Mean-field entanglement transitions in random tree tensor networks

Javier Lopez-Piqueres, Brayden Ware, and Romain Vasseur
Department of Physics, University of Massachusetts, Amherst, Massachusetts 01003, USA

(Received 19 March 2020; revised 12 June 2020; accepted 27 July 2020; published 6 August 2020)

Entanglement phase transitions in quantum chaotic systems subject to projective measurements and in random tensor networks have emerged as a new class of critical points separating phases with different entanglement scaling. We propose a mean-field theory of such transitions by studying the entanglement properties of random tree tensor networks. As a function of bond dimension, we find a phase transition separating area-law from logarithmic scaling of the entanglement entropy. Using a mapping onto a replica statistical mechanics model defined on a Cayley tree and the cavity method, we analyze the scaling properties of such transitions. Our approach provides a tractable, mean-field-like example of an entanglement transition. We verify our predictions numerically by computing directly the entanglement of random tree tensor network states.

DOI: 10.1103/PhysRevB.102.064202

Quantum entanglement has become an invaluable tool for studying the equilibrium and nonequilibrium properties of many-body quantum systems [1,2]. Recently, a new class of phase transitions separating phases with dramatically different entanglement features has been discovered. An example of such an *entanglement transition* that has attracted a lot of attention is the many-body localization transition, where many-body eigenstates change from volume-law to area-law entanglement scaling as disorder is increased [3–6]. A fundamentally different class of models also exhibiting entanglement transitions is realized by chaotic quantum systems subjected to random local projective measurements [7–20]. As a function of the rate of measurements, the entanglement entropy of individual quantum trajectories goes from volume-law to area-law entanglement scaling: Enough local measurements can collapse the many-body wave function into an area-law entangled state. Numerical studies in $1+1$ dimensions indicate that this transition is continuous with emergent conformal invariance at the critical point [11]. A closely related transition was proposed earlier by tuning the bond dimension of a state obtained at the boundary of a two-dimensional (2D) random tensor network [21,22]. In all such instances the entanglement entropy in the scaling limit takes the universal form $S - S_c = F((g - g_c)L^{1/\nu})$, with g the parameter driving the transition—either the bond dimension D in the case of random tensor networks, or the measurement rate p for random quantum circuits—and S_c the entanglement entropy at criticality.

Theories of such entanglement transitions have been proposed using a replica approach for both random tensor networks [22], and (Haar) random quantum circuits [23–31] combined with generalized measurements [16,17]. The calculation of the entanglement entropies in such circuits/networks can then be mapped onto a two-dimensional statistical mechanics model: The area- to volume-law entanglement transition corresponds to an ordering transition in the statistical mechanics model [16,17,22]. While this approach explains the general scaling properties of entanglement transitions, the resulting statistical mechanics models cannot be solved in the replica limit except in some special cases. Computing the scaling properties and the critical exponents of entanglement transitions remains a formidable challenge.

In this paper, we propose a *mean-field theory* of entanglement transitions by studying random tree tensor networks. (We expect that related transitions can also be realized in certain models of random unitary dynamics with projective measurements [32]). As a function of bond dimension D, we show that random tree tensor network wave functions go from area-law to logarithmic entanglement scaling. The calculation of the entanglement entropy maps exactly onto a replica statistical mechanics model defined on a *Cayley tree*, which we argue has mean-field-like behavior. This allows us to study this phase transition in detail using the so-called cavity method [33]. Remarkably, the absence of loops on the Cayley tree allows us to analyze analytically the universality class of this transition in the replica limit. We verify our predictions numerically by working directly with the quantum states defined as random tree tensor networks.

Random tree tensor networks. We consider one-dimensional quantum wave functions $|\psi\rangle$ given by tree tensor networks (Fig. 1)—see Refs. [34–37] and references within. The physical degrees of freedom are qudits of dimension d, which live at the boundary of the tree tensor network. Let q be the coordination number of the tree, and D the bond dimension of the tensor network. We choose the tensors to be random [21], obtained by drawing the tensor for each node of the tree independently from a featureless Gaussian distribution characterized by zero mean and unit variance. Because of the tree geometry, such wave functions can have logarithmic entanglement scaling, contrary to matrix-product states, for example.

Our main goal is to study the entanglement properties of wave functions generated from this random ensemble. This approach is inspired in spirit by random matrix theory, but it

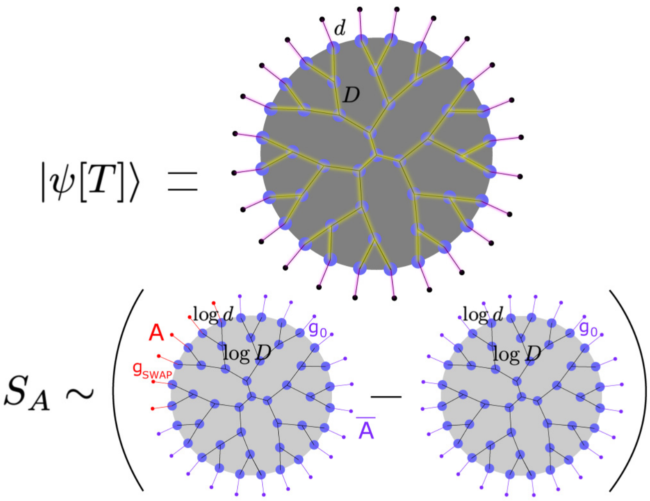

FIG. 1. Random tree tensor networks. Top: Tree tensor network geometry—the physical quantum degrees of freedom live at the boundary ("leaves") of the tree. Bottom: The entanglement entropy of a region A at the boundary can be expressed as the free-energy cost of a domain wall of a classical statistical mechanics model defined on the Cayley tree.

allows us to include some locality structure in the geometry of the "bulk" tensor network, controlling the entanglement of the boundary physical system. We will focus on the tensor-averaged Renyi entropies

$$S_A^{(n)} = \frac{1}{1-n} \log \overline{\frac{\mathrm{tr}\,\rho_A^n}{(\mathrm{tr}\,\rho)^n}}, \tag{1}$$

where $\overline{(\cdots)}$ refers to averaging over random tensors, and ρ_A is the reduced density matrix in some contiguous interval A of size L_A obtained from tracing out the complement of A in $\rho = |\psi\rangle\langle\psi|$ (Fig. 1).

Statistical mechanics model. In order to compute these Renyi entropies, we follow Refs. [16,17,22,29] and use a replica trick $\log \mathrm{tr}\,\rho_A^n = \lim_{m\to 0}[(\mathrm{tr}\,\rho_A^n)^m - 1]/m$. This allows us to express (1) as

$$S_A^{(n)} = \frac{1}{n-1} \lim_{m\to 0} \frac{1}{m} (\mathcal{F}_A - \mathcal{F}_0), \tag{2}$$

with $\mathcal{F}_{A,0} = -\log \mathcal{Z}_{A,0}$ and $\mathcal{Z}_0 \equiv \overline{(\mathrm{tr}\,\rho^n)^m}$, $\mathcal{Z}_A \equiv \overline{(\mathrm{tr}\,\rho_A^n)^m}$. Using this exact identity, the calculation of the Renyi entropies reduces to computing $\mathcal{Z}_0$ and $\mathcal{Z}_A$, and to evaluate the replica limit (2). When m and n are integers, the averages in $\mathcal{Z}_0$ and $\mathcal{Z}_A$ can be evaluated analytically using Wick's theorem. One can then express the partition functions $\mathcal{Z}_A$ and $\mathcal{Z}_0$ in terms of a classical statistical mechanics model, whose degrees of freedom are *permutations* $g_i \in S_{Q=nm}$ labeling different Wick contractions at each vertex of the tensor networks. Since the degrees of freedom of this statistical mechanics model live on the nodes of the tree tensor network, they form a *Cayley tree*, and $\mathcal{Z}_A$ and $\mathcal{Z}_0$ differ only in their boundary conditions. Using the results of Ref. [22], we find that $\mathcal{Z}_0 = \sum_{\{g_i\}} e^{-\mathcal{H}}$, with the following nearest-neighbor Hamiltonian,

$$\mathcal{H} = -\sum_{\langle i,j\rangle} J_{\langle i,j\rangle} C\left(g_i^{-1} g_j\right), \tag{3}$$

where $C(g)$ counts the number of cycles in the permutation g, $J_{\langle i,j\rangle} = \log D$ with D the bond dimension for links connecting bulk tensors, and $J_{\langle i,j\rangle} = \log d$ (with d the dimension of the Hilbert space of the boundary physical qudits) for boundary couplings involving physical degrees of freedom. This Hamiltonian is invariant under global left/right multiplication of the degrees of freedom g_i by any permutation $h \in S_Q$, so it has a $S_Q \times S_Q$ symmetry. In this mapping, the trace over physical degrees of freedom in $\mathcal{Z}_0 = \overline{(\mathrm{tr}\,\rho^n)^m}$ forces the permutations on the boundary sites corresponding to the physical qudits to be fixed to the identity permutation $g_\partial = g_0 = (\,)$ in $\mathcal{Z}_0$. Meanwhile, boundary permutations in $\mathcal{Z}_A$ are fixed to identity if they belong to $\overline{A}$ (the complement of A), whereas they are fixed to a different permutation $g_{\mathrm{SWAP}} = (12\ldots n)^{\otimes m}$ if they belong to A. The permutation g_{SWAP} arises from enforcing the partial trace in $\mathcal{Z}_A \equiv \overline{(\mathrm{tr}\,\rho_A^n)^m}$. Note that $C(g)$ is maximum for the identity permutation, so that the Hamiltonian (3) corresponds to ferromagnetic interactions.

In the language of this statistical mechanics model, the Renyi entropies (2) can be computed from the free-energy cost of inserting a *domain wall* between the boundary permutations g_0 and g_{SWAP} at the entanglement interval. This provides a very simple picture of the scaling of the entanglement entropy as a function [38] of bond dimension D. If D is small (near 1), we expect the statistical mechanics model (3) to be disordered (paramagnetic), and the free-energy cost in (2) will not scale with L_A: This corresponds to an area-law phase. If on the other hand D is large, the statistical mechanics model is in an ordered (ferromagnetic) phase with all bulk permutations aligned and equal to g_0, and the free-energy cost in (2) will be given by the energy penalty of the bonds frustrated by the domain wall minimizing this energy ("minimal cut" through the network). For large L_A and generic intervals A, this minimal domain wall cuts $\sim \log L_A$ bonds of the tensor network (Cayley tree) corresponding to logarithmic entanglement scaling $S_A \sim (\log D) \log L_A$. This implies that the ordering transition of (3) at a critical coupling $J_c = \log D_c$ corresponds to an area-law to logarithmic scaling of the Renyi entropies of the random tree tensor networks.

$Q = 2$ replicas and cavity method. In order to gain some insight into the scaling of the entanglement entropy, we start by analyzing the simpler case of $Q = 2$ replicas. As we will argue below, the mean-field nature of the statistical mechanics model on the Cayley tree will make critical properties mostly independent of Q, allowing us to generalize this insight to the replica limit $Q \to 0$. For $Q = 2$, Eq. (3) is simply an Ising model. If we let $g_i = \pm 1$ be the two elements of $S_2 \cong \mathbb{Z}_2$, (3) reads $\mathcal{H} = -\sum_{\langle i,j\rangle} J_{\langle i,j\rangle}(3 + g_i g_j)/2$, which up to an irrelevant additive constant, is an Ising model with coupling $K = J/2 = (\log D)/2$. To proceed, we use the so-called *cavity method* [33,39,40], which is a standard approach for solving statistical mechanics problems on treelike graphs. We start from an Ising model with coupling K, and generic boundary fields h_i acting on the boundary sites of the Cayley tree. It is straightforward to show that all boundary spins can be decimated, at the price of introducing new effective fields acting on the next layer of the tree, which now forms the new boundary. This process can then be iterated, and the resulting recursion ("cavity") equations for uniform boundary fields are then given by $[\sum_{\sigma_i=\pm 1} \exp(K\sigma_i\sigma_j + h^{(k+1)}\sigma_i)]^{q-1} =$

$\mathcal{C}\exp(h^{(k)}\sigma_j)$, for some constant $\mathcal{C}$. Here, we have assumed that we are working with $\mathcal{Z}_0$ for simplicity so that the boundary fields are uniform, but this approach can be readily extended to arbitrary inhomogeneous boundary fields. The critical behavior of this model is then easily deduced from solving for the cavity fields recursively [41]. Approaching the transition from the paramagnetic phase, we find that the magnetization at the root of the tree decays exponentially with the number of layers N, $\langle\sigma_0\rangle \sim \exp(-N/\xi)$ with a correlation length $\xi = -1/\log[(q-1)\tanh K]$ that diverges at the critical coupling $K_c = \text{arctanh}[1/(q-1)]$, which is finite for coordination number $q > 1$. Expanding near the critical point yields $\xi \sim |K-K_c|^{-\nu}$, with $\nu = 1$. On the ferromagnetic side, we have $\langle\sigma_0\rangle \sim h \sim (K-K_c)^{\beta}$, with $\beta = 1/2$. (A procedure to access this exponent was proposed in Ref. [18] in the context of random circuits). The order parameter exponent $\beta = 1/2$ takes the mean-field value for a transition in the Ising universality class, a general feature of statistical mechanics on the Cayley tree [42]. On the other hand, the correlation length exponent $\nu = 1$ is inherited from quasi-one-dimensional physics, as has been noted previously [43].

Entanglement scaling. The cavity method above can readily be applied to arbitrary configurations of the boundary fields, and can be used to evaluate Eq. (2) in the case of $Q = 2$ replicas [41]. We denote the averaged free-energy cost of a domain wall $S(L_A) = \mathcal{F}_A - \mathcal{F}_0$, which is the quantity which becomes entanglement entropy in the limit $Q \to 0$ from Eq. (2). On the paramagnetic side of the transition [small $K = (\log D)/2$], the Ising order decays ξ layers into the bulk, so we expect the entanglement to saturate to a constant value $S(L_A \to \infty) \propto \log\xi$, corresponding to area-law scaling. This is consistent with our numerical results [41], which indicate a divergence $S(L_A \to \infty) \sim -\alpha \log(K_c - K)$ as $K \to K_c^-$. The saturation to this area-law value occurs for $L_A \gg \xi_\star$ with the crossover scale $\xi_\star = e^{\xi} = e^{C/|K-K_c|}$. Therefore, while $\nu = 1$ in the bulk, in terms of the entanglement scaling the relevant diverging length scale diverges exponentially near the transition, due to the tree geometry. On the ordered side of the transition ($K > K_c$), $S(L_A)$ is proportional to the energy cost of the domain wall which scales as the number of layers into the bulk $\sim \log L_A$. As expected from general scaling arguments, the prefactor is set by ξ, and we find $S(L_A) \sim \frac{\log L_A}{\xi}$. Finally, scaling at the critical point is required to be $S(L_A) \sim \alpha \log\log L_A$ by general scaling considerations from the behavior in the phases, in good agreement with our numerical solution to the cavity equations for the system sizes we can access (Fig. 2). In summary, we have

$$S \sim \begin{cases} \frac{\log L_A}{\xi} + \alpha \log\log L_A, & K \to K_c^+, \\ \alpha \log\log L_A, & K = K_c, \\ \alpha \log\xi, & K \to K_c^-. \end{cases} \tag{4}$$

We find that our results are consistent with the entanglement scaling at entanglement transitions in quantum chaotic systems subject to projective measurements or in wave functions given by square random tensor networks upon replacing $\log L_A \to L_A$ [8,11,22]. This is because geodesics (minimal cut minimizing the domain wall energy at large bond dimension) in flat 2D Euclidean space are given by straight lines, whereas they scale with the logarithm of the size of region A on the Cayley tree. These different regimes can be summarized by the universal scaling form $S - S_c = F((K - K_c)(\log L_A)^{1/\nu})$ with $\nu = 1$ shown in Fig. 2.

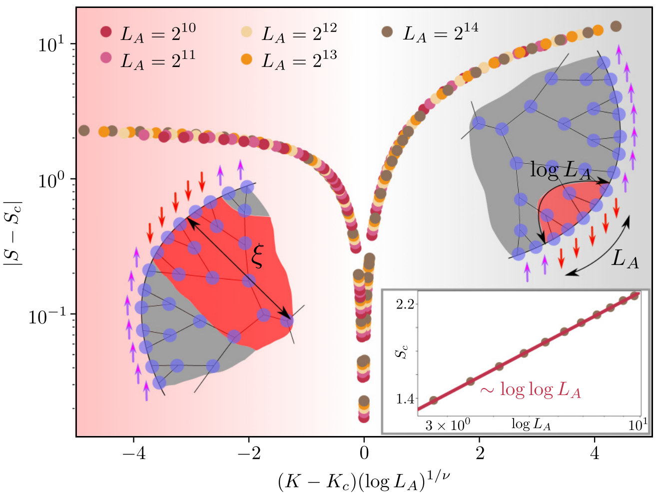

FIG. 2. Entanglement scaling. Collapse of the boundary domain wall free-energy cost for $Q = 2$ replicas, as a proxy for the entanglement entropy in the replica limit $Q \to 0$. For $K = (\log D)/2 > K_c$ the domain wall mostly follows a minimal cut through the bulk, so its energy scales logarithmically with the interval size L_A. For $K < K_c$, the domain wall fluctuates through the bulk over a number of layers given by the correlation length, which diverges as $\xi \sim |K-K_c|^{-\nu}$ with $\nu = 1$. Inset: At criticality, the entanglement scales as $S \sim \log\log L_A$.

Replica limit. So far our results for the bulk critical exponent and for the entanglement scaling (4) were inferred from the case of $Q = 2$ replicas for simplicity. We now discuss how one can obtain the critical properties in the replica limit $Q \to 0$ of Eq. (2). It is possible [41] to apply the cavity method to the model (3), but the number of cavity fields is then given by the number of irreducible representations of S_Q. As a result, the replica limit $Q \to 0$ is still out of reach on the Cayley tree. To proceed, we use the following trick based on universality: We modify the Boltzmann weights of the model Eq. (3) while preserving the $S_Q \times S_Q$ symmetry of the Hamiltonian (3). Therefore, we introduce a different statistical mechanics model,

$$\mathcal{H}_{\text{modified}} = -\sum_{\langle i,j\rangle} \log\left[1 + K\bar{\chi}\left(g_i^{-1}g_j\right)\right], \tag{5}$$

where $\bar{\chi}(g) = \frac{Q-1}{Q!}\chi(g)$ with χ the character of the *standard* representation of the symmetric group S_Q. This model is still invariant under left/right multiplication by elements of S_Q, and since the standard representation is faithful and well defined for any Q, we do not expect this modified model to have an enlarged symmetry. [This is inspired by the $O(N)$ model, whose critical behavior in 2D was understood by Nienhuis [44] by introducing a different model with the same symmetry group].

Remarkably, for uniform boundary conditions $g_\partial = g_0 = ()$ (corresponding to $\mathcal{Z}_0$), the modified model (5) is still solvable on the Cayley tree with coordination number $q = 3$

using a single cavity equation for any Q. The cavity equation reads $\sum_{g_i}[1+h^{(k)}\bar{\chi}(g_i)]^2[1+K\bar{\chi}(g_i^{-1}g_j)] = C(1+h^{(k-1)}\bar{\chi}(g_j))$. Using standard representation theory results, we find the following recursion relation for the boundary cavity fields,

$$h^{(k-1)} = \frac{K}{Q!}\frac{2h^{(k)} + (h^{(k)})^2\frac{Q-1}{Q!}}{1+(h^{(k)})^2\left(\frac{Q-1}{Q!}\right)^2}. \qquad (6)$$

We can now analytically continue Q in this equation, and study the critical behavior as a function of Q. We analyzed the fixed points of this recursion relation and their stability as a function of Q. For $Q > 1$, we find first-order transitions (with $Q = 2$ being special), while for $Q < 1$ there is a second-order transition for $K_c = Q!/2$. For $K < K_c$, the correlation length reads $\xi^{-1} = \log\frac{K_c}{K}$, so we find $\nu = 1$ as in the Ising ($Q = 2$) case. For $K > K_c$, the cavity fields flow to a nonzero value which scales as $\sim(K - K_c)$, so we find $\beta = 1$, which is the mean-field magnetization exponent of the n-state Potts model with $n < 2$. In the replica limit, we thus find $\nu = \beta = 1$, which coincide with the critical exponents of the n-state Potts model on the Cayley tree (for $n < 2$). Those exponents *do not* depend on the replica number Q, as expected from mean-field critical behavior in general—the only exception is the exponent β which happens to be different for $Q = 2$ for symmetry reasons. We expect these exponents to control the critical behavior of our model (3) in the replica limit $Q \to 0$, and while we unfortunately cannot solve the modified model (6) with inhomogeneous boundary conditions, we also expect the general scaling (4) with $\nu = 1$ to hold for $Q \to 0$.

Numerical results. We verify our predictions by generating tree tensor network states and computing their entanglement properties numerically. Each state consists of random, Gaussian-distributed tensors of dimension D on each node of the Cayley tree. By tuning the bond dimension we find a phase transition from area-law to logarithmic scaling of the entanglement entropy, with $D = 1$ (trivially) showing area-law scaling and $D = 3$ showing clear logarithmic scaling. As the dimension of tensors must be integer, we augment these states with additional tensors on each bond of the tree to interpolate between integer bond dimensions $D = 1$ and $D = 3$. With the size of the tensors on the nodes fixed at $D = 3$, we insert on each bond diagonal tensors with elements $(1, \gamma, \gamma^2)$, with the parameter γ tuned continuously from $\gamma = 0$, corresponding to $D = 1$, to $\gamma = 1$, corresponding to $D = 3$ (see Ref. [41]). Upon tuning γ, we find a phase transition from area-law to logarithmic scaling of the entanglement entropy

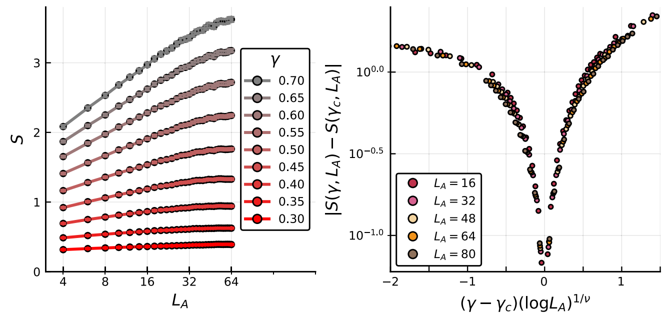

FIG. 3. Numerical results. Left panel: Averaged von Neumann entropy for random tree tensor network states of size $L = 256$ as a function of the subsystem size L_A for various values of γ, where $\gamma \in [0, 1]$ is a parameter tuning continuously the bond dimension between $D = 1$ and $D = 3$ (see text). Right panel: Collapse of the data with $\gamma_c = 0.47$ and $\nu = 1$.

(Fig. 3), consistent with the mean-field theory results detailed above. We estimate the location of the critical point γ_c to be in the interval [0.4, 0.6] and the critical exponent ν to take a value in [1, 1.5]. The precision is limited due to the rather small depth of the Cayley tree that is accessible numerically; however, we find that the quality of the collapse improves with system size and is comparable to our results for the Ising model on equally small Cayley trees [41].

Discussion. We have studied a new entanglement transition from area-law to logarithmic scaling of entanglement in random tree tensor networks. This transition can be analyzed using a mapping onto a replica statistical mechanics model on the Cayley tree which exhibits mean-field critical behavior. We computed exactly the critical exponents $\nu = \beta = 1$ relevant to this entanglement transition, and inferred the general scaling properties of the entanglement near criticality. We checked our predictions numerically by computing directly the entanglement of random tree tensor network states. It would be interesting to find other mean-field examples of entanglement transitions, especially in the context of measurement-induced transitions in random quantum circuits.

Acknowledgments. We thank Sarang Gopalakrishnan, Michael Gullans, David Huse, Sid Parameswaran, and Jed Pixley for useful discussions. R.V. also thanks Chao-Ming Jian, Andreas Ludwig, Andrew Potter, and Yi-Zhuang You for collaborations on related matters. This work was supported by the U.S. Department of Energy, Office of Science, Basic Energy Sciences, under Early Career Award No. DE-SC0019168. R.V. is supported by the Alfred P. Sloan Foundation through a Sloan Research Fellowship.

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