# Cavity evolution and the Rayleigh-Plesset equation in superfluid helium

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Based on the two-fluid hydrodynamics, an analog of the famous Rayleigh-Plesset equation for the dynamics of a spherical vapor bubble in superfluid helium is derived. The two-fluid nature of He II and the specific form of the momentum flux density tensor give rise to a number of effects in the evolution of the boundary position $R(t)$, absent in ordinary fluids. One of them is the abnormal attenuation of the boundary oscillation, which exceeds the usual viscous damping by several orders of magnitude. There is also an additional term proportional to the squared velocity of the normal component, which is independent of the derivative $dR/dt$, and therefore can be included in the pressure drop. Its physical meaning is related to the dependence of pressure on the relative velocity between the normal and superfluid components. One more effect renormalizes the coefficient in front of $(dR/dt)^2$. The dissipative part of the momentum flux tensor is also being upgraded to take into account the two-fluid hydrodynamics. As an illustration of the stated theory, a numerical solution of the obtained master equation for the evolution of a vapor film on spherical heaters in He II is presented. The obtained results declare that some early issues and conclusions on the dynamics of the cavity in superfluid helium should be reviewed.

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## I. INTRODUCTION AND SCIENTIFIC BACKGROUND

The study of the cavity dynamics is an important part of the problems of continuum mechanics, including the hydrodynamics of superfluids. These may be problems related to the evolution of bubbles created by electrons [1–3], bubbles caused by sound [4], cavitation due to negative pressure [5], collapse of bubbles [6,7], sonoluminescence [8], etc. Closely important applications related to the formation of new phases and the evolution of interphase surfaces are the quantum kinetics of phase transitions at temperatures close to absolute zero [9], and also the kinetics of nucleation and stratification of dilute $^3$He-$^4$He solutions at low temperatures [10], or the growth of helium crystal facets [11]. One more series of examples is related to the heat transfer in He II and to the possibility of utilizing superfluid helium as a coolant in cryogenic systems, which has been discussed extensively recently [12]. Knowledge of the laws governing the formation and development of vapor films on the surfaces of heaters is important for solving corresponding problems [13–16].

Studying the dynamics of the cavity, the authors of the above-cited works appeal to the Rayleigh-Plesset problem on the evolution and oscillations of an air or vapor bubble, elaborated initially for an ordinary fluid (see, e.g., Ref. [17] and references therein). Such treatment, however, is justified in the case when superfluid helium behaves as an ordinary fluid and moves as a whole with a mass velocity $\mathbf{v} = \mathbf{j}/\rho$ (see the notations below). This situation occurs when helium is driven in the motion under the action of a pressure gradient (or gravity). However, whenever heat fluxes occur, a flow of the normal component appears with a velocity $\mathbf{v}_n$, different from the mass velocity $\mathbf{v}$, and the problem requires a fundamentally different treatment related to the two-fluid nature of He II.

In the present paper the problem of the evolution of a spherically symmetric vapor bubble in superfluid helium is considered. The equation playing the role of the famous Rayleigh-Plesset problem in an ordinary fluid is derived. The obtained equation results in a number of effects absent in ordinary fluids. They include effects such as an abnormal attenuation of the boundary oscillations, exceeding the usual viscous damping by several orders, or additional pressure caused by the relative velocity between normal and superfluid components. These effects essentially influence the dynamics of cavities and should be taken into account in the relevant works.

Loosely speaking, the dynamics of a bubble is determined by the inertia (mass) of ambient liquid and the elastic properties due to the pressure inside the cavity. The latter is the result of many factors including surface tension, Coulomb pressure, hydrostatic pressure, the viscous contribution into the stress tensor, etc. The corresponding task is a complex problem requiring a careful analysis of many ingredients.

In this work, we will focus on hydrodynamics processes inside the fluid, not taking into account the involved phenomena inside the bubble. In addition, for clarity and for deductive purposes, we take a pure two-fluid Landau-Khalatnikov model, excluding more complex phenomena, such as quantum turbulence [18,19] or the existence of a thermal boundary layer near the interphase liquid-vapor boundary [20]. In addition, we omit the effects, related to the compressibility of the fluid, and, respectively, we omit the radiation of sound

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[21–23]. We also do not consider radiation of the second sound arising from nonzero relative velocity between the normal and superfluid components. Realizing the importance of the above processes for the whole evolution of the cavities, we nonetheless omit them in order to emphasize the fascinating features of the two-fluid Landau-Khalatnikov model of superfluid hydrodynamics in the process under consideration. All listed effects such as compressibility or quantum turbulence should be introduced as further approximations to the master equation (19).

In the next section we perform a detailed derivation of the Rayleigh-Plesset-like equation for the evolution of the boundary position of a spherical bubble in superfluid helium. The main features of the derived equation are also described there. Remarks on the obtained results are made and possible developments are discussed in the conclusion.

## II. THE RAYLEIGH-PLESSET PROBLEM IN SUPERFLUID HELIUM

### A. Treatment of the momentum flux density tensor $\Pi_{rr}$

It is known from the solution of the Rayleigh-Plesset problem in a classical fluid that the master variable, which controls the whole process, is the mass flow velocity $\mathbf{v}(\mathbf{r}, t)$, for the simple reason that in a spherical case the variable $v(r, t)$ at the boundary points $R(t)$ coincides with the rate of change $dR/dt$. The governing equation for $R(t)$ is derived from the equation for momentum density, which, due to a noncompressibility condition, is simply the Euler equation for velocity $v(r, t)$.

In superfluid helium the situation is more complicated. The momentum density $j_i = \rho_s v_{si} + \rho_n v_{ni}$ consists of two ingredients - superfluid and normal ones. The evolution equation for the momentum density $\mathbf{j}$ reads

$$\frac{\partial j_i}{\partial t} + \frac{\partial \Pi_{ik}}{\partial x_k} = 0. \tag{1}$$

The momentum flux density tensor $\Pi_{ik}$ has also two ingredients and it is equal to

$$\Pi_{ik} = \rho_s v_{si} v_{sk} + \rho_n v_{ni} v_{nk} + \delta_{ik} p. \tag{2}$$

It does not explicitly include the mass flow velocity $\mathbf{v}(r, t) = \mathbf{j}/\rho$, as it takes place in the case of ordinary fluids. In other words, in superfluid helium, the momentum flux density tensor $\Pi_{ik}$ is not equal to $\rho v_k \partial v_i/\partial r_k$, as it is sometimes used or implied in the relevant works. On the contrary, because of the two-fluid hydrodynamics, the structure of the momentum flux density tensor is more involved and should reflect the presence of two ingredients, the superfluid and normal parts.

In fact, the superfluid and normal velocities $\mathbf{v}_s$, $\mathbf{v}_n$ are not convenient for solving our problems. More suitable variables are the mass flow velocity and the velocity $\mathbf{v}_n$ of the normal component. Indeed, the mass flow velocity $\mathbf{v}(r, t)$ is responsible for the inertia of the fluid and the normal velocity $\mathbf{v}_n$ is tightly related to the thermal processes. Further, we assume that the total density $\rho$, as well as the superfluid and normal densities $\rho_s$ and $\rho_n$ are individually constant.

The next, crucial step is to treat the equation for momentum density (1). We have to express the momentum flux density tensor $\Pi_{ik}$ via quantities $\mathbf{v}(r, t)$ and $\mathbf{v}_n(r, t)$, which were selected as the primary variables. To get rid of the velocity of the superfluid component entering in Eq. (2) for $\Pi_{ik}$, we use the relation known from classical superfluid hydrodynamics [20,24],

$$\mathbf{v}_s = \frac{\rho \mathbf{v}}{\rho_s} - \frac{\rho_n}{\rho_s} \mathbf{v}_n. \tag{3}$$

The momentum flux density tensor $\Pi_{ik}$ [see Eq. (2)] can be rewritten in terms of the chosen variables $\mathbf{v}(r, t)$ and $v_n(r, t)$ as

$$\Pi_{ik} = \frac{\rho^2}{\rho_s} v_i v_k + \frac{\rho_n^2}{\rho_s} v_{nk} v_{ni} - \frac{\rho \rho_n}{\rho_s} v_i v_{nk} - \frac{\rho \rho_n}{\rho_s} v_k v_{ni} + \rho_n v_{ni} v_{nk} + p\delta_{ik}. \tag{4}$$

Now we have to transform the momentum flux density tensor $\Pi_{ik}$ (4) into spherical coordinates. The simplest way to do this is as follows. We can represent expressions of the type $A_k \partial B_i/\partial x_k$ as an $i$ component of combination $(\mathbf{A} \cdot \nabla)\mathbf{B}$. In accordance with the well-known mathematical relation, we have $(\mathbf{A} \cdot \nabla)\mathbf{B} = \nabla(\mathbf{A} \cdot \mathbf{B}/2)$ [25]. The latter operation is possible, since due to the spherical symmetry and incompressibility of both components, $\nabla \cdot \mathbf{A} = 0$, $\nabla \times \mathbf{A} = 0$, and the same for vector $\mathbf{B}$. Using this rule we rewrite the radial component of the momentum flux density tensor $\Pi_{rr}$ (4) in spherical coordinates as

$$\Pi_{rr}(r) = \frac{1}{2\rho_s}\left(v^2\rho^2 - 2\rho\rho_n v v_n + \rho_n \rho v_n^2\right). \tag{5}$$

Then, the equation for the mass velocity $v(r, t)$, following from (1), has the form

$$\frac{\partial v}{\partial t} + \frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{\rho}{\rho_s} v \frac{\partial v}{\partial r} - \frac{\rho_n}{\rho_s}\left(v_n \frac{\partial v}{\partial r} + v \frac{\partial v_n}{\partial r}\right) + \frac{\rho_n}{\rho_s} v_n \frac{\partial v_n}{\partial r} = 0. \tag{6}$$

If $v_n = v$ (the coflow case), the momentum flux density tensor $\Pi_{ik}$ (5) transforms to

$$\Pi_{rr} = \frac{1}{2} v^2 \frac{\rho}{\rho_s}(\rho - \rho_n) = \frac{1}{2}\rho v^2, \tag{7}$$

as it should be in the ordinary fluid.

### B. Treatment of $\mathbf{v}_n$, flux of energy

Equation (6) is the basis for obtaining the equation for the dynamics of the cavity boundary $R(t)$. However, this equation is not closed. In addition to the main variable velocity $v$, the equation also contains the velocity of the normal component $v_n$, and our goal now is to get rid of it. The simplest way to do this is based on a consideration of the energy flux density $\mathbf{W}$, which is [20,24]

$$\mathbf{W} = \left(\mu + \frac{\mathbf{v}_s^2}{2}\right)\mathbf{j} + ST\mathbf{v}_n + \rho_n \mathbf{v}_n[(\mathbf{v}_n - \mathbf{v}_s) \cdot \mathbf{v}_n]. \tag{8}$$

Here, $S$ is the entropy density, and $\mu$ the chemical potential. It should be noted at once that we observe a macroscopic energy flux $ST\mathbf{v}_n$ even in the case when the total mass flow $\mathbf{j}$ is equal to zero (the so-called counterflow). Neglecting the nonlinear effects of the third order and taking that the energy flux $\mathbf{W}$ in this case is simply the heat flux $\mathbf{Q}$, we arrive at the formula

$$\mathbf{Q} = ST\mathbf{v}_n. \tag{9}$$

In case $\mathbf{v}_n = \mathbf{v}$, or, which is equivalent, $\mathbf{v}_n = \mathbf{v}_s = \mathbf{v}$, that is, the fluid moves as a whole (the so called co-flow), the energy flux $\mathbf{W}$ has a form

$$\mathbf{W} = \left(\mu + \frac{\mathbf{v}_s^2}{2}\right)\rho\mathbf{v} + ST\mathbf{v} = \left(h + \frac{\mathbf{v}^2}{2}\right)\mathbf{v} \tag{10}$$

(here, $h = \mu + TS$ is the enthalpy), as it should be in a one-phase fluid or in the case of coflow [26].

A variant of the above approach was used in Ref. [16] where the authors applied relation (9) to express $\mathbf{v}_n$ via heat flux $\mathbf{Q}$, released by the heater. Since, however, the total mass flow $\mathbf{j}$ is not equal to zero, part of the total energy released by the heater is converted into mechanical energy, associated with the motion of helium as a whole. Thus, if the flow is not a pure counterflow, the normal velocity $\mathbf{v}_n$ is not uniquely determined by the heat flux $Q$ and the situation requires a more thorough investigation. We have to use the full expression for the energy flux (8). Neglecting again nonlinear effects of the third order as well as the irreversible energy flux $\mathbf{W}_{\text{irr}}$, we arrive at the following formula,

$$v_n = \frac{W}{ST} - \frac{\mu\rho}{ST}v \tag{11}$$

(since we work for a purely spherically symmetric case we omit the vector notations). The origin and nature of the energy flow $W$ can be different—for example, it can be a spherical heater forming a (spherical) vapor region around itself, or it can be an external pressure drop that causes the cavity to either collapse or oscillate.

Substituting $v_n$ from (11) in the expression for the momentum flux density tensor $\Pi_{rr}$ (5), we obtain

$$\Pi_{rr} = \frac{\rho}{2S^2T^2\rho_s}(S^2T^2v^2\rho + 2\rho_n STv^2\mu\rho - 2\rho_n STvW + \rho_n v^2\mu^2\rho^2 - 2\rho_n v\mu\rho W + \rho_n W^2). \tag{12}$$

Further, we will use the enthalpy $h = \mu\rho + ST$ instead of the chemical potential $\mu$, since the enthalpy $h$ is more reliably measured and tabulated quantity. After that, the expression for $\Pi_{rr}$ takes the form

$$\Pi_{rr} = \left(\frac{\rho}{2S^2T^2\rho_s}(h^2\rho_n + S^2T^2\rho - S^2T^2\rho_n)\right)v^2 - \left(\frac{\rho\rho_n h}{S^2T^2\rho_s}\right)Wv + \frac{\rho\rho_n}{2S^2T^2\rho_s}W^2. \tag{13}$$

We grouped the terms as follows: The first term contains the squared velocity $v$; the second term contains the cross term $Wv$; and, finally, the third term does not contain the mass velocity $v$ at all. Then, substituting (13) into Eq. (6), we get

$$\frac{\partial v}{\partial t} + \frac{1}{\rho}\frac{\partial p}{\partial r} + Av\frac{\partial v}{\partial r} + \frac{1}{2}Bv\frac{\partial W}{\partial r} + \frac{1}{2}BW\frac{\partial v}{\partial r} + C\frac{\partial}{\partial r}\frac{1}{2}W^2 = 0. \tag{14}$$

The coefficients $A$, $B$, and $C$ are equal to

$$A = \frac{1}{S^2T^2\rho_s}(h^2\rho_n + S^2T^2\rho - S^2T^2\rho_n), \tag{15}$$

$$B = -\frac{\rho_n}{S^2T^2\rho_s}h, \quad C = \frac{1}{S^2T^2}\frac{\rho_n}{\rho_s}. \tag{16}$$

We will discuss further the physical meaning of such a grouping.

### C. Equation for the boundary position $R(t)$

We proceed to derive equations for the boundary position evolution $R(t)$. Due to the incompressibility condition, the radial component of the mass flow velocity $v = j/\rho$ with the classical solution is

$$v(r,t) = \frac{R^2}{r^2}\frac{dR}{dt}. \tag{17}$$

To move further we have to work with the equation for the radial mass flow velocity $v(r,t)$ (14). For definiteness, at this stage we specify the problem and consider a purely thermal case: the development of a vapor film created by a spherical heater of the radius $R_H$, immersed in superfluid helium. Then the only source of energy is the heat released on the heater, and the energy flux $W(r)$ into the surrounding space has the form

$$W = \frac{R_H^2}{r^2}Q. \tag{18}$$

Here, $Q$ is the heat flux density, released on the surface of the heater. Combining Eq. (18) with the expression for mass velocity $v(r,t)$ (17), accomplishing (where appropriate) differentiation with respect to $r$ and integration over $r$ in limits from $R$ to $\infty$, we obtain the following equation for the evolution of the boundary position $R(t)$,

$$R\frac{d^2R}{dt^2} + \left(2 - \frac{A}{2}\right)\left(\frac{dR}{dt}\right)^2 - B\left(\frac{R_H^2}{R^2}Q\right)\frac{dR}{dt} - \frac{C}{2}\left(\frac{R_H^2}{R^2}Q\right)^2 = \frac{1}{\rho}[p(R) - p(\infty)]. \tag{19}$$

The classical Rayleigh-Plesset equation includes the contribution from the surface tension $2\gamma/R$ into the pressure $p(R)$ and the contribution in the momentum flux density tensor due to the fluid viscosity $\eta$. In the case of superfluid helium, the contribution from the surface tension retains its form. The situation with a contribution from the viscous momentum flux density tensor is a bit more complicated. To find it we have put the expression for the radial velocity $v(r)$ (17), as well as the radial energy flux $W(r)$ (18), in the dissipative momentum flux tensor [20,24] and apply it at point $r = R$. Corresponding calculations show that there is the viscous contribution into pressure $\Delta_{\text{diss}}p(R)$, equal to

$$\Delta_{\text{diss}}p(R) = 4\eta_n\left(\frac{\mu\rho}{ST}\right)\frac{1}{R}\frac{dR}{dt} - 4\eta_n\frac{1}{ST}\frac{R_H^2}{R^3}Q, \tag{20}$$

which should be incorporated into Eq. (19).

Resuming this section, we conclude that the master equation for the evolution of the cavity boundary position in superfluid helium, which plays the role of the Rayleigh-Plesset equation, is Eq. (19), with the dissipative additives

(20). We note again that we considered the basic variant of the Landau-Khalatnikov two-fluid hydrodynamics, without quantum turbulence [18] and a thermal boundary layer [20]. In this sense, Eq. (19) can be considered as the first step.

## III. ANALYSIS OF THE MASTER EQUATION

### A. Qualitative analysis

The resulting master equation (19) has a structure similar to the structure of the Rayleigh equation for a classical fluid [17]. Namely, this is a nonlinear differential equation of the second order describing decaying oscillations, or a relaxation-type evolution to a stationary state. At the same time Eq. (19) is very different from the classical Rayleigh-Plesset equation. It includes additional terms, absent in the classical case. This difference arises due to the two-fluid model and a specific form of the momentum flux density tensor $\Pi_{rr}$ (2). In the case when the fluid moves as a whole, i.e., $\mathbf{v}_n = \mathbf{v}_s = \mathbf{v}$, Eq. (19) reduces to the classical equation. Indeed, if we use $W = hv$ [neglecting the nonlinear effects of third order—see Eq. (10)], then the quantity $\Pi_{rr}$ (13) transforms to its classical value $v^2/2$. As a result, the well-known problems such as isothermal oscillations of the gas bubble, or the collapse of an empty cavity, coincide (except for the dissipative terms—see the last paragraph in this section) with the classical solution. The real and essential difference appears when $\mathbf{v}_n$ is not equal to $\mathbf{v}_s$.

The terms in the momentum flux tensor $\Pi_{rr}$ (13) and hence in Eq. (19) are combined into groups with different physical meanings. In particular, the terms containing the derivative $dR/dt$ are important for nonstationary processes, such as a transient process, or oscillatory motion. The remaining terms that do not contain $dR/dt$ determine the stationary solution $R(t \to \infty)$, e.g., the thickness of the vapor film, or the size of the vapor bubble. So, the third term on the right-hand side of Eq. (19), which does not depend on the velocity of the boundary position $dR/dt$, can be included into the pressure term $p(R)$. That can be additionally justified by the fact that this term is proportional to squared velocities and, hence, it is related to the dependence of pressure on the relative velocity between the normal and superfluid components (see also Refs. [20,24]). Further, we will call it a "Bernoulli-like" pressure. In many cases, this additional pressure is small—for example, in experiments [13,27] this Bernoulli-like pressure is of the order of 10% of the hydrostatic pressure $\rho g h$, however, for smaller values of $h$, and especially under microgravity conditions, it can be extremely important.

The second term on the right-hand side of Eq. (19) is of particular interest. It has the same structure as the dissipative viscous term (20), which is responsible for the attenuation of bubble oscillations. At the same time it essentially (by several orders of magnitude) exceeds this viscous damping. For this reason, we will call it the "extra-damping" term. The extra-damping term can be the source of the abnormally strong attenuation of bubble oscillations, observed in many works [13,27]. The authors of Ref. [14] called this phenomenon the abnormal "suppression of oscillations of the vapor-liquid phase boundary in superfluid helium." To our knowledge, the authors could not explain this phenomena and referred to purely experimental obstacles.

The second term on the left-hand side of Eq. (19) differs from the classical case in that the coefficient 3/2 is replaced with the quantity $2 - A/2$. In the limit of an Euler fluid when $\rho_n = 0$, the quantity $A = 1$ and the classical Rayleigh case are recovered. A preliminary numerical analysis of the solution of Eq. (19) shows that this term does not greatly affect the final results.

The dissipative additive (20) to the pressure $p(R)$ at the interface requires a special comment. In the case of thermal problems, this addition is much smaller than the cross extra-damping term and it can be neglected. In the case of a mechanical flow, when the coflow regime is realized, for example, with the collapse of the cavity, the dissipative additive (20) in the pressure at the boundary can play a significant role. However, it should be noted that the corresponding contribution differs from the quantity $4\eta_n \frac{1}{R}\frac{dR}{dt}$, which is sometimes used in the relevant works.

### B. Simple example: Evolution of the vapor film on the spherical heater

As a simple illustration to the qualitative analysis, outlined above, we will apply the approach developed in this paper to discuss a series of works on boiling superfluid helium on spherical heaters [13,14,16,27]. In the listed works, both experimental studies and theoretical calculations based on the classical equation Rayleigh-Plesset are performed. In principle, for some special cases, the Rayleigh-Plesset equation has an analytical solution. For example, in Ref. [28], such a solution was found for the Rayleigh equation when the gas in the bubble was an ideal and obeyed the polytropic law. In general cases, the pressure inside the bubble obeys a more complex law, so we use a numerical solution.

One of the principal and delicate questions for studying the dynamics of a vapor film concerns the value of the pressure near the boundary $R(t)$. As discussed earlier, there are many factors which contribute to the quantity $p(R)$. In addition to the surface tension pressure $2\gamma/R$, and the contribution to the stress tensor arising from the viscosity of the normal component, which is already present in Eq. (19), there should be also contributions due to the thermal processes inside the film. The corresponding pressure has to be determined from energy equations [7,16,17,29]. There is also the Bernoulli-like pressure, proportional to the squared normal velocity [the fourth term on the left-hand side of Eq. (19)]. A very important contribution to pressure can appear from processes of evaporation (condensation) at the boundary.

Resuming the above discussion, we conclude that the pressure inside the bubble $p(R)$ is a poorly controlled quantity that should be determined from a separate thorough analysis. That problem certainly goes beyond the scope of the task posed in our work. Our goal is to highlight the specific feature of the superfluid hydrodynamics in the evolution of a vapor film. Therefore, with the aim to find the difference in the dynamics of the cavity in a normal liquid and superfluid helium, we simply use the expression for the pressure that has been used in the above-cited articles [13,14,16,27]. The authors of these articles accept that the main ingredient in vapor pressure $p(R)$ arises due to nonequilibrium vaporization and condensation of helium at the interphase boundary. According to the theory,

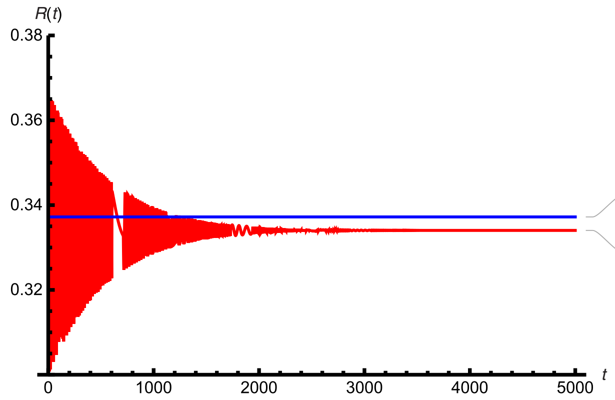


FIG. 1. Long-time behavior of $R(t)$ (in cm). See explanations in the text.

the relationship between pressure and heat flux, obtained by solving the Boltzmann kinetic equation, has the form

$$p(R) = \frac{\sqrt{\pi}}{4}\frac{Q_h}{\sqrt{2R_{\mathrm{He}}T}}\frac{R_h^2}{R(t)^2}. \tag{21}$$

Here, $R_h$ is the radius of a spherical heater, $Q_h$ is the heat flux density released, and $R_{\mathrm{He}}$ the individual gas constant.

To compare our approach with the results on He II boiling on a sphere, we take the parameters used in Ref. [13]: The radius of the spherical heater is $R_h = 3.0\,\mathrm{mm}$, the averaged immersion depth $h = 90\,\mathrm{mm}$, and the heat flux density released on the heater is $\dot{Q}_h = 30.5\ \mathrm{kW/m^2}$. As for the initial condition for velocity of the interphase boundary $dR/dt$, it can be obtained from the following considerations. Let us assume that the formation of a film and the motion of the interface at the initial moment are associated with evaporation. Then, comparing the applied heat load $4\pi R_h^2\dot{Q}_h$ with the latent heat $L$ per unit volume, we estimate the initial speed as $\dot{R}(0) = \dot{Q}_h/L \approx 1$ cm/s. It should be noted, however, that the final results are not very sensitive to the choice of initial condition (except of the initial period).

The results of the calculations on the evolution of the radius of the vapor film $R(t)$, formed after the stepwise switching on of the heat load on the spherical heater, are shown in Figs. 1–3. Figure 1 shows the behavior of the function $R(t)$

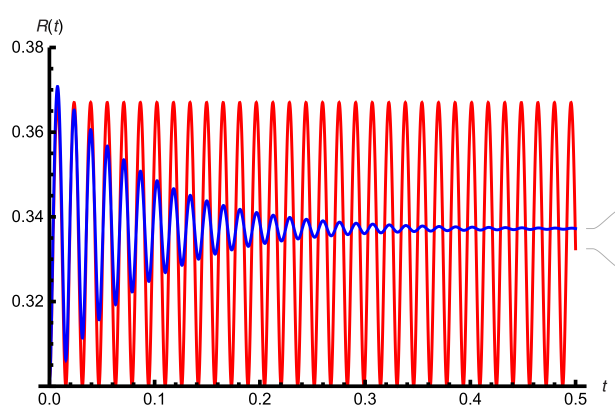


FIG. 2. Middle-time behavior of $R(t)$ (in cm). See explanations in the text.

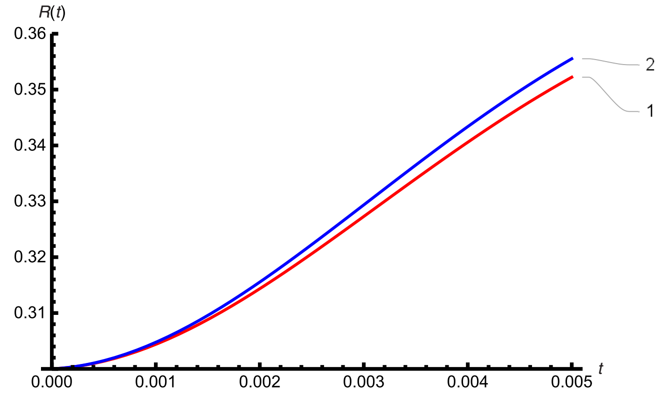


FIG. 3. initial behavior of $R(t)$ (in cm). See explanations in the text.

over a long period of time, about 5000 s. It can be seen that the solution based on the classical Rayleigh-Plesset equation (curve 1) represents very slowly damping oscillations (with decrements of the order of $1/2000\ \mathrm{s}^{-1}$), which is associated with the very low viscosity of the normal component $\nu_n$. Curve 2 corresponds to Eq. (19) for the motion of interface $R(t)$ in superfluid helium. It can be seen that, on a long-term scale, the size of the vapor film reaches a stationary state almost instantly. The asymptotic stationary states for these two curves are different, since in superfluid helium there is an additional pressure associated with the fourth term in Eq. (19).

Figure 2 shows the same curves over a short period of time (about 0.5 s). Curve 1, corresponding to the classical Rayleigh-Plesset equation, represents (practically) undamped oscillations. This is consistent with the numerical results of some works (see Fig. 7 in Ref. [13] and Fig. 2 in Ref. [14]). Curve 2, corresponding to the case investigated in our work, describes a fast approach to a stationary solution. The time to reach a stationary solution (about 0.2 s) agrees with the experimental results (see, for example, Fig. 6 in Ref. [13]), except for the oscillations, absent in the experiment. However, as stated by the authors of Ref. [13], "such oscillations were not registered in our experiments, possibly, because of the low frequency of video."

Finally, Fig. 3 shows the initial period of vapor film development. The evolutionary curves $R(t)$ for the classical Rayleigh-Plesset equation and for Eq. (19) are close to each other. This is due to the fact that the mechanisms specific to superfluid helium require more time.

The performed numerical solution of the problem demonstrates that the approach developed in the present paper (i) explains the abnormal suppression of oscillations, (ii) gives a correct quantitative description of the relaxation of the film to a stationary state, and (iii) predicts a change in the stationary film thickness associated with additional pressures in the superfluid liquid.

## IV. CONCLUSION

The problem of the cavity dynamics in superfluid helium is considered on the basis of Landau-Khalatnikov two-fluid hydrodynamics. The equation governing the evolution of the boundary position (19) significantly differs from the classical

Rayleigh-Plesset equation. The difference arises from the special form of the momentum flux density tensor, including its dissipative part. Due to the two-fluid nature of superfluids, the developed approach generates several different effects, such as the Bernoulli-like pressure term, the extra-damping term, or the dissipative additive (20) into the boundary pressure $p(R)$. These effects essentially influence the dynamics of cavities in comparison with ordinary fluids and can influence the results and conclusions made in the relevant works.

Equation (19) with dissipative additives (20) is intended to investigate the problems associated with the evolution of cavities in superfluids. Once again we would like to emphasize that this hydrodynamic description is part of the general problem and maybe not the primary part. Probably the more important ingredient is the correct analysis of the pressure drop, due to the involved thermal or/and electric processes inside the bubbles.

Thus, in works on boiling helium [13,27] the authors determine the vapor pressure $p(R)$ from the Boltzmann kinetic equation for evaporation and condensation processes. Studying multielectron bubbles in liquid helium [1–3], the authors find the electron density and its contribution to the pressure inside the bubble using the Poisson equation.

The study of corresponding processes is a separate involved problem that goes beyond the scope of this work. Therefore, we deliberately limited ourselves to the hydrodynamic part, since our goal was to emphasize the role of two-fluid hydrodynamics. Moreover, consequently pursuing that goal, we simplified the situation by taking the pure two-fluid Landau-Khalatnikov model and omitting other hydrodynamic phenomena such as the quantum turbulence [18].

Resuming, just as the classical Rayleigh-Plesset equation was a starting point for the study of a large a number of various application, Eq. (19) should be the basic relation for investigations on the problems of cavity evolution in quantum fluids. Any modifications of this equation, designed to investigate specific problems, taking into account the specific features of quantum fluids, such as a radiation of the first and/or second sound, or the appearance of a tangle of vortex filaments (quantum turbulence), should be included as further approximations and should be studied in the future.

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