to the extensive breaking of standard group symmetries and, in addition, to the puzzling lack of degeneracies in its entanglement spectrum [20] that would be expected in the presence of boundary spins and quantum group symmetry. With this paper we aim to close this gap. We explain why and in which sense the quantum group symmetry  $U_q[su(2)]$  is capable of protecting the nontrivial topology of the  $q$ AKLT model and we revisit the entanglement calculations, showing that the existing literature has been more concerned about implementing periodic boundary conditions than about actually enforcing the quantum group symmetry. We expect that our findings will trigger a systematic search for other SPT phases with nonstandard symmetries and that our methods will be helpful in identifying, characterizing, and classifying them.

### II. THE $q$ AKLT HAMILTONIAN AND ITS SYMMETRIES

The  $q$ AKLT model is defined on the Hilbert space of spin-1 quantum spins. It may be expressed in terms of standard  $SU(2)$  spins  $\vec{S}_i$  and in that formulation the  $q$ AKLT Hamiltonian reads [14,16]

$$\begin{aligned} \mathcal{H} = & b \sum_j \left\{ c \vec{S}_j \cdot \vec{S}_{j+1} + \left[ \vec{S}_j \cdot \vec{S}_{j+1} \right. \right. \\ & + \frac{1}{2}(1-c)(q+q^{-1}-2)S_j^z S_{j+1}^z \\ & \left. \left. + \frac{1}{4}(1+c)(q-q^{-1})(S_{j+1}^z - S_j^z) \right]^2 \right. \\ & + \frac{1}{4}c(1-c)(q+q^{-1}-2)^2 (S_j^z S_{j+1}^z)^2 \\ & \left. + \frac{1}{4}c(1+c)(q-q^{-1})(q+q^{-1}-2)S_j^z S_{j+1}^z \right. \\ & \left. \times (S_{j+1}^z - S_j^z) \right\} \end{aligned}$$

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$$\begin{aligned}
 & + \frac{1}{4}(c-3) \left[ \left[ c-1 + \frac{1}{2}(1+c)^2 \right] S_{j+1}^z S_j^z \right. \\
 & + 2 \left[ c - \frac{1}{8}(1+c)^2 \right] \left[ (S_{j+1}^z)^2 + (S_j^z)^2 \right] \\
 & \left. + (c-1) + \frac{1}{2}c(q^2 - q^{-2})(S_{j+1}^z - S_j^z) \right\}, \quad (1)
 \end{aligned}$$

with  $c = 1 + q^2 + q^{-2}$  and  $b = [c(c-1)]^{-1}$ . Except as stated otherwise, we will consider open boundary conditions or an infinite chain. The Hamiltonian is Hermitian for real values of  $q$ . To keep the exposition simple,  $q > 0$  will be assumed throughout this paper. We note that both parameters  $b$  and  $c$  are invariant under the substitution  $q \rightarrow q^{-1}$ .

The standard AKLT model (obtained by setting  $q = 1$ ) has a certain number of symmetries that are known to protect its topological properties [1]. These symmetries are (i)  $SO(3)$  spin-rotation symmetry [22], (ii) its  $\mathbb{Z}_2 \times \mathbb{Z}_2$  subgroup of  $\pi$  rotations around the principal axes, (iii) inversion of the chain, and (iv) time-reversal symmetry, an antilinear symmetry implementing the transformation  $\vec{S} \mapsto -\vec{S}$ .

In contrast, the Hamiltonian (1) is anisotropic for  $q \neq 1$  and hence breaks  $SU(2)$  spin-rotation symmetry. The only continuous symmetry that is left is a  $U(1)$  symmetry by rotations around the  $z$  axis. The anisotropy also breaks the  $\mathbb{Z}_2 \times \mathbb{Z}_2$  symmetry group of  $\pi$  rotations. Finally, inversion symmetry and time-reversal symmetry are broken by the term  $(S_j^z)^2 S_{j+1}^z - S_j^z (S_{j+1}^z)^2$ . To summarize, all the discrete symmetries known to protect the Haldane phase are broken explicitly. According to the general classification [1–3], the Hamiltonian (1) should thus not be regarded as residing in an SPT phase.

As will be discussed in the following section, the Hamiltonian (1) is the parent Hamiltonian of an MPS that is constructed using the representation theory of  $\mathcal{U}_q[su(2)]$ . As such, it is naturally invariant under the action of this quantum group that defines a  $q$  deformation of the  $su(2)$  spin algebra [23]. This deformation is defined in terms of  $q$ -spin generators  $\vec{S}$  satisfying the relations

$$[S^z, S^\pm] = \pm S^\pm \quad \text{and} \quad [S^+, S^-] = \frac{q^{2S^z} - q^{-2S^z}}{q - q^{-1}}. \quad (2)$$

For the spin-1 representation these commutation relations are satisfied with the identification

$$S^z = S^z \quad \text{and} \quad S^\pm = \sqrt{\frac{q + q^{-1}}{2}} S^\pm. \quad (3)$$

Even though these expressions are just rescaled versions of the standard spin operators, there is another important signature of the  $q$  deformation: The action of the  $q$ -spins  $\vec{S}$  on tensor products is not simply given by adding up the  $q$ -spins for individual factors but rather by applying a so-called coproduct. For two tensor factors this coproduct reads

$$\begin{aligned}
 \Delta(S^z) &= S^z \otimes \mathbb{I} + \mathbb{I} \otimes S^z, \\
 \Delta(S^\pm) &= S^\pm \otimes q^{S^z} + q^{-S^z} \otimes S^\pm, \quad (4)
 \end{aligned}$$

and the action on multiple tensor factors (such as the whole spin chain) is obtained by iterating this action appropriately. We note that the standard action of  $su(2)$  is only recovered in the limit  $q \rightarrow 1$ . A summary of useful informa-

tion about  $\mathcal{U}_q[su(2)]$  and its representations can be found in Appendixes A and B. The main result of this paper is that the quantum group symmetry just sketched (or rather an incarnation adapted to the fact that we are dealing with a spin-1 chain) is capable of protecting the topological properties of the  $q$ AKLT state.

It needs to be emphasized though that this symmetry is only present when the chain is (semi-)infinite or considered with open boundary conditions. If periodic boundary conditions are used, as is the case in most of the literature on this subject, the Hamiltonian and also its ground state are *not* invariant under  $\mathcal{U}_q[su(2)]$ . This symmetry is only restored if specific twisted boundary conditions are used. These facts are well known in the community working on quantum group invariant integrable models (see, e.g., Refs. [24–27]), and will also play an important role for establishing our main result. However, to keep the present discussion focused we will restrict our attention to open boundary conditions and infinite systems.

We would also like to stress that the breaking of some of the original discrete symmetries when deforming away from  $q = 1$  is rather mild. The  $\pi$  rotations around the  $x$  and  $y$  axis as well as inversion and time reversal still leave the Hamiltonian (1) invariant if they are accompanied by the transformation  $q \rightarrow q^{-1}$ . However, these operations are not symmetries, strictly speaking, but rather duality transformations, mapping one model to another one that is physically equivalent. Such duality symmetries are not covered by the standard classification of SPT phases [1–3] and hence require another approach [28]. We defer this interesting question to future research and content ourselves with a detailed discussion of the continuous  $\mathcal{U}_q[su(2)]$  symmetry. Judging from the special case  $q = 1$  we expect that continuous and discrete duality-type symmetries lead to the same  $\mathbb{Z}_2$  classification of SPT phases in the present situation.

### III. THE $q$ AKLT STATE

The starting point of the  $q$ AKLT construction is a state (or rather a set of four states) that can be represented in the form of an MPS. More precisely, we define

$$|q\text{AKLT}\rangle_{\alpha\beta} = (B_1 B_2 \cdots B_L)_{\alpha\beta} \quad (5)$$

for a finite chain of length  $L$  with open boundary conditions. In this expression,  $\alpha, \beta = \pm 1/2$  denote the degrees of freedom associated with a left and right spin- $\frac{1}{2}$  boundary spin and the rest of the product is a mixed matrix/tensor product of matrices  $B_i$  that all have the form [16] (see Appendix C)

$$B_i = \sqrt{\Omega} \begin{pmatrix} -\frac{q^{-1}}{\sqrt{q+q^{-1}}} |0\rangle_i & q^{\frac{1}{2}} |+\rangle_i \\ -q^{-\frac{1}{2}} |-\rangle_i & \frac{q}{\sqrt{q+q^{-1}}} |0\rangle_i \end{pmatrix}, \quad (6)$$

where  $\Omega = (q + q^{-1}) / (1 + q^2 + q^{-2})$ . The index  $i$  indicates on which site the physical states  $|0\rangle$  and  $|\pm\rangle$  live. The  $q$ AKLT state (5) arises from a valence-bond construction involving the spin-1 representation as the physical spin and two spin- $\frac{1}{2}$  spins as auxiliary spins [16] (see Appendix C). The associated transfer matrix has a nondegenerate (in modulus) eigenvalue 1 and this translates into the existence of a mass gap (see, e.g., Ref. [11]).

For completeness we note that the  $q$ AKLT state should be defined by

$$|q\text{AKLT}\rangle = \text{tr}(q^{2S^z} B_1 B_2 \cdots B_L) \quad (7)$$

when considering closed boundary conditions. This is explained in Appendix E based on equivariance properties of the MPS tensor  $B$  that are derived in Appendix D. The insertion of the twist  $q^{2S^z}$  acting on the auxiliary space guarantees invariance under the quantum group  $\mathcal{U}_q[su(2)]$  in this setting. As will be discussed in the following section, it is precisely this twist (and other closely related ones) that enable us to derive entanglement properties that support the interpretation of the  $q$ AKLT model as a representative of an SPT phase. We expect that closing the chain requires nonlocal terms in the Hamiltonian (1) if the  $\mathcal{U}_q[su(2)]$  symmetry is meant to be preserved. At least this is known to be the case in integrable quantum spin chains with quantum group symmetry [25,26].

#### IV. THE ENTANGLEMENT SPECTRUM REVISITED

The entanglement properties of the  $q$ AKLT state were discussed in Ref. [20]. These considerations were based on an MPS with periodic boundary conditions which, as we have pointed out, is not invariant under the action of  $\mathcal{U}_q[su(2)]$ . It is thus no surprise that the results do not reflect the degeneracies appropriately that would be expected as a result of  $\mathcal{U}_q[su(2)]$  invariance and the presence of fractionalized spin- $\frac{1}{2}$  boundary spins.

To avoid finite-size effects we work in an infinite chain where the MPS can be interpreted as a translation invariant iMPS [13,29,30]. The MPS tensor (6) is in right-canonical form and normalized, i.e., it satisfies  $BB^\dagger = \mathbb{I}$  where the product is taken both over the physical and the auxiliary index [13]. It may be checked that the alternative MPS tensor  $A = \Lambda B \Lambda^{-1}$  is in left-canonical form, i.e., it satisfies  $A^\dagger A = \mathbb{I}$ , where we introduced  $\Lambda = \text{diag}(q^{-\frac{1}{2}}, q^{\frac{1}{2}})/\sqrt{q+q^{-1}}$ . The importance of these tensors lies in the fact that the two sets of semi-infinite states,

$$|\alpha\rangle_L = (\cdots A_{-3} A_{-2} A_{-1})_{\bullet\alpha}, \quad (8)$$

$$|\alpha\rangle_R = (B_0 B_1 B_2 \cdots)_{\alpha\bullet}, \quad (9)$$

with  $\alpha = \pm 1/2$  are orthonormal on the left and right semi-infinite Hilbert space, respectively. For this reason the expression

$$|q\text{AKLT}\rangle_\infty = \sum_{\alpha=\pm 1/2} \Lambda_\alpha |\alpha\rangle_L \otimes |\alpha\rangle_R \quad (10)$$

is a Schmidt decomposition and permits to read off the (nondegenerate) entanglement spectrum  $\epsilon_\alpha = -\log \Lambda_\alpha^2$  and entanglement entropy  $S_{\text{EE}} = -\sum_\alpha \Lambda_\alpha^2 \log \Lambda_\alpha^2$  from the diagonal matrix  $\Lambda$ . We note in passing that this result precisely corresponds to the entanglement present in the normalized  $\mathcal{U}_q[su(2)]$  singlet,

$$|\text{singlet}\rangle = (q+q^{-1})^{-\frac{1}{2}} (q^{\frac{1}{2}}|\uparrow\downarrow\rangle - q^{-\frac{1}{2}}|\downarrow\uparrow\rangle). \quad (11)$$

This could of course be expected since the latter is precisely what is used to describe the singlet bonds in the valence-bond construction of the MPS. It is clear that this state has lower

entanglement than a Bell state, at least for generic values of  $q$ . However, the two spins are still fully entangled in the sense that they form a singlet with respect to the quantum group symmetry, so the states are always entangled, regardless of the value of  $q$ , and all coefficients are completely fixed up to normalization.

This type of entanglement can be captured by a  $q$ -deformed definition of the reduced density matrix. Following Ref. [31] we define

$$\rho_R^{(q)} = \text{tr}_L(q^{-2S_L^z} \rho), \quad (12)$$

where  $\rho$  is the density matrix associated with  $|q\text{AKLT}\rangle_\infty$  and  $S_L^z$  corresponds to the action of  $S^z$  on the left part of the chain which is traced out. The MPS tensor satisfies the equivariance property

$$q^{-2S^z} \triangleright B = q^{-2S^z} B q^{2S^z}, \quad (13)$$

where the symbol  $\triangleright$  on the left-hand side denotes an action on the physical space and the conjugation on the right-hand side acts on the virtual spins by means of  $\tilde{S} = \vec{S}$  where  $\vec{S}$  are the standard spin operators in the spin- $\frac{1}{2}$  representation. Using the (trivial) coproduct for  $S^z$  it can then be shown that

$$q^{-2S_L^z} |q\text{AKLT}\rangle_\infty = \sum_{\alpha=\pm 1/2} \Lambda_\alpha q^{2\alpha} |\alpha\rangle_L \otimes |\alpha\rangle_R, \quad (14)$$

i.e., the action of  $q^{-2S^z}$  can be pushed to the auxiliary level. The  $q$ -deformed entanglement spectrum thus reads

$$\epsilon_\pm^{(q)} = -\log[\Lambda_{\pm\frac{1}{2}}^2 q^{\pm 1}] = \log(q+q^{-1}), \quad (15)$$

and shows a twofold degeneracy which arises from the presence of virtual fractionalized spin- $\frac{1}{2}$  boundary spins.

Given the  $q$ -deformed reduced density matrix it is then straightforward, using again the equivariance property (13), to calculate the associated  $q$ -deformed entanglement entropy [31] which is given by

$$S_{\text{EE}}^{(q)} = -\text{tr}_R(q^{2S_R^z} \rho_R^{(q)} \log \rho_R^{(q)}) = \log(q+q^{-1}). \quad (16)$$

We note that the result is just the logarithm of the so-called quantum dimension of the spin- $\frac{1}{2}$  representation describing the virtual fractionalized boundary spin. As will be shown in the next section, the full degeneracy of the  $q$ -deformed entanglement spectrum and the reduction of the  $q$ -deformed entanglement entropy to the logarithm of the quantum dimension are easily verified to generalize to singlet bonds between arbitrary spin- $S$  representations of  $\mathcal{U}_q[su(2)]$ .

#### V. ENTANGLEMENT OF SINGLET BONDS BETWEEN HIGHER SPINS

In the previous sections we focused on the  $q$ AKLT state for the spin-1 representation of  $\mathcal{U}_q[su(2)]$ . This state is obtained from a valence-bond construction involving two spin- $\frac{1}{2}$  auxiliary spins. It is straightforward to generalize this construction to spin- $S$  auxiliary spins, resulting in a spin- $2S$  analog of the  $q$ AKLT state [32]. Just as for the ordinary  $q$ AKLT state, the correlation functions and entanglement properties of these states have been discussed in great detail [33–35]. As these entanglement considerations were based on periodic boundary

conditions which are not compatible with invariance under  $\mathcal{U}_q[su(2)]$ , we now revisit this issue from an iMPS perspective.

To keep notation simple, let us consider the singlet bond between two spin- $S$  representations which serves as a higher-spin model for the Schmidt decomposition (10). Using the explicit action (B1) of the quantum group generators it may easily be verified that the normalized singlet state can be written as

$$|\text{singlet}\rangle = \frac{1}{\sqrt{\dim_q(S)}} \sum_{m=-S}^S (-q)^{-m} |m\rangle \otimes |-m\rangle, \quad (17)$$

where the quantum dimension  $\dim_q(S)$  has been defined in Eq. (B3). If  $\rho$  denotes the associated density matrix, then the  $q$ -deformed reduced density matrix in this state is given by

$$\rho_R^{(q)} = \text{tr}_L[q^{2S_L} \rho] = \frac{1}{\dim_q(S)} \mathbb{I}. \quad (18)$$

We see that the resulting  $q$ -deformed entanglement spectrum exhibits a full degeneracy of its  $\dim(S)$   $q$ -deformed entanglement energies

$$\epsilon_m^{(q)} = \log \dim_q(S), \quad (19)$$

where  $m = -S, \dots, S$ . It is then also straightforward to determine the associated  $q$ -deformed entanglement entropy which is given by

$$S_{EE}^{(q)} = \text{tr}_R[q^{-2S_R} \rho_R^{(q)} \log \rho_R^{(q)}] = \log \dim_q(S). \quad (20)$$

We recognize that the  $q$ -deformed entanglement entropy precisely captures the quantum dimension of the two auxiliary spins forming the singlet bond.

## VI. CLASSIFICATION

Let us finally adopt a slightly more general perspective. It is known that any gapped ground state  $|\psi\rangle$  can be well approximated by means of an MPS [36]. With  $\mathcal{U}_q[su(2)]$  symmetry and integer physical spins there are two distinct classes of MPS, just as for the standard  $su(2)$  case. This is due to the fact that the representation theory of  $\mathcal{U}_q[su(2)]$  at real values  $q \neq 0$  precisely mimics the representation theory of  $su(2)$ , including labeling and dimensions of irreducible representations and tensor product decompositions [37].

In particular, integer physical spins can only arise from either two integer or two half-integer auxiliary spins. For  $su(2)$ , all of these representations lift to  $SU(2)$  while only integer spin representations lift to  $SO(3) = SU(2)/\mathbb{Z}_2$ . In contrast, half-integer spins are only projective representations of  $SO(3)$  since they have a nontrivial action of the central subgroup  $\mathbb{Z}_2 \subset SU(2)$ . This representation of the center  $\mathbb{Z}_2$  can be interpreted as a topological invariant [38]. As discussed in more detail in Ref. [39], similar statements hold true for the quantum group  $\mathcal{U}_q[su(2)]$  which allows us to define two associated mathematical structures  $SU_q(2)$  and  $SO_q(3)$  which should be interpreted as distinct exponentiated versions of  $\mathcal{U}_q[su(2)]$ .

Just as for the undeformed case, the entanglement in MPS with integer auxiliary spins can be removed while this is not the case for half-integer auxiliary spins if we insist on the

preservation of  $SO_q(3)$  symmetry. We thus expect a  $\mathbb{Z}_2$  classification of  $\mathcal{U}_q[su(2)]$ -invariant quantum spin chains based on integer spins, the  $q$ AKLT model being a representative of the nontrivial phase [40].

We recall from Sec. V that singlet bonds between two spin- $S$  representations lead to a  $(2S + 1)$ -dimensional degeneracy in the  $q$ -deformed entanglement spectrum. For half-integer  $S$  this degeneracy is even while it is odd for integer spins. Just as for the ordinary Haldane phase [8] there is thus a characteristic entanglement signature of the topologically nontrivial phase which is protected by  $SO_q(3)$  symmetry.

## VII. CONCLUSIONS AND OUTLOOK

We have presented overwhelming evidence that the  $q$ AKLT model (with  $q > 0$ ) should be regarded as a representative of another type of SPT phase, protected by the  $q$ -deformed symmetry  $SO_q(3)$ . Even though the  $q$ AKLT state on an infinite chain shows no degeneracy in the standard midcut entanglement spectrum, a twofold degeneracy is recovered when defining the reduced density matrix with an appropriate quantum trace. This statement remains true even if the entanglement in the chain is very low (i.e., for  $q \ll 1$  or  $q \gg 1$ ). In addition, the state exhibits fractionalized spin- $\frac{1}{2}$  boundary spins and nontrivial string order, as already found in earlier studies [18].

Our findings open many directions for further investigations. It has been shown recently that the  $q$ AKLT model can be obtained from the standard AKLT model by means of a procedure called ‘‘Witten’s conjugation’’ [41]. It would be interesting to understand whether this mathematical procedure may also be used to infer conclusions about the topological properties of the  $q$ AKLT state.

Also, some of the models of recent physical interest involve quantum group symmetries with  $q$  a root of unity. This is, for instance, the case for the anyonic chains that have been introduced in Ref. [42] and for the abstract classification of topological field theories that arise from intertwiner dynamics [43]. In both cases, AKLT-like states are known to exist. It would thus be interesting to investigate whether our results and methods carry over to the case  $|q| = 1$  and, in particular, roots of unity where the representation theory of  $\mathcal{U}_q[su(2)]$  becomes considerably more intricate [37]. Obviously, it would also be natural to revisit higher-spin instances of the  $q$ AKLT model [32–35].

However, the most important and far-reaching question is whether there are other kinds of generalized symmetries that are capable of protecting topological order in one-dimensional (1D) or even higher-dimensional systems. In a companion paper [39] we show that the ideas presented in this paper readily generalize to spin chains with arbitrary quantum group symmetry  $\mathcal{U}_q[\mathfrak{g}]$  (with real  $q \neq 0$ ), where  $\mathfrak{g}$  is a finite-dimensional simple Lie algebra such as  $su(N)$ ,  $so(N)$ , or  $sp(2N)$ . These results of Ref. [39] extend the classification of Ref. [38] which has been established for simple Lie groups.

A similar and hence natural class of symmetries deserving further investigation are elliptic quantum groups, i.e., two-parameter deformations of Lie algebras. On general grounds one would expect arbitrary Hopf- $*$  algebras to be good

candidates for generalized symmetries, potentially with additional restrictions on their structure.

In this context one should also ask about the role of the discrete duality symmetries that were discussed in the main text and that involve the transformation  $q \rightarrow q^{-1}$ . Are these capable of protecting topological phases by themselves and, if yes, how does the resulting classification relate to the one obtained with respect to the continuous quantum group? For the undeformed case these questions were answered in Refs. [44,45].

Since even the examples of generalized symmetries that have been discussed here and in Ref. [39] are not amenable to the standard definition of projective representations, an alternative mathematical framework will need to be developed to describe the topological invariants in full generality. It is likely that this framework will employ tools from noncommutative geometry, the natural generalization of geometric objects to an abstract algebraic setting.

Finally, thinking of implications of our findings beyond one dimension it should be noted that matrix product operators played a significant role in the treatment of two-dimensional topological phases [46,47]. Our findings may help to generalize some of these considerations.

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#### APPENDIX A: THE QUANTUM GROUP $\mathcal{U}_q[su(2)]$

There exist different conventions for the quantum group  $\mathcal{U}_q[su(2)]$  and hence it is useful to be explicit about the convention we use. Our definition follows Ref. [37] even though it should be noted that what we call  $\mathcal{U}_q[su(2)]$  is called  $\check{\mathcal{U}}_q(sl_2)$  in that book.

The quantum group  $\mathcal{U}_q[su(2)]$  carries the structure of a Hopf algebra and defines a  $q$  deformation of the Lie algebra  $su(2)$  which is recovered as  $q \rightarrow 1$ .  $\mathcal{U}_q[su(2)]$  has an algebra structure that is encoded in the commutation relations (2) and a compatible coalgebra structure that is defined in Eq. (4) in terms of a coproduct  $\Delta : \mathcal{U}_q[su(2)] \rightarrow \mathcal{U}_q[su(2)] \otimes \mathcal{U}_q[su(2)]$ . We note that the first two relations of Eq. (2) can be written as

$$q^{\alpha S^z} \mathbb{S}^{\pm} q^{-\alpha S^z} = q^{\pm\alpha} \mathbb{S}^{\pm}. \quad (\text{A1})$$

From a practical perspective the coproduct permits us to define the notion of tensor product representations. To complete the characterization of  $\mathcal{U}_q[su(2)]$  we also need to define the unit  $\eta : \mathbb{C} \rightarrow \mathcal{U}_q[su(2)]$ , the counit  $\epsilon : \mathcal{U}_q[su(2)] \rightarrow \mathbb{C}$ , and the antipode  $S : \mathcal{U}_q[su(2)] \rightarrow \mathcal{U}_q[su(2)]$ . The latter is given by

$$S(\mathbb{S}^z) = -\mathbb{S}^z, \quad S(\mathbb{S}^{\pm}) = -q^{\pm 1} \mathbb{S}^{\pm}. \quad (\text{A2})$$

The other functions have the form  $\epsilon \equiv 0$  (on the generators  $\mathbb{S}^z$  and  $\mathbb{S}^{\pm}$ ) and  $\eta(1) = \mathbb{I}$ .

Since quantum spin chains are defined on a Hilbert space we also need to introduce a suitable notion of Hermitian conjugation. This is captured by a Hopf-\* structure. With our choice of coproduct and for real values of  $q$  the Hopf-\* structure is given by

$$(\mathbb{S}^z)^* = \mathbb{S}^z, \quad (\mathbb{S}^{\pm})^* = \mathbb{S}^{\mp}. \quad (\text{A3})$$

We note that a different choice of coproduct (as is sometimes preferred in the mathematics literature) would lead to different and physically rather unnatural expressions for the Hermitian conjugation.

#### APPENDIX B: REPRESENTATIONS OF $\mathcal{U}_q[su(2)]$

The representation theory of  $\mathcal{U}_q[su(2)]$  for real values of  $q \neq 0$  very much mimics the well-known representation theory of  $su(2)$  [37]. All finite-dimensional representations are fully reducible. The quantum group has irreducible representations  $\mathcal{V}_j$  that are labeled by a spin  $j = 0, \frac{1}{2}, 1, \dots$  and have dimension  $2j + 1$ . Moreover, the decomposition of a tensor product  $j_1 \otimes j_2$  precisely corresponds to the well-known decomposition for  $su(2)$ . In view of the nontrivial (and  $q$ -dependent) coproduct, concrete expressions for Clebsch-Gordan coefficients are different though.

The representation  $\mathcal{V}_j$  is spanned by (orthonormal) vectors  $|m\rangle$  with  $m = -j, \dots, j$  on which the generators of  $\mathcal{U}_q[su(2)]$  act by

$$\begin{aligned} \mathbb{S}^z |m\rangle &= m |m\rangle, \\ \mathbb{S}^+ |m\rangle &= \sqrt{[j-m]_q [j+m+1]_q} |m+1\rangle, \\ \mathbb{S}^- |m\rangle &= \sqrt{[j+m]_q [j-m+1]_q} |m-1\rangle. \end{aligned} \quad (\text{B1})$$

These expressions make use of so-called  $q$  numbers  $[x]_q$  that are defined via

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}} \xrightarrow{q \rightarrow 1} x. \quad (\text{B2})$$

An important invariant of the representation  $\mathcal{V}_j$  is the quantum dimension  $\dim_q(j)$  that is defined by

$$\dim_q(j) = \text{tr}_j(q^{2S^z}) = [2j+1]_q \quad (\text{B3})$$

and approaches the standard dimension  $\dim(j) = 2j + 1$  in the limit  $q \rightarrow 1$ . In the context of the  $q$ AKLT model one naturally encounters the representations  $\mathcal{V}_0$ ,  $\mathcal{V}_{\frac{1}{2}}$ , and  $\mathcal{V}_1$  which describe singlet bonds, auxiliary spins, and physical spins, respectively.

#### APPENDIX C: CONSTRUCTION OF THE MPS

In the main text we used an expression for the MPS that is different from others that can be found in the literature. We therefore include the simple derivation here. For the construction of the MPS tensor we look at one physical site which comprises two auxiliary spins plus the left auxiliary spin of its neighbor to the right. We thus consider the tensor product  $\mathcal{V}_{\frac{1}{2}} \otimes \mathcal{V}_{\frac{1}{2}} \otimes \mathcal{V}_{\frac{1}{2}}$  and start by inducing a singlet in the right two

factors. This is done by means of the map  $I_0 : \mathcal{V}_0 \rightarrow \mathcal{V}_{\frac{1}{2}} \otimes \mathcal{V}_{\frac{1}{2}}$ ,

$$I_0(1) = \frac{1}{\sqrt{q+q^{-1}}} (q^{\frac{1}{2}}|\uparrow\downarrow\rangle - q^{-\frac{1}{2}}|\downarrow\uparrow\rangle). \quad (\text{C1})$$

Up to normalization this yields the two states

$$|\alpha\rangle (q^{\frac{1}{2}}|\uparrow\downarrow\rangle - q^{-\frac{1}{2}}|\downarrow\uparrow\rangle) = q^{\frac{1}{2}}|\alpha\uparrow\rangle|\downarrow\rangle - q^{-\frac{1}{2}}|\alpha\downarrow\rangle|\uparrow\rangle,$$

where  $\alpha \in \{\uparrow, \downarrow\}$ . On these states we act with the projector onto the  $S = 1$  component in the first two factors which can easily be confirmed to be given by

$$P_1 = |+\rangle\langle\uparrow\uparrow| + |-\rangle\langle\downarrow\downarrow| + \frac{q^{-\frac{1}{2}}|0\rangle\langle\uparrow\downarrow| + q^{\frac{1}{2}}|0\rangle\langle\downarrow\uparrow|}{\sqrt{q+q^{-1}}}.$$

Writing the resulting states in matrix form in the standard basis of  $\mathcal{V}_{\frac{1}{2}}$  we find

$$g = \begin{pmatrix} -\frac{q^{-1}}{q+q^{-1}}|0\rangle & \frac{q^{\frac{1}{2}}}{\sqrt{q+q^{-1}}}|+\rangle \\ -\frac{q^{-\frac{1}{2}}}{\sqrt{q+q^{-1}}}|-\rangle & \frac{q}{q+q^{-1}}|0\rangle \end{pmatrix}. \quad (\text{C2})$$

This expression turns out to be right canonical but not normalized. Including the correct normalization leads to the MPS tensor (6) used in the main text.

#### APPENDIX D: EQUIVARIANCE OF THE MPS TENSOR

The MPS tensor defined in Eq. (6) satisfies the equivariance relations

$$\begin{aligned} \mathbb{S}^z \triangleright B &= \mathbb{S}^z B - B \mathbb{S}^z, \\ q^{\alpha \mathbb{S}^z} \triangleright B &= q^{\alpha \mathbb{S}^z} B q^{-\alpha \mathbb{S}^z}, \\ \mathbb{S}^{\pm} \triangleright B &= \mathbb{S}^{\mp} B q^{-\mathbb{S}^z} - q^{\mp 1} q^{-\mathbb{S}^z} B \mathbb{S}^{\mp}, \end{aligned} \quad (\text{D1})$$

as can easily be checked case by case. We note that these relations can be expressed as  $X \triangleright B = B(\mathbb{I} \otimes S)\Delta(X^*)$  (with a suitable interpretation of the action on  $B$  on the right-hand side) as expected from the general Hopf algebra structure of  $\mathcal{U}_q(\mathfrak{sl}_2)$ . The first two lines reflect what one would have for the action of an ordinary Lie algebra and a group, respectively.

(See Ref. [48] for a discussion of equivariance properties of general MPS tensors in the group case.)

The identities above are valid for a single site. In an MPS one has mixed tensor/matrix products of the form

$$|\text{MPS}\rangle = B_1 B_2 \cdots B_L, \quad (\text{D2})$$

where the index is referring to the site of the physical spin. This product has the equivariance properties

$$\begin{aligned} \mathbb{S}^z \triangleright (B_1 \cdots B_L) &= \mathbb{S}^z B_1 \cdots B_L - B_1 \cdots B_L \mathbb{S}^z, \\ q^{\alpha \mathbb{S}^z} \triangleright (B_1 \cdots B_L) &= q^{\alpha \mathbb{S}^z} B_1 \cdots B_L q^{-\alpha \mathbb{S}^z}, \\ \mathbb{S}^{\pm} \triangleright (B_1 \cdots B_L) &= \mathbb{S}^{\mp} B_1 \cdots B_L q^{-\mathbb{S}^z} \\ &\quad - q^{\mp 1} q^{-\mathbb{S}^z} B_1 \cdots B_L \mathbb{S}^{\mp}, \end{aligned} \quad (\text{D3})$$

which are an immediate consequence of the relations (D1) for the individual tensors. Indeed, the terms created between two  $B$ 's simply drop out. When verifying these relations it is important to work with the correct coproduct for a multiple tensor product.

#### APPENDIX E: PERIODIC BOUNDARY CONDITIONS

An important consequence of the relations (D3) is that an MPS with periodic boundary conditions is not invariant under the action of  $\mathcal{U}_q[\mathfrak{su}(2)]$ . Instead, one will need to work with the quantum trace

$$|\text{MPS}\rangle = \text{tr}[q^{2\mathbb{S}^z} B_1 \cdots B_L]. \quad (\text{E1})$$

It is evident that this state is invariant under the action of  $\mathbb{S}^z$  and  $q^{\alpha \mathbb{S}^z}$ . When acting with  $\mathbb{S}^{\pm}$  we find

$$\begin{aligned} \mathbb{S}^{\pm} |\text{MPS}\rangle &= \text{tr}[q^{2\mathbb{S}^z} \mathbb{S}^{\mp} X q^{-\mathbb{S}^z}] - q^{\mp 1} \text{tr}[q^{2\mathbb{S}^z} q^{-\mathbb{S}^z} X \mathbb{S}^{\mp}] \\ &= \text{tr}[q^{\mathbb{S}^z} \mathbb{S}^{\mp} q^{-\mathbb{S}^z} q^{\mathbb{S}^z} X] - q^{\mp 1} \text{tr}[\mathbb{S}^{\mp} q^{\mathbb{S}^z} X] \\ &= q^{\mp 1} \text{tr}[\mathbb{S}^{\mp} q^{\mathbb{S}^z} X] - q^{\mp 1} \text{tr}[\mathbb{S}^{\mp} q^{\mathbb{S}^z} X] \\ &= 0, \end{aligned} \quad (\text{E2})$$

where we have introduced the abbreviation  $X = B_1 \cdots B_L$ . The necessity for the use of quantum traces in a quantum group context has been known for a long time in the context of quantum integrable systems (see, e.g., Refs. [27,31]).

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