Editors' Suggestion

Supersymmetry method for interacting chaotic and disordered systems: The Sachdev-Ye-Kitaev model

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The nonlinear supermatrix σ model is widely used to understand the physics of Anderson localization and the level statistics in noninteracting disordered electron systems. In contrast to the general belief that the supersymmetry method applies only to systems of noninteracting particles, we adopt this approach to the disorder averaging in the interacting models. In particular, we apply supersymmetry to study the Sachdev-Ye-Kitaev (SYK) model where the disorder averaging has so far been performed only within the replica approach. We use a slightly modified time-reversal invariant version of the SYK model and perform calculations in real time. As a demonstration of how the supersymmetry method works, we derive saddle-point equations. In the semiclassical limit, we show that the results are in agreement with those found using the replica technique. We also develop the formally exact superbosonized representation of the SYK model. In the latter, the supersymmetric theory of original fermions and their superpartner bosons is reformulated as a model of unconstrained collective excitations. We argue that the supersymmetry description of the model paves the way for precise calculations in SYK-like models used in condensed matter, gravity, and high-energy physics.

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I. INTRODUCTION

The study of disordered and chaotic systems is a prevalent topic in condensed-matter physics, and various models of interacting particles have been under intensive investigation for more than half a century. Less expected has been a recent application of models with disorder to gravity and quantum field theory [1–3]. This field of research is fast growing, and the study of disorder and chaos can nowadays be considered as an interdisciplinary. The latter, in particular, means that methods of calculations developed in condensed-matter theory can be used in gravitation and high-energy physics.

Of course, one can simply use diagrammatic expansions in both the interaction and disorder [4] and sum the most important diagrams as it has been performed in Ref. [3]. However, this approximation does not generally give full information about the system, and one has to use nonperturbative methods.

Quantum phenomena in disordered or chaotic systems can efficiently be investigated analytically using methods of quantum field theory. Three most popular approaches are based on the replica trick [5], the Keldysh technique [6–9], and the supersymmetric σ -model approach originally developed by one of the authors [10,11]. The necessity of applying these techniques stems from the fact that physical correlation functions of interest are expressed in terms of functional integrals containing weight denominators whereas averaging over quenched disorder has to be performed at the end of calculations. This makes a direct application of methods of quantum field theory difficult. All the methods of Refs. [5–12] allow one to eliminate the weight denominator Z–the partition function of the system—and average over disorder just at the beginning of all calculations. As a result of this manipulation, one obtains an effective field theory for "interacting" particles and application of well-developed methods and approximations become feasible.

Although the replica, Keldysh, and supersymmetry techniques look similar to each other, their efficiency when being applied to different problems is very different. The replica approach allows one to avoid explicitly calculating Z by introducing an integer number of copies of the system and making use of the replica trick. It can be used for various systems of interacting particles, spins, etc., but the method requires an analytical continuation to noninteger numbers of replicas and assumes the existence of the replica limit when the number of copies $n \to 0$. A general procedure of this continuation does not exist, and one obtains very often unphysical results in certain situations, although one can also obtain important results using this method [12]. Within the Keldysh technique, one doubles the degrees of freedom to obtain a normalized theory with partition function $Z = 1$. The Keldysh σ-model representation of disordered systems is formally exact, but it can be quite complicated for some specific cases. Both approaches have been successfully applied to interacting theories with the disorder, but their efficiency in making essentially nonperturbative calculations is rather limited.

The supersymmetry approach makes use of the fact that the partition function of noninteracting fermions is always the inverse of that of the analogous bosonic theory. Therefore, if one introduces additional bosonic degrees of freedom that replicate the fermionic action, the overall partition function of the supersymmetric theory will be reduced to one. The approach is proven to be a handy tool for studies in various fields of physics and, in particular, in models of quantum chaos involving random matrix theory (RMT) and various models of disorder [10,11,13].

One of the prominent methods employing supersymmetry is the nonlinear supersymmetric σ model [10,11] description of disordered metallic conductors. According to this standard formalism, effective field theory is described by action with coordinate-dependent supermatrix field $Q(r)$, obeying the constraint $Q^2(r) = 1$. This method has a broad range of applications, including the study of Anderson localization, mesoscopic fluctuations, levels statistics in a limited volume, quantum chaos. The limitation of the supersymmetric approach was that it was deemed to be inapplicable to systems of interacting particles.

However, it turns out that there are important nontrivial models of interacting particles with the disorder that can be written in a supersymmetric form, and one can average over the disorder at the beginning of calculations. The main goal of this paper is to identify such models and develop the supersymmetry approach to the disorder averaging. To be more specific, we will apply this approach to study the Sachdev-Ye-Kitaev (SYK) model [1,14], originally considered in Refs. [15,16]. In this model, the disorder averaging was so far performed only within the replica trick approach. Our mapping of the SYK model onto a supersymmetric model containing both fermion and boson degrees of freedom, and subsequent averaging over the disorder is exact. Moreover, we demonstrate that the new supersymmetric model with an effective particle-particle interaction can be reformulated in terms of some generalized supermatrix σ model (superbosonization). This procedure is also exact. Leaving investigation of new nontrivial regimes of the SYK model for the future, we concentrate, here, on analyzing the semiclassical limit of the model. The results obtained in the semiclassical limit within this new approach are in agreement with those found earlier using the replica technique. The applicability of the supersymmetry method to the SYK model opens a new way of calculations for a certain class of models in condensed-matter, gravity and high-energy physics.

The SYK model exhibits inherently non-Fermi-liquid behavior and quantum many-body chaotic eigenspectrum [2,3,17–22]. This suggests that the two-point correlation function of the original fields of the model does not fully capture the many-body level statistics. The reason is that these are the many-body states that entirely determine the close energy levels. Thus, the many-body level statistics of the SYK model that follows the universal behavior of Wigner-Dyson random-matrix ensembles is inaccessible to original single-particle fields. To account for many-body effects of the model, we perform the superbosonization transformation and rewrite the model in terms of the collective many-body excitations. To show the workability of the representation, we reproduce earlier established results. We also demonstrate that the developed superbosonized description of the SYK model is capable of producing novel nonperturbative many-body effects.

The paper is organized as follows. In Sec. II, we introduce the SYK model. In Sec. III, we develop a new supersymmetric σ-model representation for interacting disordered fermion systems and apply it to SYK model. To derive it, we decouple the interaction Hamiltonian using the conventional Hubbard-Stratonovich approach. Then, we notice that the Hubbard-Stratonovich field can, in some situations, be gauged out from the denominator. This enables one to supersymmetrize the interacting theory. In Sec. IV, the new formalism is tested by calculating the fermion Green's function in the SYK model at long times and is argued to be efficient for other interacting models with the disorder.

In Sec. V, we rewrite the supersymmetric SYK model as a model describing unconstrained supermatrices representing collective many-body excitations. Such a representation where the partition function is represented in a supermatrix action formulation without any constraints is dubbed superbosonization. Since the transformation is exact, it is fully capable describing the many-body modes instead of the original fermions of the SYK model. As such, it represents the first step towards derivation of the Wigner-Dyson eigenvalue statistics and the calculation of the Thouless time at which the universal random-matrix behavior sets in. Our conclusions and the possible directions for the future research are discussed in Sec. VI.

II. MODEL

The study of out-of-time correlation functions [23] in the SYK model shows [18,24–27], that it exhibits chaotic behavior at all timescales. At short times, it has exponentially decaying correlators, whereas at ultralong times, otherwise nearly zero temperatures when the energy scale is less than the many-body level spacing, one has maximal chaoticity in the large system size limit. This happens because Lyapunov exponent saturates to the conjectured upper bound [25]. One of the important problems here is the test of eigenstate thermalization hypothesis (ETH) [28,29], which is a conjecture about the nature of matrix elements of physical observables that, if holds, reconciles the predictions of statistical physics of equilibrating states with those of quantum mechanics in the long-time limit.

The study of the low-energy (long time) scale [30,31] shows ETH behavior because the eigenstates exhibit volume law entanglement [32,33] suggesting that system becomes ergodic. However, one of the major questions, here, is associated with finding the intermediate time/energy scale at which the system transfers to a thermalized state. The characteristic timescale that leads to ergodicity in the SYK system is analogous to Thouless time in dirty metals, whereas the states are analogous to diffusive modes there. In the intermediate stage, one does not have an ergodic state. To study the latter, in Refs. [31,34–36], the local two fermions hopping term (SYK_2) with random coupling was added to the four-fermion long-range randomly interacting SYK_4 Hamiltonian. Here, the thermalization properties, including the Lyapunov exponent (or scrambling rate) and the so-called butterfly velocity, were analyzed. The butterfly speed is the speed at which the impacts of a local perturbation proliferate, whereas the scrambling rate is a proportion of the rate at which the local perturbation is mixed into nonlocal degrees of freedom. It has been demonstrated that in a general quantum framework, the Lyapunov exponent is limited by the temperature.

Another development in this direction was reported in Refs. [37–53], where d-dimensional generalization of SYK model was proposed by taking a number of SYK droplets in real space and including fermion hopping terms between them. This line of investigations is, however, out of the scope of the present paper.

The level statistics in the generalized $\mathrm{SYK}_4 + \mathrm{SYK}_2$ model was studied recently using exact diagonalization [34]. The results suggest that upon fixing the range of two-fermion hopping and keeping the four-fermion interaction sufficiently long ranged, the spectral correlations will not change substantially compared to the random-matrix prediction, which is typical for chaotic quantum systems. However, by reducing the range of the two-fermion terms, one will see a transition into an insulating state, characterized by Poisson statistics. It appeared, that in the vicinity of the many-body metal-insulator transition point, the spectral correlations share all the features that had been previously found in systems at the Anderson transition and in the proximity of the many-body localization transition. This indicates the potential relevance of generalized SYK models in the context of many-body localization and exhibits itself as a starting point for the exploration of a gravity dual of this phenomenon.

An important demonstration of the SYK model being maximally chaotic is the fact of having a finite entanglement entropy at zero temperature [30,33], indicating that, at long timescales, there is maximal mixing in the ground state. Basic features of the SYK model, that, in turn, support the existence of a gravity dual, include maximal chaos in the strong-coupling limit, finite zero-temperature entropy, linear specific heat in the low-temperature limit, the exponential growth of low-energy excitations, and the short-range spectral correlations given by random-matrix theory.

In its simplified version [2], the complex SYK model is a system of randomly interacting N (originally Majorana) spinless fermions represented by their annihilation (creation) operators $\hat{c}_i$ ($\hat{c}_i^\dagger$), $i = 1, \ldots, N$ with random all-to-all interactions given by the Hamiltonian,

$$\hat{H} = \sum_{ij,kl} J_{ij,kl} \hat{c}_i^\dagger \hat{c}_j \hat{c}_k^\dagger \hat{c}_l - \mu \sum_i \hat{c}_i^\dagger \hat{c}_i. \tag{2.1}$$

The coupling constant $J_{ij,kl}$ was assumed to be a random complex number,

$$J_{ij,kl}^* = J_{lk,ji}, \tag{2.2}$$

with a Gaussian distribution characterized by the following average and variance:

$$\langle J_{ij,kl} \rangle = 0, \quad \langle J_{ij,kl} J_{i'j',k'l'}^* \rangle = \frac{J^2}{N^3} \delta_{ii'} \delta_{jj'} \delta_{kk'} \delta_{ll'}. \tag{2.3}$$

Averages of the type $\langle J_{ij,kl} J_{i'j',k'l'} \rangle$ are equal to zero unless they can be reduced to Eq. (2.3) using the symmetry relation (2.2). The generalization of the SYK model to the case of random q-fermion interaction with even q is dubbed in the literature SYK_q model. In the latter, instead of four fermion interactions with random coupling, one has a q-fermion interaction.

At long timescales (low temperatures), the SYK model is conformal because the term that contains a time derivative in the Lagrangian can be ignored. The action of the model can be written using the so-called G, Σ representation and the Schwarzian theory [1,3] can describe its soft-mode fluctuations. It has been shown that this theory is equivalent [17,18,54] to two-dimensional dilaton gravity and the Jackiw-Teitelboim model [55–57]. This fact points out the link between AdS_2 black-hole physics and the SYK model.

The spectral form factor in the SYK model was studied numerically in Ref. [58] (analogous two-point correlation functions were studied using the random-matrix approach in Refs. [36,59]). Roughly, the spectral form factor is the Fourier transform of the connected two-point density-density correlation function $\langle \rho(E) \rho(E') \rangle$ in random-matrix theory. The question of the precise window of universality in which random-matrix theory is applicable is still unknown.

The supersymmetric reformulation of the SYK model we propose below, may give a possibility to derive it theoretically. The method may also allow one to study corrections beyond this universality regime. It is worth emphasizing that developing the supersymmetric representation we start with a fermionic SYK model analogous to the one given by Eq. (2.1). Bosons appear after certain transformations and are somehow "fictitious bosons," such as those that appear in the supersymmetry technique for electron systems [10,11].

All this clearly contrasts works on supersymmetric generalizations of the SYK model. For example, Ref. [60] reported a supersymmetric generalization of the SYK. In that model, however, the four-fermion coupling constants J_{ijkl} are not entirely random (they are correlated and defined by free-coupling constants in supercharge Q). The bosonic field appears, here, as a nondynamical field to linearize the supersymmetry transformation and realize the supersymmetry algebra off shell. Similar supersymmetric lattice models were reported in Refs. [61–67]. Specific correlations of the random couplings of these models lead to $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetries. Previous attempts to use a supersymmetry technique for the k-body matrix models were reported in Ref. [68]. Supersymmetric models with random couplings that include both bosons and fermions were considered in Refs. [69,70], whereas Ref. [71] explored the possibility of extending the $(1+1)$-dimensional bosonization technique to $(0 + 1)$-dimensional SYK-type systems. Reference [72] suggested that the SYK model with Majorana fermions and without fine-tuned couplings has the capacity of possessing some hidden supersymmetry, which may also be present in the complex SYK model when the chiral symmetry is present [73].

III. SUPERSYMMETRY REFORMULATION OF THE SYK MODEL: AVERAGING OVER QUENCHED DISORDER

Now, we apply the supersymmetry approach to the interacting SYK model. We believe that such an approach opens the door to analyzing the many-body effects and exponentially small bulk level spacing of the model. The formalism could also be adapted to study the effects in generalized SYK models, such as $\mathrm{SYK}_4 + \mathrm{SYK}_2$ and establish a fruitful connection between complex and Majorana models.

Although the original model Eq. (2.1) has been written in the Hamiltonian representation, it is more convenient to use the functional integral representation with fermionic fields $\chi_i(t)$, $\chi_i^*(t)$. They obey the anticommutation relations,

$$\{\chi_i, \chi_j\} = \{\chi_i^*, \chi_j^*\} = \{\chi_i, \chi_j^*\} = 0, \quad (3.1)$$

and we use the convention $(\chi_i^*)^* = -\chi_i$.

In order to develop the supersymmtery approach for the model with the fermion-fermion interaction, we slightly modify the original model specified by Eq. (2.1). Using the anticommuting Grassmann fields χ, we write correlation functions in terms of a functional integral over these fields as

$$G_{ij}(t,t') = -\frac{i\int \chi_i(t)\chi_j^*(t')\exp(iS[\chi,\chi^*])D\chi D\chi^*}{\int \exp(iS[\chi,\chi^*])D\chi D\chi^*}. \quad (3.2)$$

In Eq. (3.2), the product of the fields $\chi_i(t)$ and $\chi_j^*(t)$ for arbitrary i, j and times t defines the Green's function G_{ij}. Here, we start with the action $S[\chi,\chi^*]$, which is slightly different from the field representation of the model given by Eq. (2.1). Namely, we consider

$$S[\chi,\chi^*] = \int_{-\infty}^{\infty}\left[\sum_{i=1}^{N}\chi_i^*(i\partial_t+\mu)\chi_i(t) - \sum_{ij,kl=1}^{N} J_{ij,kl}[\chi_i^*(t)\chi_j(t) - \chi_j^*(t)\chi_i(t)] \times [\chi_k^*(t)\chi_l(t) - \chi_l^*(t)\chi_k(t)]\right]dt. \quad (3.3)$$

The random coupling constants $J_{ij,kl}$ in Eq. (3.3) are assumed to be real and obey the symmetry relations,

$$J_{ij,kl} = -J_{ji,kl} = -J_{ij,lk} = J_{kl,ij}. \quad (3.4)$$

Their distribution is Gaussian with zero average,

$$\langle J_{ij,kl}\rangle = 0, \quad (3.5)$$

and the variance,

$$\langle J_{ij,kl}J_{i'j',k'l'}\rangle = \frac{J^2}{8N^3}[(\delta_{ii'}\delta_{j,j'} - \delta_{ij'}\delta_{ji'})(\delta_{k,k'}\delta_{ll'} - \delta_{kl'}\delta_{lk'}) + (\delta_{ik'}\delta_{j,l'} - \delta_{il'}\delta_{jk'})(\delta_{k,i'}\delta_{lj'} - \delta_{kj'}\delta_{li'})]. \quad (3.6)$$

One can interpret the model described by Eq. (3.3) as a time-reversal invariant symmetrized version of the SYK model. The model was also recently considered in Ref. [74].

First, under the functional integral, we introduce a time-dependent Hubbard-Stratonovich real antisymmetric matrix field $M_{ij}^F(t)$ and decouple the four-fermion interaction of the SYK Hamiltonian (2.2) by inserting the identity operator,

$$\mathbb{1} \equiv \int \frac{\mathcal{D}M^F}{\mathrm{Det}[J_{ij,kl}]}\exp\Bigg\{i\int dt \sum_{ij,kl=1}^{N}\sum_{i'j',k'l'=1}^{N} \times \left[M_{ij}^F - i(\chi_l^*\chi_k - \chi_k^*\chi_l)J_{kl,ij}\right] \times (J^{-1})_{ij,i'j'}\left[M_{j'i'}^F - J_{i'j',k'l'}i(\chi_{k'}^*\chi_{l'} - \chi_{l'}^*\chi_{k'})\right]\Bigg\}, \quad (3.7)$$

into the functional integrals over χ, χ^* in Eq. (3.2). Here, $(J^{-1})_{ij,kl}$ is the inverse of $J_{ij,kl}$, namely,

$$\sum_{kl}(J^{-1})_{ij,kl}J_{kl,mn} = \delta_{im}\delta_{jn}. \quad (3.8)$$

Using the last property in Eq. (3.4) of the coupling $J_{ij,kl}$ and the Hermiticity of the matrix $M_{ij}(t)$, we see that the exponent in Eq. (3.7) is purely imaginary and the integral over matrix $M_{ij}(t)$ converges. Then, the action for the time-reversal symmetric modification of the SYK model is now equivalent to that of a system of electrons moving in a fluctuating real antisymmetric field $M_{ij}(t)$ with random $J_{ij,kl}$,

$$\begin{aligned} S[\chi,\chi^*,M^F] &= S_0[\chi,\chi^*,M^F] + S_{\text{fluct}}[M^F] \\ &= \int_{-\infty}^{\infty} dt \sum_{ij=1}^{N}\Big\{\chi_i^* t(t)[(i\partial_t+\mu)\delta_{ij} - 2iM_{ij}^F(t)]\chi_j(t) \\ &\quad + \sum_{ijkl} M_{ij}^F(t)(J^{-1})_{ij,kl}M_{lk}^F(t)\Big\}. \end{aligned} \quad (3.9)$$

Here $S_{\text{fluct}}[M^F]$ represents the Gaussian fluctuations of $M_F(t)$. This action, therefore, defines the Green's function G_{ij} of fermionic fields $\chi_i(t)$, $\chi_j^*(t)$ as

$$G_{ij}(t,t') = -\frac{i\int \chi_i(t)\chi_j^*(t')\exp(iS[\chi,\chi^*,M^F])D\chi D\chi^* DM^F}{\int \exp(iS[\chi,\chi^*,M])D\chi D\chi^* DM}. \quad (3.10)$$

The random coupling $J_{ij,kl}$ enters both the numerator and the denominator in Eq. (3.10), and one cannot average over this coupling directly. This situation is typical for problems with quenched disorder. The standard supersymmetry approach of Refs. [10,11] relies on the fact that the system is initially noninteracting. In that case, one replaces the denominator by an integral over bosonic fields in the numerator. Since, here, we deal with an inherently interacting system, we generate a field M_{ij} which enters both numerator and denominator in Eq. (3.10) and seemingly invalidates the possibility of supersymmetrizing the action.

Although this obstacle cannot be generally overcome, the SYK model considered here is, in this respect, exceptional. Now, we make a crucial observation. We show, now, that the integral over the fermionic fields χ, χ^* in the denominator of Eq. (3.10) does not, in fact, depend on the Hubbard-Stratonovich field $M(t)$. The reason is that the real antisymmetric matrix $M(t)$ can be reduced to a time-independent constant matrix M_0 by a gauge transformation $2M(t) = 2U^TM_0U - U^T\partial_tU$ of the orthogonal group $U^TU = 1$. Here, the constant matrix M_0 is block diagonal with real 2×2 antisymmetric blocks along the diagonal $\hat{\mu}_i = \begin{pmatrix} 0 & \mu_i \\ -\mu_i & 0 \end{pmatrix}$ with $i = 1, 2, \ldots, N/2$. The matrix M_0 represents the zero mode of $M(t)$ and appears due to periodicity of restrictions on $U(t)$. At zero temperature, it vanishes, $M_0 = 0$, and the transformation reduces to a pure gauge transformation $2M(t) \to -U^T\partial_tU$. Since the gauge transformation of free fermions is not anomalous [75], it helps us to simplify the integral in the denominator of Eq. (3.10),

$$\int \exp(iS_0[\chi,\chi^*,M])D\chi D\chi^* \\ = \mathrm{Det}[(i\partial_t+\mu)\delta_{ij}-2iM_{ij}] \\ = \mathrm{Det}[U^T(i\partial_t+\mu)U] = \mathrm{Det}[(i\partial_t+\mu)]. \quad (3.11)$$

So, what we end up having in the denominator is just a determinant $\mathrm{Det}[(i\partial_t+\mu)]$, which is independent of the fluctuating field $M(t)$ and random coupling constants J_{ijkl}. This point is crucial, and it allows one to express the integral over the fermionic fields in the denominator in Eq. (3.10) via additional bosonic *superpartner* fields. This is a standard procedure of the supersymmetric approach developed in Refs. [10,11].

Following this approach, we introduce complex bosonic fields $s_i(t)$, $i = 1, 2\cdots N$ and a new bosonic model with the action,

$$S^B[s,s^*] = \int_{-\infty}^{\infty}\Bigg[\sum_{i=1}^{N} s_i^*(t)(i\partial_t+\mu)s_i(t) \\ - \sum_{ij,kl=1}^{N} J_{ij,kl}[s_i^*(t)s_j(t)-s_j^*(t)s_i(t)] \\ \times[s_k^*(t)s_l(t)-s_l^*(t)s_k(t)]\Bigg]dt. \quad (3.12)$$

The action $S^B[s,s^*]$ looks identical to action $S[\chi,\chi^*]$, Eq. (3.3), and it is real. Moreover, the coupling constant $J_{ij,kl}$ obeys the same symmetry relations (3.4). Now, we write the bosonic partition function,

$$Z_B = \int \exp(iS[s,s^*])Ds\,Ds^*. \quad (3.13)$$

As the action $S^B[s,s^*]$ is real, the integral over $s(t)$ in Eq. (3.13) converges. Then, we make the same decoupling of the interaction in Eq. (3.13) as we have performed for the fermionic model and write the partition function Z_B in the form

$$Z_B = \int \exp(iS^B[s,s^*,M^B])Ds\,Ds^*DM^B. \quad (3.14)$$

Here, the action $S^B[s,s^*,M^B]$ equals to

$$S[s,s^*,M^B] \\ = \int_{-\infty}^{\infty} dt \sum_{i,j=1}^{N}\Big\{s_i^*(t)\big[(i\partial_t+\mu)\delta_{ij}-2iM_{ij}^B(t)\big]s_j(t) \\ + \sum_{ij,kl} M_{ij}^B(t)(J^{-1})_{ij,kl}M_{kl}^B(t)\Big\}. \quad (3.15)$$

The matrix $M^B(t)$ in Eqs. (3.14) and (3.15) has the same symmetry as the matrix $M^F(t)$ in Eqs. (3.7)–(3.11), and we can calculate the Gaussian integrals over the bosonic field $\mathbf{s}(t)$ in the same manner as previously,

$$\int \exp\left[i\int_{-\infty}^{\infty}\big[s_i^*(t)(i\partial_t+\mu)\delta_{ij}-2M_{ij}^B(t)\big]s_j(t)dt\right]Ds\,Ds^* \\ = \big\{\mathrm{Det}\big[(i\partial_t+\mu)\delta_{ij}-2M_{ij}^B\big]\big\}^{-1} = [\mathrm{Det}(i\partial_t+\mu)]^{-1}. \quad (3.16)$$

We see that the matrix $M^B(t)$ is gauged out, and the result of the integration over $s(t)$, $s^*(t)$is performed exactly in the same way as in the fermionic determinant. This matrix is also real and antisymmetric. However, in contrast to Eq. (3.11), one obtains the inverse of the determinant. It is this property of bosonic determinants that allows one to get rid of the denominator in Eq. (3.2).

Combining the fermionic and bosonic degrees of freedom, one can form a supervector $\Phi \equiv (\{\chi_i\};\{s_i\}) \in U(N,1|N,1)$ and its Hermitian conjugate supervector $\Phi^\dagger \in U(N,1|N,1)$. This allows us to write a supersymmetric action for the time-reversal invariant SYK model as

$$\tilde{S}[\Phi,\Phi^\dagger,\hat{M}] \\ = \int dt\Bigg[\sum_{i,a}\Phi_i^\dagger(t)[(i\partial_t+\mu)\delta_{ij}-2\hat{M}_{ij}(t)]\Phi_j(t) \\ + \sum_{ijkl}\mathrm{Tr}[\hat{M}_{ij}(t)(J^{-1})_{ij,kl}\hat{M}_{kl}(t)]\Bigg], \quad (3.17)$$

where the two-component supervectors have the following structure:

$$\Phi_i(t) = \begin{pmatrix}\chi_i(t)\\ s_i(t)\end{pmatrix}, \quad \Phi_i^\dagger(t) = [\chi_i^*(t) \quad s_i^*(t)]. \quad (3.18)$$

and

$$\hat{M}_{ij}(t) = \begin{pmatrix} M_{ij}^F(t) & 0\\ 0 & M_{ij}^B(t)\end{pmatrix} \quad (3.19)$$

is a diagonal matrix in the space of the supervectors Φ. Having set up this structure, one can readily write the fermion Green's function G_{ij} in Eq. (3.2) as

$$G_{ij}(t,t') \\ = -i\int \Phi_i^1(t)\Phi_j^{1\dagger}(t')\exp(iS[\Phi,\Phi^\dagger,\hat{M}])D\Phi\,D\Phi^\dagger D\hat{M}. \quad (3.20)$$

Importantly, the absence of the weight denominator in Eq. (3.20) allows one to average over the random coupling $J_{ij,kl}$ in the beginning of all calculations.

We see that, although we have started with an interacting theory, the supersymmetry approach to quenched averaging [11] works in this case as well due to the fact that the spatial dimension in this problem is effectively zero. We emphasize that all the transformations reducing Eq. (3.2)–(3.20) are formally exact. Now, we integrate in Eq. (3.20) over the matrix $\hat{M}(t)$ to obtain

$$G_{ij}(t,t') = \int \Phi_i^1(t)\Phi_j^{1\dagger}(t')\exp(iS[\Phi,\Phi^\dagger])D\Phi\,D\Phi^\dagger, \quad (3.21)$$

Here, in Eq. (3.21), the action $S[\Phi, \Phi^\dagger]$ equals

$$S[\Phi, \Phi^\dagger] = \int_{-\infty}^{\infty} dt \Bigg[\sum_i \Phi_i^\dagger(t)(i\partial_t + \mu)\Phi_i(t) - \sum_{a=1}^{2} \sum_{ij,kl} J_{ij,kl} \big[\Phi_i^{a\dagger}(t)\Phi_j^a(t) - \Phi_j^{a\dagger}(t)\Phi_i^a(t)\big] \times \big[\Phi_k^{a\dagger}(t)\Phi_l^a(t) - \Phi_l^{a\dagger}(t)\Phi_k^a(t)\big] \Bigg], \tag{3.22}$$

where $a = 1$ denotes the fermionic component of the supervector $\Phi(t)$ defined in Eq. (3.18), whereas $a = 2$ stands for the bosonic one.

Before performing disorder averaging, it is convenient to use more compact notations via introducing four-component supervectors $\Psi(t)$ as

$$\Psi_i(t) = \frac{1}{\sqrt{2}} \begin{pmatrix} \chi_i^*(t) \\ \chi_i(t) \\ s_i^*(t) \\ s_i(t) \end{pmatrix},$$

$$\bar{\Psi}_i = \frac{1}{\sqrt{2}}[\chi_i(t)\chi_i^*(t) - s_i(t)s_i^*(t)]. \tag{3.23}$$

The supervector $\bar{\Psi}$ is related to Ψ by a charge conjugation,

$$\bar{\Psi} = (C\Psi)^T, \tag{3.24}$$

where T stands for transposition and the matrix C is given by

$$C = \begin{pmatrix} c_2 & 0 \\ 0 & c_1 \end{pmatrix}, \quad c_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad c_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

One can note that $\bar{\Psi}$ has a simple connection to the Hermitian-conjugated supervector $\Psi^\dagger$,

$$\bar{\Psi} = \Psi^\dagger \tau_3, \tag{3.25}$$

where

$$\tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.26}$$

is the Pauli matrix in the "particle-hole" space of matrices c_2 and c_1.

Furthermore, the square of the modulus of the supervector Ψ is equal to

$$|\Psi|^2 = \Psi^\dagger\Psi = \bar{\Psi}\tau_3\Psi. \tag{3.27}$$

It is also seen that

$$\bar{\Psi}_i\Psi_j = -\bar{\Psi}_j\Psi_i. \tag{3.28}$$

Substituting Eqs. (3.23)–(3.28) into Eqs. (3.21) and (3.22), we rewrite the fermion Green's function in a more compact form

$$G_{ij}(t,t') = \int \Psi_{2i}^1(t)\Psi_{2j}^{1\dagger}(t') \exp(iS[\Psi, \Psi^\dagger]) D\Psi\, D\Psi^\dagger. \tag{3.29}$$

In Eq. (3.29), superscripts numerate blocks in the superspace, whereas first subscripts numerate elements in the particle-hole space. The action $S[\Psi, \Psi^\dagger]$ entering Eq. (3.29) is given by

$$S[\Psi, \Psi^\dagger] = \int_{-\infty}^{\infty} dt \Bigg[\sum_i \bar{\Psi}_i(t)(i\partial_t + \tau_3\mu)\Psi_i(t) - 4\sum_{a=1}^{2}\sum_{ij,kl} J_{ij,kl}\big[\bar{\Psi}_i^a(t)\Psi_j^a(t)\big]\big[\bar{\Psi}_k^a(t)\Psi_l^a(t)\big] \Bigg], \tag{3.30}$$

where $\Psi^a(t),\ a = 1, 2$ stand for the fermion and boson components of the supervectors Ψ. Substituting Eq. (3.30) into Eq. (3.29), one can easily average over the random $J_{ij,kl}$ using Eq. (3.6). The expression for the disorder-averaged Green's function, thus, will read as

$$\langle G_{ij}(t,t')\rangle = \int \Psi_{2i}^1(t)\Psi_{2j}^{1\dagger}(t') \exp(i\bar{S}[\Psi, \Psi^\dagger]) D\Psi\, D\Psi^\dagger, \tag{3.31}$$

where the nonlocal action action $\bar{S}[\Psi, \Psi^\dagger]$ equals

$$\bar{S}[\Psi, \Psi^\dagger] = \int_{-\infty}^{\infty} dt \sum_{i=1}^{N} \bar{\Psi}_i(t)(i\partial_t + \tau_3\mu)\Psi_i(t) \times \frac{iJ^2}{N^3} \sum_{a,b=1}^{2} \sum_{ij,kl}^{N} \int_{-\infty}^{\infty} \big[\bar{\Psi}_i^a(t)\Psi_j^a(t)\big]\big[\bar{\Psi}_k^a(t)\Psi_l^a(t)\big] \times \big[\bar{\Psi}_l^b(t')\Psi_k^b(t')\big]\big[\bar{\Psi}_j^b(t')\Psi_i^b(t')\big] dt\, dt'. \tag{3.32}$$

We see that the action $\bar{S}[\Psi, \Psi^\dagger]$ in Eq. (3.32) does not contain disorder anymore, and the integral over the supervectos $\Psi_i^\dagger(t)$ and $\Psi_j(t)$ in Eq. (3.31) is clearly convergent.

Of course, in Eq. (3.32), the addition of extra bosonic degrees of freedom comes at the price of introducing additional integrals. However, the resultant theory Eqs. (3.31) and (3.32) does not contain disorder and is fully supersymmetric. As such, it has many simplifications. One simplification is the cancellation of a variety of Feynman diagrams in the perturbation theory in interactions due to the supersymmetry. Another simplification follows from the superbosonization of this supersymmetric action discussed in Sec. V. In the superbosonized representation, instead of the functional integral over supervectors, one deals with an integral over supermatrices. In that approach, the number of integration variables can significantly be reduced upon the diagonalization of the supermatrices.

However, let us fist make a saddle-point approximation that has to become exact in the limit $N \to \infty$. This is performed in the next section. Comparison of the hereby obtained results with those obtained within the replica approach in Refs. [2,14] can be performed, but one cannot expect a full coincidence because we consider a somewhat different model. In contrast to the calculations presented there, we use the real-time representation.

IV. SADDLE-POINT APPROXIMATION

The saddle-point approximation is expected to become exact in the limit $N \to \infty$. In order to see this property explicitly and proceed with the calculations, let us introduce

2×2 supermatrices $W^{ab}(t,t')$ as

$$W^{ab}(t,t')=\frac{2}{N}\sum_{i=1}^{N}\Psi_i^a(t)\bar{\Psi}_i^b(t'), \tag{4.1}$$

where supervectors Ψ and $\bar{\Psi}$ are specified in Eq. (3.18). The supermatrix $W(t,t')$ has the evident symmetry,

$$W^{\dagger}(t,t')=W(t',t). \tag{4.2}$$

Using Eqs. (4.1) and (4.2), and the disorder averaging procedure resulting in Eq. (3.32), we explicitly reduce Eq. (3.32) to a considerably more compact form

$$\begin{aligned}\bar{S}[\Psi,\Psi^{\dagger}]\\ &=\int_{-\infty}^{\infty}dt\sum_{i=1}^{N}\bar{\Psi}_i(t)(i\partial_t+\tau_3\mu)\Psi_i(t)\\ &+\frac{iNJ^2}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dt\,dt'(2\{[W^{21}(t,t')W^{12}(t',t)]\}^2\\ &+\{[W^{11}(t,t')W^{11}(t',t)]\}^2+\{\mathrm{Tr}[W^{22}(t,t')W^{22}(t',t)]\}).\end{aligned} \tag{4.3}$$

where 2×2 matrices W^{ab} have matrix-valued entries. Elements of matrices $W^{21}(t,t')$ and $W^{12}(t',t)$ are anticommuting fields, whereas those of the matrices $W^{11}(t,t')$ and $W^{22}(t,t')$ contain products of two anticommuting fields or are conventional complex functions.

Here, we would like to invite the reader's attention to the resemblance of the action (3.32) with the replicated imaginary time action of the SYK model outlined in Ref. [2] [see Eq. (16) there]. However, now we have the formally exact supersymmetric representation of the model where no replica limit $n\to 0$ (see, e.g., Refs. [76–79]), has to be taken. It is also worth emphasizing that, here, we have 4×4 supermatrices $W(t,t')$ instead of $n\times n$ matrices in the replica approach. We emphasize that Eqs. (3.31) and (4.3) are still exact for any N.

Now, one can explicitly see that the interaction term in Eq. (4.3) is proportional to N, and the accuracy of the saddle-point approximation should follow from the assumption that this number is large. Although details are different, we use the general chain of transformations suggested in Refs. [2,14] and analyze the behavior of the fermion Green's function.

First, we decouple the interaction terms in Eq. (4.3) by introducing auxiliary functions $P^{ab}(t,t')$ and integrating over them. We write

$$\begin{aligned}&\exp\left[-J^2N\sum_{a,b=1}^{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{2}\{\mathrm{Tr}[W^{ab}(t,t')W^{ba}(t',t)]\}^2dt\,dt'\right]\\ &=Z_0\int DP\exp\left\{-N\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dt\,dt'\sum_{a,b=1}^{2}\left[\frac{P^{ab}(t,t')P^{ba}(t',t)}{2J^2}-iP^{ab}(t,t')\mathrm{Tr}[W^{ab}(t',t)W^{ba}(t,t')]\right]\right\},\end{aligned} \tag{4.4}$$

where

$$Z_0=\int DP\exp\left[-N\sum_{a,b=1}^{N}\int_{-\infty}^{\infty}\frac{P^{ab}(t,t')P^{ba}(t',t)}{2J^2}dt\,dt'\right]. \tag{4.5}$$

In Eqs. (4.4) and (4.5), $P^{11}(t,t')$ and $P^{22}(t,t')$ are real symmetric functions, whereas $P^{12}(t,t')=[P^{21}(t',t)]^*$. The contribution of off-diagonal elements $W^{ab}(t,t')$, $a\neq b$ to Eq. (4.4) is subleading at $N\gg 1$. The reason for this is that these elements are Grassmann variables, and upon expanding the exponent in (4.4), one generates only first-order and mixed second-order terms that come with small powers of N. Thus, in the main approximation in N, contributions coming from $W^{aa}(t,t')$ are most important, and we concentrate on them.

To simplify the action and analyze its equations of motion, we have to decouple the terms $\{\mathrm{Tr}[W(t,t')W(t',t)]\}^2$ by one more Gaussian decoupling. To do this, we introduce a new diagonal matrix-field $Q^{aa}(t,t')$, $a=1,2$ and use the following identities:

$$\begin{aligned}&\exp\left[Ni\int_{-\infty}^{\infty}P^{aa}(t,t')W^{aa}(t,t')W^{aa}(t',t)dt\,dt'\right]\\ &=\int DQ\exp\left[-Ni\sum_{ij}^{N}\int_{-\infty}^{\infty}P^{aa}(t,t')\{\mathrm{Tr}[Q^{aa}(t,t')Q^{aa}(t',t)+2Q^{aa}(t,t')\Psi^a(t')\bar{\Psi}^a(t)]\}dt\,dt'\right]Z_a[P]\\ &=Z_a[P]\int DQ\exp\left[-Ni\sum_{ij}\int_{-\infty}^{\infty}\{\mathrm{Tr}[P^{aa}(t,t')Q^{aa}(t,t')Q^{aa}(t',t)]-2(-1)^{a-1}\bar{\Psi}^a(t)P^{aa}(t,t')Q^{aa}(t,t')\Psi^a(t')\}dt\,dt'\right],\end{aligned} \tag{4.6}$$

where $a=1,2$ and we introduced the following notation:

$$Z_a[P]=\int DQ\exp\left[iN\sum_{ij}\int_{-\infty}^{\infty}P^{aa}(t,t')Q^{aa}(t,t')Q^{aa}(t',t)\right]. \tag{4.7}$$

In Eq. (4.6), the new matrices,

$$Q(t,t') = \begin{pmatrix} Q^{11}(t,t') & 0 \\ 0 & Q^{22}(t,t') \end{pmatrix} \tag{4.8}$$

have the following symmetry:

$$\bar{Q}(t,t') = CQ^T(t,t')C^T = Q^\dagger(t,t'). \tag{4.9}$$

All these decouplings and notations allow us to write the full partition function Z of the model in the form

$$Z = \int \exp(iS[\Psi,\Psi^\dagger,P,Q])Z[P]D\Psi\, DP\, DQ, \tag{4.10}$$

where the integrant contains a factor $Z[P]$ given by

$$Z[P] = Z_a[P]Z_b[P]Z_0. \tag{4.11}$$

In Eq. (4.10), the functional $S[\Psi,\Psi^\dagger,P,Q]$ is given by

$$\begin{aligned} S[\Psi,\Psi^\dagger,P,Q] &= \int_{-\infty}^{\infty} dt\, dt' \Bigg[\sum_{i=1}^{N} \bar{\Psi}_i(t)[\delta_{t,t'}(i\partial_{t'} + \tau_3\mu) \\ &\quad + 2P(t,t')Q(t,t')]\Psi_i(t') \\ &\quad - N\sum_{i,j=1}^{N} \mathrm{Tr}[P(t,t')Q(t,t')Q(t't)] + \frac{iN}{2J^2}\mathrm{Tr}[P^2(t,t')]\Bigg], \end{aligned} \tag{4.12}$$

where

$$P(t,t') = \begin{pmatrix} P^{11}(t,t') & 0 \\ 0 & P^{22}(t,t') \end{pmatrix}. \tag{4.13}$$

Integrating out the supervectors $\Psi,\Psi^\dagger$, one obtains, using Eq. (4.10), the following formula for the partition function Z:

$$Z = \int Z[P,Q]DP\, DQ, \tag{4.14}$$

with the integrant $Z[P,Q]$ being equal to

$$\begin{aligned} Z[P,Q] = \exp\Bigg\{ N\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dt\, dt' \Bigg[-\frac{\mathrm{Tr}\, P^2(t,t')}{2J^2} \\ + \mathrm{Tr}[k\ln[\delta(t-t')(i\partial_{t'} + \tau_3\mu) + 2P(t,t')Q(t,t')] \\ - i\,\mathrm{Tr}[P(t,t')Q(t,t')Q(t',t)]\Bigg]\Bigg\}. \end{aligned} \tag{4.15}$$

Here, we introduced a 2×2 matrix k, that differentiates between bosonic and fermionic superpartners,

$$k = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The presence of the large N in the exponential in Eq. (4.15) allows one to calculate the integral over $P(t,t')$ and $Q(t,t')$ using the saddle-point method. Minimizing action $-\ln Z[P,Q]$ with respect to the matrices $Q(t,t')$ and $P(t,t')$, we obtain the following saddle-point equations:

$$Q(t,t') = -ik[\delta(t-t')(i\partial_{t'} + \tau_3\mu) + 2P(t,t')Q(t,t')]^{-1}, \tag{4.16}$$

$$\begin{aligned} P(t,t') = -iJ^2Q(t,t')Q(t',t) + 2J^2Q(t,t') \\ \times k[(i\partial_t + \tau_3\mu)\delta(t-t') + 2P(t,t')Q(t,t')]^{-1}. \end{aligned} \tag{4.17}$$

Using Eq. (4.16), we rewrite Eq. (4.17) in a simpler form

$$P(t,t') = iJ^2Q(t,t')Q(t',t). \tag{4.18}$$

As the next step, substituting Eq. (4.18) into Eq. (4.16), one will obtain a closed equation for $Q(t,t')$,

$$\begin{aligned} Q(t,t') = -ik[(i\partial_t + \tau_3\mu)\delta(t-t') \\ + 2iJ^2Q(t,t')Q(t',t)Q(t,t')]^{-1}. \end{aligned} \tag{4.19}$$

Note that, Eq. (4.19) can also be written in a form of a differential equation,

$$\begin{aligned} [(i\partial_t + \tau_3\mu)Q(t,t') + 2iJ^2\int Q(t,t'')Q(t'',t)Q(t,t'')] \\ \times Q(t'',t')dt'' = -ik\delta(t-t'). \end{aligned} \tag{4.20}$$

As the function $Q(t,t')$ is diagonal, one can solve Eq. (4.20) separately for the fermion and boson parts. At small energies (long-time limit) and $\mu = 0$, one can neglect the first line in Eq. (4.20). Assuming that the solutions depend on the time difference, one comes to the following set of equations:

$$2J^2\int [Q^F(t-t'')]^2Q^F(t''-t)Q^F(t''-t')dt'' = -\delta(t-t'), \tag{4.21}$$

$$2J^2\int (Q^B(t-t''))]^2Q^B(t''-t)Q^B(t''-t')dt'' = \delta(t-t'),$$

where $Q^F(t,t')$ and $Q^B(t,t')$ are fermion and boson parts the matrix $Q(t,t')$. The structure of Eqs. (4.22) is similar that of equations obtained in Ref. [2], although there are small differences due to a fact that we considered, here, a slightly different model Eq. (3.3) written in real time.

From Eqs. (4.18)–(4.20), we can find the Green's function of fermions and bosons in energy space,

$$G(\omega) = [\omega\mathbb{1}_4 - \Sigma(\omega)]^{-1},$$

where $\mathbb{1}_4-$ is a four-dimensional identity matrix and $\Sigma(\omega)$ is the Fourier image of the electron/boson self-energy,

$$\Sigma(t-t') = -2J^2kG^2(t-t')G(t'-t). \tag{4.22}$$

In Eq. (4.22), Σ and G are 4×4 diagonal matrices. For the fermionic part, this relations fully coincide with ones obtained in Ref. [2], whereas bosonic self-energy has the opposite sign as it should be. One can see easily that this sign difference gives the unity partition function $Z[P,Q]$ given by Eqs. (4.14) and (4.15). Indeed, writing the derivative of the logarithm of the partition function Z and using the saddle-point equations (4.16) and (4.18), we obtain

$$-\frac{\partial}{\partial J}\ln Z[P_J,Q_J] = -\frac{N}{J^3}\mathrm{Tr}\, P_J^2(t,t'), \tag{4.23}$$

where Q_J and P_J are solutions of the saddle-point equations (4.16) and (4.18). Using Eqs. (4.18), (4.22), and (4.23) and reconstructing the partition function $Z[P_J, Q_J]$ from its derivative we conclude that it is equals one. This confirms that the saddle-point solution does not contradict the supersymmetry. Although study of the solution of Eqs. (4.22) at arbitrary time is also interesting, we do not perform it here.

In the scaling low-energy limit $\omega \ll J$ and zero chemical potential, the expression for the Green's function $G_a(\omega)$ (where $a = 1, 2$ corresponds to fermions and $a = 3, 4$ to bosons) has a one-dimensional time reparametrization, $t = f(\sigma)$, and emergent $U(1)$ gauge invariance, defined by Sachdev in Ref. [2] for imaginary time,

$$G_a(t,t') = [f'(\sigma)f'(\sigma')]^{-1/4}\frac{g(\sigma)}{g(\sigma')}G_a(\sigma,\sigma')$$

$$\Sigma_a(t,t') = [f'(\sigma)f'(\sigma')]^{-3/4}\frac{g(\sigma)}{g(\sigma')}\Sigma_a(\sigma,\sigma'). \quad (4.24)$$

Here, $f(\sigma)$ and $g(\sigma)$ are arbitrary functions. These symmetries impose strong restrictions on G and Σ and lead to the following asymptotic expression for the Green's function at zero temperature,

$$G_1(t) = \begin{cases} \frac{Ce^{3i\pi/4}\sin[\pi/4+\theta]}{\sqrt{\pi t}}, & t \gg 1/J, \\ \frac{Ce^{-3i\pi/4}\sin[\pi/4+\theta]}{\sqrt{-\pi t}}, & -t \gg 1/J, \end{cases}$$

with constant C. These expressions were first obtained in Ref. [2]. Therefore, at least, in the asymptotic regime of long times (low energies), we do not expect a difference between our supersymmetric formulation of the SYK model and the replica approach to it. However, at intermediate times, when we cannot ignore the kinetic term for supersymmetric (fermionic) fields in action (4.12), a difference may be essential.

V. SUPERBOSONIZATION OF THE SYK MODEL

In Sec. IV, we explicitly developed a supersymmetry method for the interacting SYK model, which produced nonperturbative results. Remarkably, in the above-developed approach, the supersymmetry is explicit at the level of the saddle-point equations. These equations are very interesting and may potentially provide some more new information about the system behavior at various energy scales. At the same time, as we can see from saddle-point equations (4.22), the fermion and boson sectors of the diagonal matrix field $Q(t,t')$ are decoupled. Thus, bosons and fermions do not interfere with each other in this formulation.

Interestingly enough, there is an alternative conceptually similar but technically different way of formulating the SYK model as a supersymmetric σ model. It is the superbosonization procedure, which will be developed in this section. We will show that at the level of the saddle-point equations in the superbosonized description, bosonic degrees of freedom interfere with fermions. This interference effect can be accounted for analytically. It may potentially become crucial for revealing novel modes in correlation functions—the advantage of the supersymmetric approaches as compared to replica and imaginary time methods is that they allow for controlled analysis of the intermediate-time regime.

Consider a function $F(\Phi \otimes \Phi^\dagger)$ of the tensor product of a supervector Φ and its conjugate $\Phi^\dagger$ given by Eq. (3.18). Generally, after ensemble averaging of disordered single-particle systems, one deals with integrals of type $\int D\Phi\, D\Phi^\dagger F(\Phi \otimes \Phi^\dagger)$. The superbosonization formula essentially allows evaluating such a supervector integral to an integral over a supermatrix Q, where Q has no constraints (unlike direct product $\Phi \otimes \Phi^\dagger$).

Being formally exact, the superbosonization approach [80–82] proved to be very efficient in producing nonperturbative results, for example, in the theory of almost diagonal random matrices [82–84] where the standard supersymmetry method [10,11,85] was also instrumental [86,87]. To derive the superbosonized representation of the SYK model, here, we will follow a slightly different path from the one outlined in Sec. III. In contrast with Eq. (3.17), wherein the joined fermion-boson action contained two different Hubbard-Stratonovich fields $M_F(t)$ and $M_B(t)$ defined in (3.19) for fermions and bosons, respectively, here, we introduce a unique field $M(t)$ [88]. This procedure is allowed because of the property that the determinant in the denominator Eq. (3.11) is independent of the fluctuating Hubbard-Stratonovich field. Then, this procedure will lead to to the action,

$$S = \int dt \left[\sum_{i,a} \Phi^{+}_{i,a}[(i\partial_t + \mu)\delta ij - 2M_{ij}]\Phi_{i,a} + \sum_{ijkl,a,b} M_{ij}[J^{-1}]M_{kl} \right]. \quad (5.1)$$

Furthermore, we integrate over the Gaussian fluctuating field M. This procedure gives the following expression for the action:

$$S = \int dt \sum_{i,a} \Phi^{+}_{i,a}\left[(i\partial_t + \mu)\Phi_{i,a} + \sum_{ijkl,a,b} J_{ijkl}\Phi^{\dagger}_{i,a}(t)\Phi_{j,a}(t)\Phi^{\dagger}_{k,b}(t)\Phi_{l,b}(t) \right]. \quad (5.2)$$

As the next step, we perform disorder averaging. The integration measure of random couplings J_{ijkl} is Gaussian: $\sim\exp(-N^3 \sum_{ijkl} J^{\dagger}_{ijkl}J_{ijkl}/8J^2)$ with $J^{\dagger}_{ijkl} = J_{jilk}$. However, since the couplings have a property of $J_{ijkl} = -J_{ilkj} = -J_{kjil} = J_{klij}$, only half of them are independent. We can select the independent part of couplings J_{ijkl} by using the ordering of the indices and choosing $i > k,\ j > l$ term. Other terms with $i > k,\ j < l,\ i < k,\ \ j > l,\ i < k,\ \ j < l$ are equal to selected one with the appropriate sign. The measure over independent couplings, thus, becomes

$$\mathcal{W}(J) = \exp\left(-\frac{N^3}{2J^2} \sum_{i>k,j>l} |J_{ijkl}|^2 \right). \quad (5.3)$$

According to the ordering of indices decribed above, the interaction term on the right-hand side of (5.2) is a sum of four independent terms $J_{ijkl},\ i>k,\ j>l$,

$$\sum_{a,b}\sum_{ijkl} J_{ijkl}\Phi^{\dagger}_{i,a}(t)\Phi_{j,a}(t)\Phi^{\dagger}_{k,b}(t)\Phi_{l,b}(t) = \sum_{a,b}\sum_{i>k,j>l} 2J_{ijkl}[\Phi^{\dagger}_{i,a}(t)\Phi_{j,a}(t)\Phi^{\dagger}_{k,b}(t)\Phi_{l,b}(t) - \Phi^{\dagger}_{i,a}(t)\Phi_{l,a}(t)\Phi^{\dagger}_{k,b}(t)\Phi_{j,b}(t)]. \tag{5.4}$$

The disorder averaging (i.e., the integration over independent J_{ijkl}), thus, produces an interacting theory with action that is similar to the one in Eq. (3.32),

$$S = \int dt\Bigg\{\sum_{i,a}\Phi^{\dagger}_{i,a}(t)(i\partial_t+\mu)\Phi_{i,a}(t) + \frac{2iJ^2}{N^3}\int dt\,dt'\sum_{aba'b'}\sum_{i>k,j>l}[\Phi^{\dagger}_{i,a}(t)\Phi_{j,a}(t)\Phi^{\dagger}_{k,b}(t)\Phi_{l,b}(t) \times \Phi^{\dagger}_{j,a'}(t')\Phi_{i,a'}(t')\Phi^{\dagger}_{l,b'}(t')\Phi_{k,b'}(t') - \Phi^{\dagger}_{i,a}(t)\Phi_{j,a}(t)\Phi^{\dagger}_{k,b}(t)\Phi_{l,b}(t)\Phi^{\dagger}_{l,a'}(t')\Phi_{i,a'}(t')\Phi^{\dagger}_{j,b'}(t')\Phi_{k,b'}(t')]\Bigg\}. \tag{5.5}$$

Equation (5.5) is invariant under the supersymmetry transformation $\delta\chi_i = \epsilon b_i,\ \delta b_i = -\epsilon\chi_i,\ i = 1\cdots N$, where ϵ is an infinitesimal Grassmann parameter. The reason for this is that the building block of the action, namely, $\Phi^{+}_{i,a}\Phi_{i,a}$, is invariant.

There are two distinct approaches for superbosonization of the SYK model. A general approach is based on the introduction of identity into the partition function,

$$1 = \int_{\mathbb{H}_n} dQ_{ia,jb}(t,t')\delta[Q_{ia,jb}(t,t') - \Phi_{ia}(t)\Phi^{\dagger}_{jb}(t')]. \tag{5.6}$$

Here, $Q_{ia,jb}(t,t')$ is a nonlocal supermatrix. The second simpler way would be through introducing

$$1 = \int_{\mathbb{H}_n} dQ^i_{aa'}(t,t')\delta\big[Q^i_{aa'}(t,t') - \Phi_{ia}(t)\Phi^{\dagger}_{ia'}(t')\big] \tag{5.7}$$

imposed by the nonlocal matrix $Q^i_{aa'}(t,t')$. Here, $\mathbb{H}_n$ is the linear space of Hermitian $2n\times 2n$ supermatrices. We recall that formal sums of formal products $\Phi\otimes\Phi^{\dagger}$, where $\Phi\in U(n,1|n,1)$ and $\Phi^{\dagger}\in U(n,1|n,1)$ are supervectors, constitute a vector space. This vector space is defined, up to isomorphism, by the condition that every antisymmetric bilinear map $f\colon U(n,1|n,1)\times\bar{U}(n,1|n,1)\to\mathbb{G}$ determines a unique linear map $g\colon U(n,1|n,1)\otimes\bar{U}(n,1|n,1)\to\mathbb{G}$ with $f(\Phi,\Phi^{\dagger}) = g(\Phi\otimes\Phi^{\dagger})$. This implies that, if we consider a map, $\mathcal{F}\colon\mathbb{H}_n\to\mathbb{G}$, then, the integral $\int D\Phi\ D\Phi^{\dagger}\ F(\Phi\otimes\Phi^{\dagger})$ is now well defined. From now on, we will restrict ourselves to the case of maps $\mathcal{F}$ such that the above integral is convergent.

The δ function in Eqs. (5.6) and (5.7) is a functional defined as in Ref. [82]. Namely, for all $\mathcal{A}\in\mathbb{H}_n$, the convergent integral $\delta(\mathcal{A}) = \lim_{\eta\to 0}\int_{\mathbb{H}_n} DB\,\exp\{i\,\mathrm{Str}[\mathcal{A}B] - \tilde{\eta}\,\mathrm{Str}[B^2]\}$, taken over $\mathbb{H}_n$ with flat Berezin measure [89], where the symbol “Str” stands for supertrace, satisfies the condition $\int_{\mathbb{H}_n} D\mathcal{A}'\ \delta(\mathcal{A}'-\mathcal{A})\equiv 1$. Moreover, for any map, $\mathcal{F}\colon\mathbb{H}_n\to\mathbb{G}$, that converges exponentially (or faster), the identity $\mathcal{F}(Q)\equiv\int_{\mathbb{H}_n} D\mathcal{A}\,\mathcal{F}(\mathcal{A})\delta(\mathcal{A}-Q)$ always holds.

Using the above expression for the δ functional in Eq. (5.7), and inserting the identity to the partition function defined by Eq. (5.5), we obtain an effective action,

$$S = \int dt\,dt'\Bigg\{\sum_i\{(i\partial_t+\mu)\delta_{tt'}\mathrm{Str}[Q^i(t,t')] + \sum_a\Phi^{\dagger}_{i,a}(t)B^i_{aa'}(t,t')\Phi_{i,a'}(t') + \mathrm{Str}[B^i(t,t')Q^i(t',t)] - \eta\,\mathrm{Str}[B^i(t,t')B^i(t,t')]\} + \frac{2iJ^2}{N^3}\sum_{i>k,\,j>l}\{\mathrm{Str}[Q^i(t,t')Q^j(t',t)]\mathrm{Str}[Q^k(t,t')Q^l(t',t)] - \mathrm{Str}[Q^i(t,t')Q^j(t',t)Q^k(t,t')Q^l(t',t)]\}\Bigg\}. \tag{5.8}$$

We see that the superfield $\Phi_{i,a}(t)$ enters into this action only as a quadratic form with the matrix $B^i(t,t')$. Therefore, the integral over superfields $\Phi_{i,a}$ in the partition function can be exactly evaluated, producing the superdeterminant of $B^i(t,t')$ in the denominator of the integrand. It is worth mentioning that the supermatrix B should be considered as a matrix by its arguments $B^i(t,t') = B^i_{t,t'}$. The partition function of the model, thus, becomes

$$Z = \int\prod_i \mathcal{D}B^i(t,t')\mathcal{D}Q^i(t,t')\mathrm{Sdet}[B]\exp\{iS\}, \tag{5.9}$$

where “S det” is superdeterminant.

Now, omitting for a while the first two terms in Eq. (5.8), we introduce a notation $\tilde{S}$ for the remaining terms in the expression and write is in the form

$$\tilde{S} = \lim_{\eta\to 0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dt\,dt'\Bigg\{\mathrm{Str}[B^i(t,t')Q^i(t',t)] + i\eta\sum_i \mathrm{Str}\{[B^i(t,t')]^2\} + \frac{2iJ^2}{N^3}\times\sum_{i>k,j>l}\{\mathrm{Str}[Q^i(t,t')Q^j(t',t)]\mathrm{Str}[Q^k(t,t')Q^l(t',t)] - \mathrm{Str}[Q^i(t,t')Q^j(t',t)Q^k(t,t')Q^l(t',t)]\}\Bigg\}.$$

Then, following the method introduced in Ref. [82], we join Q with B by introducing a new supermatrix $\bar{B} = BQ$. Here, we note that the formal sums of Hermitian superbivectors (product of two supermatrices, each of them being from the linear space of complex Hermitian supermatrices $\mathbb{H}_n$), constitute a vector space $\Lambda^2(\mathbb{H}_n)$ called the second exterior power of $\mathbb{H}_n$. Then, integration over $\bar{B}\in\Lambda^2(\mathbb{H}_n)$ decouples from the partition function and produces a constant,

$$C_n = \int_{\Lambda^2(\mathbb{H}_n)}\mathcal{D}\bar{B}\mathrm{Sdet}[\bar{B}]\exp\left\{\int dt\,dt'\mathrm{Str}[\bar{B}(t,t')]\right\}. \tag{5.10}$$

The Berezinian of the transformation $\bar{B} = BQ$ is one. One can see this by explicitly writing the transformations using the matrix form of $\bar{B}$ (with first indices corresponding to fermion or boson fields). Namely, the Jacobian of the transformation $\bar{B}_{Bc} = B_{Ba}Q_{ac}$ is Sdet$[Q]$, whereas for $\bar{B}_{Fc} = B_{Fa}Q_{ac}$ the Jacobian is $1/\mathrm{Sdet}[Q]$. As a result of the transformation, these two terms cancel each other in the product. This happens because the fields $\bar{B}_{Fc}$ and $\bar{B}_{Bc}$ always have opposite fermionic parity.

Finally, the partition function Z acquires the form

$$Z = \int_{\mathbb{H}_n}\prod_i \mathcal{D}Q^i(t,t')\frac{1}{\mathrm{SDet}[Q^i(t,t')]}\exp\{i\tilde{S}(Q)\}, \tag{5.11}$$

with the action,

$$\tilde{S}(Q) = \int dt\,dt'\Bigg\{\sum_i \mathrm{Str}[(i\partial_t+\mu)\delta_{tt'}Q^i(t,t')] + \frac{2iJ^2}{N^3}\sum_{i>k,j>l}\{\mathrm{Str}[Q^i(t,t')Q^j(t',t)] \times\mathrm{Str}[Q^k(t,t')Q^l(t',t)] - \mathrm{Str}[Q^i(t,t')Q^j(t',t)Q^k(t,t')Q^l(t',t)]\}\Bigg\}, \tag{5.12}$$

It is important, now, to observe that the 2×2 matrix Green's function $\mathcal{G}_i^{ab}(t,t')$ of two-component superfields $\Phi_{ia}(t)$ and $\Phi^\dagger_{ib}(t')$ (that contains the fermion propagator in its fermion-fermion block) is equal to the vacuum average of the superbosonization matrix field $\langle Q^i_{ab}\rangle$. Indeed, using the identity Eq. (5.7), one can introduce $Q_i^{ab}(t,t')$ under the integral and obtain

$$\begin{aligned}\mathcal{G}_i^{ab}(t,t') &= -i\langle\Phi_{ia}(t)\Phi^\dagger_{ib}(t')\rangle\\ &= -i\int\mathcal{D}\Phi\,\Phi_{ia}(t)\Phi^+_{ib}(t')\exp\{iS\}\\ &= -i\int\mathcal{D}Q\,Q^i_{ab}(t,t')\exp\{i\tilde{S}(Q)\}\\ &= -i\langle Q^i_{ab}(t,t')\rangle.\end{aligned} \tag{5.13}$$

The functional SDet$[Q_i(t,t')]$ in the denominator of the expression (5.11) for Z should be understood as the superdeterminant of the supermatrix $Q^i_{aa'}(t,t')$, which acts linearly in the continuous space of time t. Namely, arguments t,t' should be considered as matrix indices. One can incorporate the preexponent $1/\mathrm{SDet}[Q^i(t,t')]$ into the effective action S_{eff}, that can be written as

$$S_{\text{eff}} = \int dt\,dt'\Bigg\{\sum_i\mathrm{Str}[(i\partial_t+\mu)\delta_{tt'}Q^i(t,t')] + \frac{2iJ^2}{N^3}\sum_{i>k,j>l}\{\mathrm{Str}[Q^i(t,t')Q^j(t',t)] \times\mathrm{Str}[Q^k(t,t')Q^l(t',t)] - \mathrm{Str}[Q^i(t,t')Q^j(t',t)Q^k(t,t')Q^l(t',t)]\} + i\sum_i\mathrm{Str}[\ln Q^i](t,t')\delta(t-t')\Bigg\}. \tag{5.14}$$

Here, $\ln Q^i$ should be understood as the formal series $\ln Q^i = (Q^i-1)+\frac{1}{2}(Q^i-1)*(Q^i-1)+\frac{1}{3}(Q^i-1)*(Q^i-1)*(Q^i-1)+\cdots$, where the symbol $*$ stands for the convolution product $[A*B](t,t') = \int dt'' A(t,t'')B(t'',t')$.

Now, let us analyze the equation of motion of the field $Q^i(\tau,\tau')$ and compare it with the analysis performed in Sec. III. The crucial point is that we have an additional $\ln Q^i(t,t')$ term, which can contribute in the asymptotic analysis. From Eq. (5.7), we see that $\langle Q^i(t,t')\rangle = \langle\Phi_i(t)\Phi_i^\dagger(t')\rangle$ gives the Green's function $\mathcal{G}^i(t,t')$ and its asymptotic behavior at long timescale $t\to\infty$ is defined by the equation of motion for matrix field $Q^i(t,t')$,

$$\begin{aligned}\frac{\delta S_{\text{eff}}}{\delta Q^l(t,t')} &= 0\\ &= (i\partial_t+\mu)\delta_{tt'} + i[Q^l(t,t')]^{-1}\\ &\quad+\frac{2iJ^2}{N^3}\sum_{i>k,j>l}\mathrm{Str}[Q^i(t,t')Q^j(t',t)]Q^k(t,t')\\ &\quad-\frac{2iJ^2}{N^3}\sum_{i>k,j>l}Q^i(t,t')Q^j(t',t)Q^k(t,t').\end{aligned} \tag{5.15}$$

This equation shows that the solutions can be independent of the index i and, therefore, we drop it. Putting now $-i\langle Q(t,t')\rangle\equiv\mathcal{G}(t,t')$ into Eq. (5.15), setting $\mu=0$, and using $\mathcal{G}(t,t') = [(i\partial_t)\delta_{tt'} - iK(t,t')]^{-1}$ with $K(t,t')$ being the self-energy, at the long timescale, we obtain

$$K(t,t') = 2J^2\mathrm{Str}[\mathcal{G}(t,t')\mathcal{G}(t',t)]\mathcal{G}(t,t') \\ -2J^2\mathcal{G}(t,t')\mathcal{G}(t',t)\mathcal{G}(t,t'). \qquad (5.16)$$

In the saddle-point approximation and due to supersymmetry, we expect that the fermion-fermion (F) and boson-boson (B) entries of the Green's function are equal: $\mathcal{G}_F(t,t') = \mathcal{G}_B(t,t')$. The implication of this fact is that $\mathrm{Str}[\mathcal{G}(t,t')\mathcal{G}(t',t)] = 0$, which leads to the relation between the self-energy and the Green's functions for fermions and bosons,

$$K(t,t') = -2J^2\mathcal{G}(t,t')\mathcal{G}(t',t)\mathcal{G}(t,t').$$

We note similarity with Eq. (4.22) and the similar relation for fermions obtained using the replica approach. Hence, *at long timescales*, our supersymmetric model reproduces the same asymptotics for the Green's function as the replica method provides. However, at intermediate times, suppersymmetric action is essentially different from the replica field theory, and we expect that this method will provide new results at intermediate timescales. In order to see this, we rewrite the supertrace over the supermatrices Q_i in the interaction terms of the action S_{eff}, defined by (5.14), using fermion-boson (FB) and boson-fermion (BF) components of the supermatrices,

$$\mathrm{Str}[Q^i(t,t')Q^j(t',t)] \\ = Q^i_{BF}(t,t')Q^j_{FB}(t',t) - Q^i_{FB}(t,t')Q^j_{BF}(t',t) \\ - Q^i_{FF}(t,t')Q^j_{FF}(t',t) + Q^i_{BB}(t,t')Q^j_{BB}(t',t). \qquad (5.17)$$

Similarly for the $\mathrm{Str}[Q^k(t,t')Q^l(t',t)]$ part and $\mathrm{Str}[Q^i(t,t')Q^j(t',t)Q^k(t,t')Q^l(t',t)]$. BF and FB components of the supermatrices are Grassmann variables, and the integration over them is easily performed. It will produce separate actions for fermions and bosons of the form of Eq. (5.14) and preexponential mixed polynomials from the BB and FF components. Appearance of these mixed polynomials is a result of the formally exact supersymmetric approach, and these terms are not captured within the replica approach. At longe timescales, they have subleading contributions to the correlation function but will have essential contributions in the intermediate finite time region. This fact is a major advantage of the supersymmetric method. More detailed and complete analysis of these effects is a subject of future investigations.

The supersymmetry method can also be used to study nonperturbative effects in general SYK_q models. The spectral correlators in the SYK_q models and their deviation from RMT are studied in Refs. [59,90]. It appears that a small number of long-wavelength modes, which can be parametrized via Q-Hermit orthogonal polynomials, describe the deviation. Moreover, the SYK model with Majorana fermions is more straightforward and can be formulated as a σ model [30,90]. The analysis of two-point spectral correlators in two-loop order and shows corrections to RMT, whose lowest-order term corresponds to scale fluctuations in good agreement with numerical results [90]. However, the question remains about other loop terms in the loop expansion, and how they should (not) contribute. In general, the range of validity of the loop expansion remains open and is expected to be detectable from the supersymmetry method. Another obvious open problem is the model at $q = 2$. In this case, one should expect Poisson statistics for the spectral correlation. This is in contrast to the replica field theory which suggests a RMT behavior with a significant Thouless scale as shown in Ref. [90].

We expect a more straightforward understanding of the RMT structure of the SYK_q model. The superbosonization technique is well developed [82] and can provide exact and nonperturbative results. In our supersymmetric formulation, there are additional bosonic modes that interact with the original fermions. In the expression (5.17), we present an example of such terms. Just long-wavelength modes of these bosons have the potential of solving the problem of the scale of Thouless transition universally for $q \leqslant 4$.

Another advantage of the superbosonized σ-model representation described above is that it is efficient for computation of correlation functions. The procedure, described in Ref. [82], consists of

(1) Diagonalization of the $m \times m$ supermatrix field Q as $Q = UQ_{\mathrm{diag}}V$ with diagonalization matrices $U \in U(m|m)$ and $V \in U(m|m)/U^{2m}(1)$ restricted to the unitary supergroup and its subspace with removed phases.

(2) After the diagonalization of the supermatrix Q, one can integrate over Q by integrating over its boson-boson eigenvalues in the interval $\mathbb{R} \equiv \{-\infty, \infty\}$, whereas the integration over the fermion-fermion eigenvalues should be performed in the interval $i\mathbb{R} \equiv \{-i\infty, i\infty\}$.

We see that this procedure significantly reduces the number of integrations one has to perform to calculate correlation functions within the superbosonized representation.

VI. CONCLUSIONS AND OUTLOOK

Despite being a standard tool for nonperturbative calculations in disordered and chaotic systems, the supersymmetric σ model has rather poorly been understood for interacting systems. Historically, it was believed that the Hubbard-Stratonovich decoupling of the interaction Hamiltonian would not help to develop a supersymmetric description of the partition function of the model. The reason is that one has to introduce two *different* Hubbard-Stratonovich bosonic fields M_1 and M_2 to decouple interaction terms both in the numerator and in the denominator of the expression for any correlation function. It was believed, for about 40 yr, that supersymmetric σ-model representation of interacting systems was impossible because fluctuating M_1 and M_2 fields are independent. And, therefore, supersymmetry cannot become manifest in a theory that is disordered, interacting, and dynamical.

In this paper, we have challenged this belief and have developed a rigorous supersymmetric σ-model framework for interacting disordered systems. The idea that helps to overcome the above-mentioned problem of independence of fluctuating fields M_1 and M_2 is the following. The partition function of the system is calculated by the functional integration of an exponentiated action functional over the space of dynamical field configurations. We showed that, for $(0+1)$-dimensional systems, such as quantum dots, the Hubbard-Stratonovich field in the denominator could be gauged out. It can also be reintroduced back to guarantee supersymmetry. In order to derive basic formulas of the supersymmetry method, we have introduced a new version of the SYK model. In contrast to the previous versions, the model is time-reversal invariant. One of the main achievements of this paper is that we have given a supersymmetric σ-model description of the SYK model. We have also developed its superbosonized description, where the functional integral is taken over unconstrained dynamical supermatrix fields representing collective many-body excitations.

It is now a conventional wisdom [1] that the SYK model exhibits many-body chaotic properties at all timescales. At short times, chaos shows up in exponentially decaying correlations as manifested in out-of-time correlation functions [18,23–27] . At long timescales, chaos manifests itself in a random-matrix ensemble due to quantum energy-level repulsion [36]. However, the nature of the transition region from nonergodic to ergodic regimes remains unclear. Moreover, the physics of nonergodic states is not yet fully understood, and dirty metals represent an excellent physically motivated playground for such studies. Here, an important development was made in Ref. [91] where the theoretical description of nonergodic extended states in a modified SYK model was put forward. The problem of finding the ergodic (Thouless) time in the SYK model was considered in Ref. [30] where the questions regarding the nature of the relaxation modes, their classification by certain effective quantum numbers, as well as the density of states, were addressed. An important correlation function, capable of detecting chaotic properties of the SYK model, is the spectral number variance $\Sigma_2(\epsilon)$. It represents the statistical variation in the number of many-body levels contained in an energy window of width E. The variance $\Sigma_2(\epsilon)$ was studied in Ref. [59] where a deviation from the random-matrix ensemble prediction was reported. This deviation demonstrates the possible breakdown of ergodicity, and this is one of the interesting points that can be investigated further using superbosonization.

The spectral form factor considered in Ref. [58], representing the Fourier transform of the energy-dependent spectral two-point correlation function $R_2(\epsilon)$ is yet another quantity of interest. Whereas the long-time profile of it showed a ramp structure characteristic for random-matrix theory ensembles, universal deviations from random-matrix theory were observed for shorter times (see Refs. [92–95] for related studies). Density-density correlators were studied in Ref. [30] within the replica approach describing the quantum chaotic dynamics of the SYK model at long times. It was observed that there are nonergodic collective modes, which relax in some time interval and become ergodic states by entering into the long-time regime. The latter modes can be described using the random-matrix theory. These interesting modes share similar properties with the diffusion modes of dirty metals and have quantum numbers which have been identified as the generators of the Clifford algebra [27]. There, each of the $2N$ different products formed from N-Majorana operators represents a mode.

Here, we propose that the superbosonization approach to the SYK model will open new possibilities to study intermediate-time regions and reveal new aspects of chaotic properties. In particular, it would be fascinating to: (i) calculate the one-point correlation function $\langle\rho(E)\rangle$ (the density of states) in a superbosonized representation of the SYK model and compare it with the universal random-matrix prediction; (ii) calculate the two-point correlation function $\langle\rho(E)\rho(E')\rangle$ in the SYK model using its superbosonized representation and compare it with numerical calculations in Refs. [30,58]; (iii) reveal the role of bosonic excitations presented in a superbosonized representation and to detect their behavior at short times.

Systematic deviations from the random-matrix predictions, for sufficiently well-separated eigenvalues, imply that the model is not ergodic at short times. The point of departure from the results of random-matrix theory increases with N, which is an indication of having a Thouless energy scale [96–99] in the system. Detection of Thouless time within a superbozonized approach is yet another exciting project. It would be also interesting to calculate moments of the spectral density within the supersymmetric σ-model approach.

On another front, it is well known that Anderson localization can be avoided under certain conditions for disorder potential supporting long-range [100–102] or short-range [103–109] correlations in low dimensions. It is, thus, very interesting to investigate the effect of introducing correlations to the disordered interaction constant J_{ijkl}. We expect that such an analysis can also be performed using the technique outlined in the present paper.

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