Relaxation of the degenerate one-dimensional Fermi gas

K. A. Matveev[1] and Zoran Ristivojevic [2]

[1]*Materials Science Division, Argonne National Laboratory, Argonne, Illinois 60439, USA*
[2]*Laboratoire de Physique Théorique, Université de Toulouse, CNRS, UPS, 31062 Toulouse, France*

We study how a system of one-dimensional spin-$\frac{1}{2}$ fermions at temperatures well below the Fermi energy approaches thermal equilibrium. The interactions between fermions are assumed to be weak and are accounted for within the perturbation theory. In the absence of an external magnetic field, spin degeneracy strongly affects relaxation of the Fermi gas. For sufficiently short-range interactions, the rate of relaxation scales linearly with temperature. Focusing on the case of the system near equilibrium, we linearize the collision integral and find the exact solution of the resulting relaxation problem. We discuss the application of our results to the evaluation of the transport coefficients of the one-dimensional Fermi gas.

I. INTRODUCTION

Relaxation of conventional Fermi liquids is well understood [1]. It is dominated by two-particle collisions of the elementary excitations of the liquid. At low temperature T the number of states available for scattering is small, resulting in a small relaxation rate $\tau^{-1} \propto T^2$. The fact that τ^{-1} is small compared with the typical energy T of the excitation is at the foundation of the Fermi liquid theory [1]. It is important to keep in mind that the above result applies only to systems of fermions in two or more spatial dimensions.

Relaxation proceeds very differently in one dimension [2]. Most importantly, the scattering processes involving only two fermions do not lead to relaxation, and thus the dominant processes involve three particles. The relaxation rate for spin-polarized one-dimensional fermions scales as $\tau^{-1} \propto T^7$ [2–5]. Such a weak relaxation at $T \to 0$ is due to the small density of states for three-particle scattering and a strong suppression of the scattering amplitude for spin-polarized fermions, which is a manifestation of the Pauli principle [6].

The goal of this paper is to explore relaxation of the one-dimensional Fermi gas in the absence of magnetic field, when the system is fully spin degenerate. We consider the low-temperature regime $T \ll \mu$, where μ is the chemical potential. At these low temperatures the dominant scattering processes involve three particles with energies near μ (see Fig. 1). The processes illustrated in Fig. 1(a) involve two fermions near one Fermi point and the third fermion near the other one. They give rise to decay of quasiparticles both at finite temperature and at $T = 0$. The decay rate of a quasiparticle with energy of order T due to scattering processes of this type was evaluated in Ref. [7]. The result, $\tau^{-1} \propto T$, is much greater than the decay rate $\tau^{-1} \propto T^7$ [2–5] for spin-polarized fermions, because the scattering amplitude, instead of being suppressed due to the Pauli principle, diverges at small momentum transfer as $|p_1 - p_1'|^{-1}$. The processes shown in Fig. 1(b) involve three particles near the same Fermi point and are not allowed at zero temperature. To our knowledge, their effect on the decay of quasiparticles in the spin-degenerate

Fermi gas has not been considered before. We show that their contribution is small compared with that of the processes in Fig. 1(a) only for interactions that fall off sufficiently fast with the distance between particles.

Focusing on the latter case, we consider the relaxation of the Fermi gas to equilibrium. When the distribution function is close to the equilibrium form, we are able to find a complete solution of the relaxation problem by diagonalizing exactly the linearized collision integral corresponding to the processes of Fig. 1(a) at small temperature. This solution enables one to obtain the time evolution of any nonequilibrium distribution function at small deviation from thermal equilibrium.

Understanding the relaxation properties of the one-dimensional Fermi gas is required for the evaluation of its transport coefficients, such as thermal conductivity. At $T \ll \mu$ one can identify two kinds of thermal conductivity [5,8,9]. The ordinary thermal conductivity κ is controlled by the exponentially rare processes involving backscattering of particles near the bottom of the band. It describes thermal transport at exponentially small frequencies. At higher frequencies the thermal transport is described by a different transport coefficient κ_{ex}, which is essentially the thermal conductivity of the gas of elementary excitations of the system [8]. Our treatment of the relaxation of the one-dimensional Fermi gas will enable us to express κ_{ex} in terms of temperature, chemical potential, and interaction strength.

The paper is organized as follows. In Sec. II we evaluate the three-particle scattering rates associated with the two types of processes illustrated in Fig. 1. In Sec. III we estimate the decay rates of quasiparticle states with energies of order T and discuss how these rates scale with the temperature for weak interaction potentials decaying with the distance as $1/|x|^\gamma$. In Sec. IV we solve the relaxation problem in the regime of short-range interactions ($\gamma > 5/2$). To leading order in small temperature and weak interaction, the corresponding linearized collision integral is diagonalized exactly in Appendix A. The spectrum of the relaxation rates is qualitatively different from that in the spin-polarized system, which is briefly discussed in Appendix B. We discuss our results and their implications for
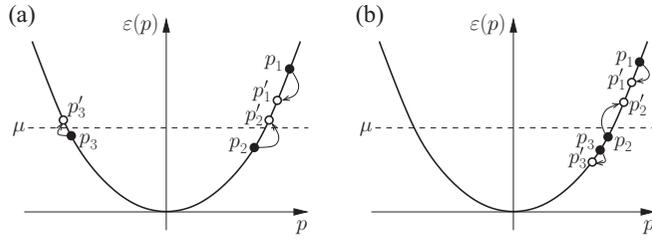
FIG. 1. At low temperature $T \ll \mu$ the dominant scattering processes involve three particles with energies within T from the chemical potential μ. (a) A typical three-particle scattering process with two fermions near one Fermi point and one near the other Fermi point. (b) A scattering process with all three fermions near the same Fermi point.

the transport coefficients of the one-dimensional Fermi gas in Sec. V.

II. THREE-PARTICLE SCATTERING RATE

We consider a system of one-dimensional spin-$\frac{1}{2}$ fermions with quadratic dispersion $\varepsilon_p = p^2/2m$ and weak two-particle interaction, which we describe by the Hamiltonian

$$\hat{V} = \frac{1}{2L} \sum_{\substack{p_1, p_2, q \\ \sigma_1, \sigma_2}} V(q) c^{\dagger}_{p_1+q,\sigma_1} c^{\dagger}_{p_2-q,\sigma_2} c_{p_2,\sigma_2} c_{p_1,\sigma_1}. \quad (1)$$

Here L is the system size, $V(q)$ is the Fourier transform of the interaction potential, and $c_{p,\sigma}$ is the operator annihilating a fermion with momentum p and z component of spin σ.

In one dimension the restrictions imposed by conservation of momentum and energy preclude relaxation by two-particle scattering processes. Thus the dominant scattering processes involve three particles. Because interaction (1) couples only two fermions, the three-particle scattering amplitude must be obtained in the second order of the perturbation theory in $V(q)$. Such a calculation was performed in Ref. [10]. The rate of scattering of three fermions with momenta p_1, p_2, p_3 and spins σ_1, σ_2, σ_3 to new states with momenta p'_1, p'_2, p'_3 and spins σ'_1, σ'_2, σ'_3, respectively, has the form

$$\mathcal{W}^{1'2'3'}_{123} = \frac{2\pi}{\hbar} \left| \mathcal{A}^{1'2'3'}_{123} \right|^2 \delta(E - E'), \quad (2)$$

where $E = \varepsilon_{p_1} + \varepsilon_{p_2} + \varepsilon_{p_3}$ and $E' = \varepsilon'_{p_1} + \varepsilon'_{p_2} + \varepsilon'_{p_3}$ are the energies of the three particles before and after the scattering event and $\mathcal{A}^{1'2'3'}_{123}$ is the scattering matrix element. The latter can be presented in the form

$$\mathcal{A}^{1'2'3'}_{123} = \sum_{\pi(1'2'3')} \text{sign}(1'2'3') \delta_{\sigma_1,\sigma'_1} \delta_{\sigma_2,\sigma'_2} \delta_{\sigma_3,\sigma'_3}$$

$$\times \left(a^{p_a,p_b}_{p_1,p_2} + a^{p_a,p_c}_{p_1,p_3} + a^{p_b,p_c}_{p_2,p_3} \right) \delta_{P,P'}. \quad (3)$$

Here the summation is performed over all the permutations of the final states of the three particles, $P = p_1 + p_2 + p_3$ and $P' = p'_1 + p'_2 + p'_3$ are total momenta before and after the

scattering event, and

$$a^{p_a,p_b}_{p_1,p_2} = \frac{1}{L^2} V(p_a - p_1) V(p_b - p_2)$$

$$\times \left(\frac{1}{E - \varepsilon_{p_1} - \varepsilon_{p_b} - \varepsilon_{P-p_1-p_b}} \right.$$

$$+ \left. \frac{1}{E - \varepsilon_{p_2} - \varepsilon_{p_a} - \varepsilon_{P-p_2-p_a}} \right). \quad (4)$$

In the absence of magnetic field the occupation numbers of all the states do not depend on the spin. Thus it is convenient to sum the scattering rate (2) over spin indices and introduce

$$W^{p'_1,p'_2,p'_3}_{p_1,p_2,p_3} = \sum_{\substack{\sigma_1,\sigma_2,\sigma_3 \\ \sigma'_1,\sigma'_2,\sigma'_3}} \mathcal{W}^{1'2'3'}_{123}. \quad (5)$$

Our goal is to evaluate the scattering rate (5) assuming that all three fermions are near the Fermi points (see Fig. 1). We start with the state described by the momenta of the three particles and notice that collisions conserve the total momentum $P = p_1 + p_2 + p_3$ and energy $E = (p_1^2 + p_2^2 + p_3^2)/2m$. Aside from P and E, a full description of a state of three particles requires one additional parameter. We denote this parameter α and introduce it via

$$p_j = \frac{1}{3}P - 2\sqrt{\frac{m\mathcal{E}}{3}} \cos\left(\alpha - \frac{2\pi j}{3}\right), \quad j = 1, 2, 3, \quad (6a)$$

where $\mathcal{E} = E - P^2/6m$ is the total energy of the three fermions in the center-of-mass frame. Thus the state of three particles is described by P, $\mathcal{E}$, and α. The momenta of the three particles after the collision are similarly parametrized by P', $\mathcal{E}'$, and α' according to

$$p'_j = \frac{1}{3}P' - 2\sqrt{\frac{m\mathcal{E}'}{3}} \cos\left(\alpha' - \frac{2\pi j}{3}\right), \quad j = 1, 2, 3. \quad (6b)$$

Conservation of momentum and energy implies that the scattering rate (5) has the form

$$W^{p'_1,p'_2,p'_3}_{p_1,p_2,p_3} = \Theta \delta(\mathcal{E} - \mathcal{E}') \delta_{P,P'}, \quad (7)$$

where Θ is in general a function of $\mathcal{E}$, α, and α'. (The dependence of Θ on the total momentum P is precluded by Galilean invariance.)

The scattering process shown in Fig. 1(a) involves two fermions near the right Fermi point $p = p_F$ and the third one near $-p_F$. In terms of our variables P, $\mathcal{E}$, and α, these conditions translate to

$$|P - p_F| \lesssim \frac{T}{\mu} p_F, \quad \left| \mathcal{E} - \frac{8}{3}\mu \right| \lesssim T, \quad |\alpha| \lesssim \frac{T}{\mu}, \quad (8)$$

where the chemical potential μ is given by the Fermi energy $p_F^2/2m$ in the low-temperature limit. For the processes of Fig. 1(b), in which all three fermions are near the right Fermi point, we have

$$|P - 3p_F| \lesssim \frac{T}{\mu} p_F, \quad \mathcal{E} \lesssim \frac{T^2}{\mu}, \quad -\frac{\pi}{3} < \alpha < \frac{\pi}{3}. \quad (9)$$

The estimate of $\mathcal{E}$ at low temperatures is obtained by noticing that the typical difference of momenta of the fermions

$|p_1 - p_2| \sim \sqrt{m\mathcal{E}}$ is of the order of T/v_F, where $v_F = p_F/m$ is the Fermi velocity.

If the interaction potential falls off with the distance sufficiently slowly, the Fourier transform $V(q)$ is not analytic at $q = 0$. For example, for the Coulomb interaction, $V(q) \propto \ln(1/|q|)$. Relaxation of the Fermi gas with such long-range interactions has a number of special features, which we leave for future study. In the following we assume that the range of interactions between fermions is short. The exact criterion for a potential to be considered short range depends on the particular result and is discussed below. An interaction potential that decays exponentially at large distances corresponds to $V(q)$ that is analytic at $q = 0$. This is sufficient to classify such potentials as short range, but in practice exponential decay is not required.

In the case of a short-range potential, the general result for the three-particle scattering rate $W_{p_1, p_2, p_3}^{p_1', p_2', p_3'}$ given by Eqs. (2)–(5) can be simplified for the two types of processes that dominate relaxation at low temperatures (see Fig. 1). For the process of Fig. 1(a) we use the simplification (8) and find the scattering rate in the form (7) with

$$\Theta = \frac{9\pi}{L^4} \frac{\hbar^3 \Lambda^2}{m^2} \frac{\alpha^2 + \alpha'^2}{(\alpha^2 - \alpha'^2)^2}, \qquad (10)$$

where Λ is a dimensionless parameter defined as

$$\Lambda = \frac{V(0)V(2p_F) - V(2p_F)^2 - 2p_F V(0)V'(2p_F)}{(\hbar v_F)^2}. \qquad (11)$$

The result (10) is applicable as long as $V(0)$ is well defined. For interactions that fall off with the distance as $1/|x|^\gamma$ this requires $\gamma > 1$.

For the scattering processes of Fig. 1(b) we have $\mathcal{E} \ll \mu$ [see Eq. (9)], which enables one to simplify the general expression for the scattering rate $W_{p_1, p_2, p_3}^{p_1', p_2', p_3'}$ given by Eqs. (2)–(5) to the form (7) with

$$\Theta = \frac{1458\pi m^2}{\hbar L^4} [V(0)V''(0)]^2 \frac{1 - \cos(3\alpha)\cos(3\alpha')}{[\cos(3\alpha) - \cos(3\alpha')]^2}. \qquad (12)$$

The applicability of this expression is limited to interaction potentials for which the second derivative of the Fourier transform $V''(q)$ is well defined at $q = 0$. For interactions that fall off as $1/|x|^\gamma$ this requires $\gamma > 3$.

It is instructive to consider a special case of $V(q) = \text{const}$, which corresponds to the interaction of fermions in the form $U(x) \propto \delta(x)$. The model of spin-$\frac{1}{2}$ fermions with interaction of this type was studied by Gaudin and Yang and shown to be integrable [11,12]. This property implies that no scattering of elementary excitations is allowed [13]. Substitution of $V(q) = \text{const}$ into Eqs. (10) and (12) indeed yields $\Theta = 0$. More generally, integrability should result in a vanishing scattering amplitude (3) for $V(q) = \text{const}$. This was verified in Ref. [10].

III. DECAY OF QUASIPARTICLE STATES

As a first step toward understanding relaxation of the one-dimensional Fermi gas we estimate the decay rates of quasiparticles due to the three-particle scattering processes.

For a quasiparticle of momentum p the decay rate is given by

$$\frac{1}{\tau} = \frac{1}{2} \sum_{\substack{p_1, p_2, p_3 \\ p_1', p_2', p_3'}} W_{p_1, p_2, p_3}^{p_1', p_2', p_3'} \theta(p_2 - p_3)\theta(p_1' - p_2')\theta(p_2' - p_3')\delta_{p, p_1}$$

$$\times n_{p_2} n_{p_3} (1 - n_{p_1'})(1 - n_{p_2'})(1 - n_{p_3'}). \qquad (13)$$

Here the unit step function $\theta(x)$ is used to limit the summations to distinct sets of momenta before and after scattering, and $1/2$ compensates for the summation over the spin of the initial particle included in Eq. (5). To estimate the rate, we convert the sum to an integral and substitute the general form (7) of the scattering rate. This yields

$$\frac{1}{\tau} = \frac{L^4 m^2}{384\pi^4 \hbar^4} \iiint d\mathcal{E}\, d\alpha\, d\alpha'\, \Theta$$

$$\times n_{p_2} n_{p_3} (1 - n_{p_1'})(1 - n_{p_2'})(1 - n_{p_3'}). \qquad (14)$$

Here we transformed the integral to the variables (6) using

$$dp_1 dp_2 dp_3 = \frac{m}{\sqrt{3}} dP\, d\mathcal{E}\, d\alpha. \qquad (15)$$

The Fermi occupation numbers, to which one should substitute the expressions for momenta using Eq. (6), effectively limit the range of integration in Eq. (14).

Assuming that the quasiparticle of interest has the energy within T from the Fermi level, its decay is controlled by the two processes shown in Fig. 1. We start with the process shown in Fig. 1(a) and substitute into Eq. (14) the expression (10) for Θ. This yields

$$\frac{1}{\tau} \sim \frac{\Lambda^2}{\hbar} \int_{|\mathcal{E} - \frac{8\mu}{3}| \lesssim T} d\mathcal{E} \iint_{|\alpha|, |\alpha'| \lesssim \frac{T}{\mu}} d\alpha\, d\alpha' \frac{\alpha^2 + \alpha'^2}{(\alpha^2 - \alpha'^2)^2}, \qquad (16)$$

with the ranges of integrations controlled by the omitted Fermi occupation numbers [see Eq. (8)]. Ignoring for the moment the singularity at $\alpha = \pm\alpha'$, we find that the integral over α and α' is of order unity, while the integral over $\mathcal{E}$ is of the order of T. We therefore conclude that the processes of Fig. 1(a) result in the relaxation rate of the order of

$$\frac{1}{\tau_a} = \frac{\Lambda^2}{\hbar} T. \qquad (17)$$

Similarly, for the processes of Fig. 1(b), substitution of Eq. (12) into Eq. (14) yields

$$\frac{1}{\tau} \sim \frac{m^4}{\hbar^5} [V(0)V''(0)]^2 \int_{0 < \mathcal{E} \lesssim \frac{T^2}{\mu}} d\mathcal{E}$$

$$\times \iint_{|\alpha|, |\alpha'| < \frac{\pi}{3}} d\alpha\, d\alpha' \frac{1 - \cos(3\alpha)\cos(3\alpha')}{[\cos(3\alpha) - \cos(3\alpha')]^2}. \qquad (18)$$

The corresponding relaxation rate is

$$\frac{1}{\tau_b} = \frac{m^4}{\hbar^5 \mu} [V(0)V''(0)]^2 T^2. \qquad (19)$$

Our estimates (17) and (19) should be understood as follows. The quasiparticle decay rates (16) and (18) diverge due to the singularities at $\alpha = \pm\alpha'$. One can see from Eq. (6) that these divergences emerge as a result of scattering processes for which the fermion with momentum p scatters to a state with momentum p' approaching p. Within our perturbative

treatment, in the lowest order in interaction strength, the decay rate is infinite. On the other hand, an infinitesimal change of momentum of the fermion from p to p' has little effect on the observable quantities. In the next section we see that the evolution of the fermion distribution function is not affected by these singularities. Thus expressions (17) and (19) give the order-of-magnitude estimates of the relaxation rates associated with the processes shown in Fig. 1.

In the above calculation we assumed that the fermion in the state with momentum p had the energy ε_p near the Fermi energy, $|\varepsilon_p - \mu| \sim T$. Decay of quasiparticles with energies larger than temperature in an electron gas with Coulomb interactions was studied in Ref. [7]. Only the processes of the type shown in Fig. 1(a) were considered. At energies of order T the corresponding results of Ref. [7] are consistent with our estimate (17) provided the logarithmic singularity of $V(q) \propto \ln(1/|q|)$ is properly cut off.

Comparison of expressions (17) and (19) shows that at low temperature relaxation is dominated by the processes of Fig. 1(a). This conclusion holds for sufficiently short-range interactions, such that $V''(0)$ is well defined. In the case of a potential that falls off as a power law $|x|^{-\gamma}$ at large distances, this condition requires $\gamma > 3$. For $1 < \gamma < 3$ the temperature dependence of the rate τ_b^{-1} can be obtained as follows. The Fourier transform of the interaction potential $V(q)$ is well defined at $q = 0$ for $\gamma > 1$. However, its second derivative diverges at $q \to 0$ as $V''(q) \propto |q|^{\gamma-3}$. For the process shown in Fig. 1(b) the typical difference of momenta in the argument of V in Eq. (4) is of the order of T/v_F. Thus one can obtain the temperature dependence of τ_b^{-1} by substituting $V''(T/v_F) \propto T^{\gamma-3}$ for $V''(0)$ in Eq. (19). This yields

$$\frac{1}{\tau_b} \propto T^{2\gamma-4}. \tag{20}$$

At $T \to 0$ the above rate is negligible compared with $1/\tau_a \propto T$ if $\gamma > 5/2$. Conversely, for $1 < \gamma < 5/2$ we expect relaxation to be dominated by the processes of Fig. 1(b).

IV. RELAXATION OF THE DISTRIBUTION FUNCTION

We now consider how the one-dimensional Fermi gas relaxes to its equilibrium state. The latter is described by the occupation numbers of the different momentum states in the Fermi-Dirac form

$$n_p^{(0)} = \frac{1}{e^{(\varepsilon_p - \mu)/T} + 1}. \tag{21}$$

The evolution of the occupation numbers n_p toward the equilibrium values (21) due to the three-particle collisions is described by the following collision integral:

$$\dot{n}_p = -\frac{1}{2} \sum_{\substack{p_1, p_2, p_3 \\ p_1', p_2', p_3'}} W_{p_1, p_2, p_3}^{p_1', p_2', p_3'} \theta(p_2 - p_3)\theta(p_1' - p_2')\theta(p_2' - p_3')$$
$$\times \delta_{p, p_1} \big[n_{p_1} n_{p_2} n_{p_3} (1 - n_{p_1'})(1 - n_{p_2'})(1 - n_{p_3'})$$
$$- (1 - n_{p_1})(1 - n_{p_2})(1 - n_{p_3}) n_{p_1'} n_{p_2'} n_{p_3'} \big]. \tag{22}$$

Here we again limit the summation to nonequivalent sets of initial as well as final momenta.

For a system near thermal equilibrium it is convenient to present occupation numbers in the form

$$n_p = n_p^{(0)} + g_p \phi_p, \tag{23}$$

where

$$g_p = \sqrt{n_p^{(0)}\big(1 - n_p^{(0)}\big)} = \frac{1}{2\cosh\frac{\varepsilon_p - \mu}{2T}}. \tag{24}$$

We then substitute Eq. (23) into Eq. (22), linearize in small ϕ_p, and obtain

$$\dot{\phi}_p = -\widehat{W}\phi_p, \tag{25}$$

where the linearized collision integral $\widehat{W}$ is defined by

$$\widehat{W}\phi_p = \frac{1}{2} \sum_{\substack{p_1, p_2, p_3 \\ p_1', p_2', p_3'}} W_{p_1, p_2, p_3}^{p_1', p_2', p_3'} \theta(p_2 - p_3)\theta(p_1' - p_2')\theta(p_2' - p_3')$$
$$\times \delta_{p, p_1} g_{p_2} g_{p_3} g_{p_1'} g_{p_2'} g_{p_3'}$$
$$\times \left(\frac{\phi_{p_1}}{g_{p_1}} + \frac{\phi_{p_2}}{g_{p_2}} + \frac{\phi_{p_3}}{g_{p_3}} - \frac{\phi_{p_1'}}{g_{p_1'}} - \frac{\phi_{p_2'}}{g_{p_2'}} - \frac{\phi_{p_3'}}{g_{p_3'}} \right). \tag{26}$$

The problem of the relaxation of the system to thermodynamic equilibrium has now been reduced to solving Eq. (25). Since $\widehat{W}$ is a real symmetric linear integral operator, one can, in principle, solve the eigenvalue problem

$$\widehat{W}\phi_p^{(l)} = \frac{1}{\tau_l}\phi_p^{(l)} \tag{27}$$

and obtain real eigenvalues τ_l^{-1}. A general solution of Eq. (25) is then obtained as a linear combination

$$\phi_p(t) = \sum_l C_l\, e^{-t/\tau_l}\phi_p^{(l)}. \tag{28}$$

Thus the eigenvalues defined by Eq. (27) are the relaxation rates associated with modes $\phi_p^{(l)}$.

Our goal is to study relaxation of the one-dimensional Fermi gas at low temperatures $T \ll \mu$. As discussed above, the relaxation is dominated by the three-particle processes shown in Fig. 1. We limit ourselves to the relatively short-range interactions that fall off faster than $1/|x|^{5/2}$. As we discussed in Sec. III, for such interactions relaxation is dominated by the processes shown in Fig. 1(a). Thus from now on the processes of Fig. 1(b) are neglected.

An important feature of the process shown in Fig. 1(a) is that while all the momenta of the initial and final states of the fermions measured from the nearest Fermi point are of the order of T/v_F, the difference of momenta $|p_3 - p_3'|$ is much smaller than T/v_F. Indeed, using Eqs. (6) and (8), we find

$$|p_3 - p_3'| \simeq \frac{2p_F}{3}|\alpha^2 - \alpha'^2| \lesssim \frac{T^2}{v_F\mu} \ll \frac{T}{v_F}. \tag{29}$$

This feature can be understood as follows. The possible values of momenta of the three particles before and after collision are restricted by the momentum and energy conservation laws. At low temperature the energy spectrum of the particles near the Fermi points is approximately linear:

$$\varepsilon_p = \frac{p^2}{2m} \simeq \mu + v_F(|p| - p_F). \tag{30}$$

In this approximation, any choice of momenta p_1, p_2, p_1', and p_2' such that $p_1 + p_2 = p_1' + p_2'$ guarantees that $\varepsilon_{p_1} + \varepsilon_{p_2} =$

$\varepsilon_{p'_1} + \varepsilon_{p'_2}$. Thus both momentum and energy are conserved if $p_3 = p'_3$. A small quadratic correction to the energy in Eq. (30) results in small $|p_3 - p'_3|$ [see Eq. (29)].

Nonlinearity of the energy spectrum must be taken into account when solving the quantum-mechanical problem of evaluation of the three-particle scattering rate (see Sec. II). Linearization of the spectrum at that stage would lead to singular scattering rates. On the other hand, the collision integral in both its original and linearized forms (22) and (26) takes finite values when the spectrum approaches linear form (30). This procedure is appropriate only for studying the relaxation of the system in the leading order at low temperature [14]. Because in this approximation $p_3 = p'_3$, the distribution function of the particles near the left Fermi point in Fig. 1(a) remains unchanged. Thus, to leading order in $T \ll \mu$ the subsystems of right- and left-moving particles relax independently of each other.

We now substitute Eqs. (7) and (10) into the definition (26) of the operator $\widehat{W}$ and use Eqs. (6) and (15) to convert the sum into an integral over P, P', $\mathcal{E}$, $\mathcal{E}'$, α, and α'. Assuming $p_3 = p'_3$, the integral over the first four of these variables is straightforward and yields

$$\widehat{W}\phi_p = \frac{3}{16\pi^3 \tau_a} \iint d\alpha\, d\alpha' \frac{\alpha^2 + \alpha'^2}{(\alpha^2 - \alpha'^2)^2}\, g_{p_2} g_{p'_1} g_{p'_2}$$
$$\times \left(\frac{\phi_p}{g_p} + \frac{\phi_{p_2}}{g_{p_2}} - \frac{\phi_{p'_1}}{g_{p'_1}} - \frac{\phi_{p'_2}}{g_{p'_2}} \right). \quad (31)$$

Here τ_a is defined by Eq. (17), the spectrum ε_p in the definition (24) of g_p is linearized according to Eq. (30),

$$g_p = \frac{1}{2\cosh \frac{v_F(|p| - p_F)}{2T}}, \quad (32)$$

and the momenta

$$p_2 = p + \frac{4p_F}{\sqrt{3}}\alpha, \quad p'_{1,2} = p + \frac{2p_F}{\sqrt{3}}(\alpha \mp \alpha'), \quad (33)$$

are evaluated to linear order in T using Eqs. (6) and (8). Given that g_p falls off exponentially away from $p = p_F$, the integrals over α and α' in Eq. (31) should be taken from $-\infty$ to $+\infty$.

Similarly to expression (16) for the quasiparticle decay rate, the integrand of Eq. (31) contains a factor $1/(\alpha^2 - \alpha'^2)^2$, which diverges at $\alpha = \pm\alpha'$. However, one can easily see from Eq. (33) that the expression in the second line of Eq. (31) vanishes at $\alpha = \pm\alpha'$. Thus the integrand is only singular as $1/(\alpha^2 - \alpha'^2)$, resulting in a finite integral that should be treated as a principal value.

The eigenvalue problem (27) with $\widehat{W}$ defined by Eqs. (31)–(33) can be solved exactly (see Appendix A). The eigenvalues and eigenfunctions are

$$\frac{1}{\tau_l} = \frac{3}{32\pi^3 \tau_a} \times \begin{cases} \displaystyle\sum_{j=1}^{l} \frac{1}{j} & \text{for even } l \\[3mm] \displaystyle\sum_{j=1}^{l} \frac{1}{j} - \frac{2}{l(l+1)} & \text{for odd } l, \end{cases} \quad (34)$$

$$\phi_p^{(l)} = \theta(p) B_l\left(\frac{v_F(p - p_F)}{\pi T} \right) g_p. \quad (35)$$

Here $l = 0, 1, 2, \ldots$, the rate τ_a^{-1} is defined by Eq. (17), and $B_l(u)$ are modified Bateman polynomials [15–17] defined by

$$B_l(u) = \frac{i^l}{\pi} \cosh \frac{\pi u}{2} \int_{-\infty}^{+\infty} dx\, e^{-iux} \frac{P_l(\tanh x)}{\cosh x}, \quad (36)$$

where $P_l(y)$ are the Legendre polynomials. In particular,

$$B_0(u) = 1, \quad B_1(u) = u, \quad B_2(u) = \frac{3u^2 - 1}{4}. \quad (37)$$

The step function $\theta(p)$ in Eq. (35) accounts for the fact that in the linearized spectrum approximation only the right-moving particles are scattered in Fig. 1(a). In Eq. (35) we omitted the normalization factor, which can be restored with the help of Eq. (A22).

In addition to the processes illustrated in Fig. 1(a) there are similar ones that involve two particles near the left Fermi point and one particle near the right one. These processes equilibrate the left-moving particles. Inversion symmetry dictates that the relaxation rates are again given by Eq. (34) with the relaxation modes

$$\phi_p^{(l)} = \theta(-p) B_l\left(-\frac{v_F(p + p_F)}{\pi T} \right) g_p. \quad (38)$$

For $l = 0$ and 1 the relaxation rates (34) vanish. The corresponding eigenfunctions (35) are

$$\phi_p^{(0)} = \theta(p) g_p, \quad \phi_p^{(1)} = \theta(p) \frac{v_F(p - p_F)}{\pi T} g_p. \quad (39)$$

Indeed, from Eq. (31) one immediately obtains $\widehat{W}\phi_p^{(0)} = 0$, as the expression in parentheses vanishes. The deviation of the distribution function from the equilibrium form (21) described by $\phi_p^{(0)}$ corresponds to a small change of the chemical potential μ. Thus this zero mode reflects the conservation of the number of particles near the right Fermi point. To verify that $\widehat{W}\phi_p^{(1)} = 0$, one should keep in mind that $p + p_2 = p'_1 + p'_2$, which follows immediately from Eq. (33). The latter condition is satisfied automatically for the linearized spectrum (30) because in this case $p_3 = p'_3$. Alternatively, the same condition can be interpreted as conservation of energy of the two particles near the right Fermi point. Correspondingly, the deviation of the distribution function from the equilibrium form (21) described by $\phi_p^{(1)}$ can be interpreted as a result of a small change of temperature.

V. DISCUSSION OF THE RESULTS

In this paper we have studied the relaxation of a gas of one-dimensional spin-$\frac{1}{2}$ fermions at low temperatures. We focused on the case of small deviations of the distribution function from the equilibrium Fermi-Dirac form (21). This enabled us to linearize the collision integral and obtain the spectrum of relaxation rates (34) in terms of the interaction potential and temperature. To leading orders in small temperature and weak interactions the result (34) is exact.

The relaxation rates (34) scale linearly with the temperature [see Eq. (17)]. This conclusion is consistent with the expectation based on the earlier results for the quasiparticle energy relaxation rate in a one-dimensional electron gas with Coulomb interactions [7]. Unlike the authors of Ref. [7], we considered both three-particle processes shown in Fig. 1.

We showed that the scattering processes of Fig. 1(b), which were neglected in Ref. [7], give subleading contribution to the relaxation rate provided that the interaction between the fermions falls off with the distance faster than $1/|x|^{5/2}$. An important example of such a one-dimensional Fermi system is the electron gas in a quantum wire with a metal gate parallel to it, in which case interactions fall off as $1/|x|^3$. The expected temperature dependence for more slowly decaying potentials is given by Eq. (20).

Our result (34) predicts a discrete spectrum of the relaxation rates. It is instructive to compare this behavior with the case of spin-polarized one-dimensional Fermi gas. The linearized collision integral analogous to Eq. (31) was obtained in Ref. [5]. It can be diagonalized numerically (see Appendix B). Importantly, the spectrum of relaxation rates is continuous. The relaxation modes are qualitatively different as well. Specifically, each mode of the continuous spectrum has a singularity at a certain value of momentum and can be associated with decay of a particular quasiparticle state. Continuous spectrum and singularities in relaxation modes were also obtained in other systems of spin-polarized fermions [18,19]. In contrast, our results (35) and (38) show smooth analytic behavior as a function of momentum.

Our expressions (35) and (38) for the relaxation modes contain step functions $\theta(\pm p)$, which limit the ranges of momentum to either positive or negative values. This should not be considered to be a singularity as a function of momentum as our approach is limited to fermion states near the Fermi points, where the linearization (30) of the spectrum is justified. Given the inversion symmetry of the problem it is natural to introduce even and odd modes

$$\phi_p^{(l,+)} = B_l\left(\frac{v_F(|p| - p_F)}{\pi T}\right)g_p, \qquad (40)$$

$$\phi_p^{(l,-)} = B_l\left(\frac{v_F(|p| - p_F)}{\pi T}\right)g_p \,\mathrm{sgn}\,p. \qquad (41)$$

In the approximation of linearized spectrum the modes $\phi_p^{(l,+)}$ and $\phi_p^{(l,-)}$ have the same relaxation rate (34) for any given l. We expect the main effect of the spectral curvature to be a small-in-T/μ splitting of the degeneracies of relaxation rates of the even and odd modes.

The relaxation properties of the one-dimensional Fermi gas determine its transport coefficients, such as the thermal conductivity and viscosity. The thermal conductivity of one-dimensional systems of spinless fermions has been recently studied in Refs. [5,9]. The dc thermal conductivity κ of these systems is controlled by the processes involving exponentially weak backscattering of particles near the bottom of the band. At frequencies above an exponentially small value $\omega^* \propto \exp(-\mu/T)$, the backscattering processes are negligible, and the thermal transport is controlled by the thermal conductivity κ_{ex} of the gas of elementary excitations [8]. A relation between κ_{ex} and the solutions of the relaxation problem has the form [5,20]

$$\kappa_{\mathrm{ex}} = \frac{1}{2m^4T^2}\sum_s \tau_s \frac{\langle\phi_p^{(s)}|\psi_p\rangle^2}{\langle\phi_p^{(s)}|\phi_P^{(s)}\rangle}. \qquad (42)$$

Here the summation is over all the eigenmodes of the relaxation problem with nonvanishing rates τ_s^{-1}; the inner product is defined as

$$\langle\alpha_p|\beta_p\rangle = \int_{-\infty}^{+\infty}\frac{dp}{2\pi\hbar}\alpha_p\beta_p, \qquad (43)$$

and ψ_p is given by

$$\psi_p = 3p_F\left[(|p| - p_F)^2 - \frac{\pi^2T^2}{3v_F^2}\right]g_p \,\mathrm{sgn}\,p. \qquad (44)$$

Interestingly, ψ_p coincides with $\phi_p^{(2,-)}$ given by Eq. (41) up to a momentum-independent factor $4\pi^2T^2m/v_F$ [see Eq. (37)]. Since the eigenmodes $\phi_p^{(s)}$ are orthogonal to each other, only the term with $s = \{2, -\}$ in the sum in Eq. (42) gives a nonvanishing contribution. This greatly simplifies the evaluation of κ_{ex}, which yields

$$\kappa_{\mathrm{ex}} = \frac{2\pi^3}{5}\frac{T^3v_F\tau_2}{\hbar\mu^2} = \frac{128\pi^6}{45}\frac{T^2v_F}{\Lambda^2\mu^2}, \qquad (45)$$

where we applied Eqs. (34) and (17). The result (45) differs dramatically from $\kappa_{\mathrm{ex}} \propto T^{-4}$ in the case of spinless fermions [5]. This is due to the slow relaxation of the spinless system, $\tau^{-1} \propto T^7$ [2–5], compared to $\tau^{-1} \propto T$ for spin-$\frac{1}{2}$ fermions. Equation (45) gives the thermal conductivity of the one-dimensional Fermi gas in a broad range of frequencies below τ_a^{-1} given by Eq. (17) [21].

An expression similar to Eq. (42) can be obtained for the bulk viscosity ζ of a one-dimensional spinless quantum liquid [22]. Unlike thermal conductivity, ζ is controlled by the relaxation modes that are even with respect to inversion. In fact, for the bulk viscosity of a spinless system the analog of ψ_p in Eq. (42) is proportional to $\phi_p^{(2,+)}$ defined by Eq. (40). Generalization of the treatment of bulk viscosity in Ref. [22] to systems with spins is not entirely straightforward and will be discussed elsewhere.

Given the special role that the modes $\phi_p^{(2,+)}$ and $\phi_p^{(2,-)}$ play in the evaluation of the transport coefficients, it is worth discussing how the corresponding relaxation rate behaves for the long-range interactions. As we saw in Sec. III, for interaction potentials that fall off at $x \to \infty$ as $1/|x|^\gamma$ with $\gamma < 5/2$, the scattering processes of Fig. 1(b) dominate the relaxation of the Fermi gas. These processes obey conservation laws of the number of right-moving particles, their momentum, and energy. Thus the corresponding collision integral must have three zero modes. It is easy to see from Eq. (26) that these three modes are $\phi_p^{(0)}$, $\phi_p^{(1)}$, and $\phi_p^{(2)}$ defined by Eqs. (35) and (37). Symmetry requires that the modes $\phi_p^{(0)}$, $\phi_p^{(1)}$, and $\phi_p^{(2)}$ defined by Eq. (38) are also zero modes of the collision integral due to the processes involving three particles on the same branch. We therefore conclude that the even and odd combinations $\phi_p^{(2,+)}$ and $\phi_p^{(2,-)}$ are not affected by the processes of Fig. 1(b) and remain eigenfunctions of the linearized collision integral even for $\gamma < 5/2$. Additionally, our result for τ_2^{-1} obtained from Eq. (34) and the result (45) for κ_{ex} remain unchanged for long-range interactions with $1 < \gamma < 5/2$. In the important case of Coulomb interaction $e^2/|x|$ with a short distance cutoff w, which corresponds to $\gamma = 1$, this result is still valid if one substitutes $\Lambda = (2e^2/\hbar v_F)^2 \ln(p_F w/\hbar)\ln(\mu/T)$ (cf. Ref. [7]).

In this paper the interactions between fermions are treated in the lowest order of the perturbation theory. This has enabled us to ignore the Luttinger liquid effects that develop in interacting one-dimensional systems at $T \to 0$, such as spin-charge separation. The latter means that instead of quasiparticles and quasiholes with Fermi statistics the elementary excitations of the system are two types of bosons, in the charge and spin sectors, propagating at different velocities. Luttinger liquid effects can be neglected if the interactions are sufficiently weak compared with the typical energy of the quasiparticles [7], which in our case is the temperature. This results in the condition $p_F V(0)/\hbar \ll T$.

APPENDIX A: SOLUTION OF THE EIGENVALUE PROBLEM (27)

Here we solve the eigenvalue problem (27) with the operator $\widehat{W}$ defined by Eqs. (31)–(33). We start by introducing dimensionless variables u, w, and w' via

$$p = p_F + \frac{\pi T}{v_F} u, \quad \alpha = \frac{\sqrt{3}\pi T}{8\mu}(w + w'),$$

$$\alpha' = \frac{\sqrt{3}\pi T}{8\mu}(w - w'), \tag{A1}$$

and denoting $\phi_p = \Phi(u)$. Substitution of Eq. (A1) into Eq. (31) yields $\widehat{W}\phi_p = (3/32\pi^3\tau_a)\widehat{\Omega}\Phi(u)$, where

$$\widehat{\Omega}\Phi(u) = \iint dw\,dw' \frac{w^2 + w'^2}{8w^2w'^2} G(u + w + w')G(u + w)$$

$$\times G(u + w')\left(\frac{\Phi(u)}{G(u)} + \frac{\Phi(u + w + w')}{G(u + w + w')}\right.$$

$$\left. -\frac{\Phi(u + w)}{G(u + w)} - \frac{\Phi(u + w')}{G(u + w')}\right), \tag{A2}$$

where the integrals extend from $-\infty$ to $+\infty$ and

$$G(u) = \frac{1}{\cosh(\pi u/2)}. \tag{A3}$$

As a result the eigenvalue problem (27) takes the dimensionless form

$$\widehat{\Omega}\Phi_l(u) = \omega_l \Phi_l(u), \tag{A4}$$

with the eigenvalues ω_l determining the relaxation rates

$$\frac{1}{\tau_l} = \frac{3\omega_l}{32\pi^3\tau_a}. \tag{A5}$$

Next, we use the symmetry $w \leftrightarrow w'$ to replace $(w^2 + w'^2)/w^2w'^2 \to 2/w^2$ in Eq. (A2) and perform the Fourier transform

$$\Phi(u) = \frac{1}{\sqrt{2\pi}} \int \varphi(x)e^{-iux}\,dx,$$

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} \int \Phi(u)e^{iux}\,du. \tag{A6}$$

Then the eigenvalue problem (A4) transforms to

$$\int_{-\infty}^{+\infty} K(x, x')\varphi_l(x')\,dx' = \omega_l\,\varphi_l(x), \tag{A7}$$

where the kernel is given by

$$K(x, x') = \iint \frac{du\,dw}{4\pi}\left[\frac{e^{iu(x-x')}}{w\sinh\frac{\pi w}{2}}\left(\frac{G(u+w)}{G(u)} - e^{-iwx'}\right)\right.$$

$$\left. -2ie^{iux}\frac{\sin(wx')}{w^2\cosh x'}G(u+w)\right]. \tag{A8}$$

Let us now split the kernel into three contributions:

$$K(x, x') = K_1(x - x') + K_2(x, x') + K_3(x, x'), \tag{A9}$$

where

$$K_1(x - x') = \int \frac{du\,dw}{4\pi} \frac{e^{iu(x-x')}}{w\sinh\frac{\pi w}{2}}\left(\frac{G(u+w)}{2G(u)} + \frac{G(u-w)}{2G(u)} - 1\right), \tag{A10}$$

$$K_2(x, x') = \int \frac{du\,dw}{4\pi} e^{iu(x-x')}\frac{1 - \cos(wx')}{w\sinh\frac{\pi w}{2}} = \ln(\cosh x)\delta(x - x'), \tag{A11}$$

$$K_3(x, x') = -\frac{i}{\cosh x'}\int \frac{du\,dw}{4\pi} e^{iux}\frac{\sin(wx')}{w^2}[G(u+w) - G(u-w)] = -\frac{|x + x'| - |x - x'|}{2\cosh x\cosh x'}. \tag{A12}$$

Evaluation of the first kernel is somewhat nontrivial. The result can be presented in the form

$$\int_{-\infty}^{\infty} K_1(x - x')\varphi(x')dx'$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \ln\left(2\tanh\frac{|x - x'|}{2}\right)\text{sgn}\,(x' - x)\frac{d\varphi}{dx'}dx'. \tag{A13}$$

In the following discussion we only use the fact that K_1 is a function of the difference $x - x'$; the explicit form (A13) will not be used. The evaluation of the integrals (A11) and (A12) is straightforward.

Note that $\Phi_0(u) = G(u)$ is an obvious solution of the eigenvalue problem (A4) with the eigenvalue $\omega_0 = 0$ [see Eq. (A2)]. Therefore, its Fourier transform

$\phi_0(x) = \sqrt{2/\pi}\, g(x)$, where

$$g(x) = \frac{1}{\cosh x}, \tag{A14}$$

must solve the eigenvalue problem (A7) with the same eigenvalue $\omega_0 = 0$. Noticing that $K_3(x, x')$ is odd in x and x', we conclude that a condition

$$\int_{-\infty}^{\infty} K_1(x - x')g(x')dx' = g(x)\ln g(x) \tag{A15}$$

must be satisfied.

We now establish some general properties of the derivatives of $g(x)$. Noticing that

$$g'(x) = -g(x)\tanh x, \quad (\tanh x)' = 1 - \tanh^2 x, \tag{A16}$$

it is straightforward to show that the lth derivative $g^{(l)}(x)$ is given by $g(x)$ multiplied by the polynomial of $\tanh x$ of lth power. Let us then consider the most general function of this form:

$$\begin{aligned} a_l g^{(l)}(x) &+ a_{l-1} g^{(l-1)}(x) + \cdots + a_0 g(x) \\ &= g(x)(b_l \tanh^l x + b_{l-1}\tanh^{l-1} x + \cdots + b_0). \end{aligned} \tag{A17}$$

We assume here that the leading coefficients a_l and b_l do not vanish. The two forms of the expression (A17) are equivalent; each set of coefficients $(a_0, a_1, \ldots, a_l)$ uniquely defines the set $(b_0, b_1, \ldots, b_l)$ and vice versa.

We now show that when the integral operator with the kernel $K(x, x')$ is applied to a function of the form (A17), the resulting function also has form (A17), with the same l. By applying the sum of K_1 and K_2 to the lth derivative of $g(x)$ and using Eq. (A15), we find

$$\begin{aligned} \int [K_1(x &- x') + K_2(x, x')]g^{(l)}(x')\, dx' \\ &= \frac{d^l}{dx^l} \int K_1(x - x')g(x')\, dx' + g^{(l)}(x)\ln\cosh x \\ &= \frac{d^l}{dx^l}[g(x)\ln g(x)] - g^{(l)}(x)\ln g(x) \\ &= \sum_{j=1}^{l} \frac{l!}{j!(l-j)!} g^{(l-j)}(x)\frac{d^j}{dx^j}\ln g(x). \end{aligned} \tag{A18}$$

As we saw earlier $g^{(l-j)}(x)$ is given by $g(x)$ multiplied by a polynomial of $\tanh x$ of power $l - j$. Taking into consideration Eq. (A16) and noticing that $[\ln g(x)]' = -\tanh x$, it is easy to see that $d^j \ln g(x)/dx^j$ is a polynomial of $\tanh x$ of power j. Thus each term in the last line of Eq. (A18) is $g(x)$ multiplied by a polynomial of $\tanh x$ of power l, and therefore the right-hand side of Eq. (A18) has the form (A17). Addition of the terms with lower-order derivatives, which are also present the left-hand side of Eq. (A17), does not change the general form of the result. Thus the application of the integral operator with the kernel $K_1 + K_2$ to a function of form (A17) gives a function of the same form.

Because $K_3(x, x')$ is odd in x and x', the corresponding integral operator gives zero when applied to $g(x)\tanh^{2m} x$ with $m = 0, 1, 2, \ldots$. Let us now apply this operator to $g(x)\tanh^{2m+1} x$:

$$\begin{aligned} \int K_3(x, x')&g(x')\tanh^{2m+1} x'\, dx' \\ &= -\int \frac{|x + x'| - |x - x'|}{2\cosh x}\tanh^{2m+1} x'\frac{dx'}{\cosh^2 x'} \\ &= \frac{g(x)}{4(m+1)}\int (\tanh^{2m+2} x' - 1) \\ &\quad [\mathrm{sgn}(x + x') - \mathrm{sgn}(x' - x)]\, dx' \\ &= \frac{g(x)}{2(m+1)}\int_{-x}^{x}(\tanh^{2m+2} x' - 1)\, dx' \\ &= -\frac{g(x)}{m+1}\sum_{j=0}^{m}\frac{\tanh^{2j+1} x}{2j+1}. \end{aligned} \tag{A19}$$

Thus the integral operator with the kernel K_3 applied to a function of the form (A17) results in a function of the same form.

We have therefore demonstrated that the action of the integral operator with the kernel (A9) on any function of the form (A17) results in a function of the same form. This enables us to find the eigenfunctions of the integral operator in Eq. (A7). We first notice that a function of the form (A17) is fully described by $l + 1$ coefficients $b_0, b_1, \ldots, b_l$. Such functions form an $(l + 1)$-dimensional subspace, and our operator in this subspace is a symmetric matrix of size $(l + 1) \times (l + 1)$. It has $l + 1$ eigenfunctions that are orthogonal to each other and have the form $g(x)p_l(\tanh x)$, where p_l is a polynomial of power l. When l is increased by 1, a new eigenfunction $g(x)p_{l+1}(\tanh x)$ appears. Thus all the solutions have polynomials p_l of different powers. The orthogonality condition

$$\begin{aligned} \int_{-\infty}^{+\infty} g(x)&p_l(\tanh x)\, g(x)p_{l'}(\tanh x)\, dx \\ &= \int_{-1}^{1} p_l(y)p_{l'}(y)\, dy = \delta_{l,l'} \end{aligned} \tag{A20}$$

indicates that $p_l(y)$ are proportional to the Legendre polynomials $P_l(y)$. The normalized eigenfunctions are

$$\varphi_l(x) = \sqrt{l + \frac{1}{2}}\,\frac{P_l(\tanh x)}{\cosh x}. \tag{A21}$$

Normalized eigenfunctions $\Phi_l(u)$ of the operator (A2) are obtained by performing the inverse Fourier transform (A6) of the above expression,

$$\Phi_l(u) = i^l \frac{\sqrt{2l + 1}}{2\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-iux}\frac{P_l(\tanh x)}{\cosh x}dx, \tag{A22}$$

where the additional factor i^l ensures that $\Phi_l(u)$ is real for all l. Taking into account the definition of u in Eq. (A1) and omitting the normalization constant one obtains our result (35).

To find the eigenvalues ω_l, we consider separately the cases of even and odd l. Because $K_3(x, x')$ is odd in x and x', it does not affect the eigenvalues for even l, when the eigenfunction (A21) is even in x. The combined effect of K_1 and K_2 on the eigenfunction φ_l can be obtained from Eq. (A18). Its right-hand side is a linear combination of $\varphi_j(x)$ with $j \leqslant l$.
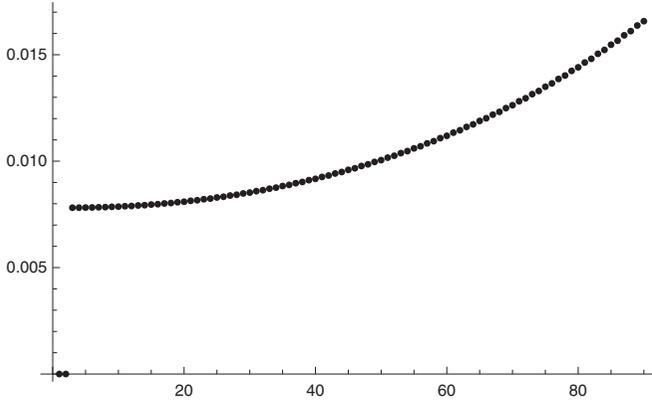
FIG. 2. Eigenvalues λ_l of the operator $\widehat{M}$ obtained by numerical solution of the integral equation (B2).

Note that the term $g(x) \tanh^l x$ appears only in $\varphi_l(x)$. Thus the coefficient of this term in the last line of Eq. (A18) is given by that in the expansion of $g^{(l)}$ in the left-hand side times ω_l. Using the relations (A16), one easily finds

$$\frac{g^{(l)}(x)}{g(x)} = (-1)^l \, l! \, \tanh^l x + \cdots,$$

$$\frac{d^j}{dx^j} \ln g(x) = (-1)^j (j-1)! \, \tanh^j x + \cdots,$$

where the omitted terms have the form $c_j \tanh^j x$ with $j < l$. Applying these results to Eq. (A18), we obtain

$$\omega_l(-1)^l \, l! \, g(x) \tanh^l x$$

$$= \sum_{j=1}^{l} \frac{l!}{j!(l-j)!}(-1)^{(l-j)}(l-j)! \, g(x) \tanh^{l-j} x \, (-1)^j$$

$$\times (j-1)! \, \tanh^j x.$$

This immediately yields

$$\omega_l = \sum_{j=1}^{l} \frac{1}{j} \tag{A23}$$

for even l.

For odd $l = 2m+1$ there is an additional contribution due to the K_3 part of the kernel. It is given by the coefficient of $g(x) \tanh^{2m+1} x$ on the right-hand side of Eq. (A19), i.e., $\delta\omega_{2m+1} = -[(m+1)(2m+1)]^{-1}$. Thus for odd l the eigenvalue is

$$\omega_l = \sum_{j=1}^{l} \frac{1}{j} - \frac{2}{l(l+1)}. \tag{A24}$$

Equations (A23) and (A24) in combination with Eq. (A5) give the result (34).

APPENDIX B: RELAXATION RATE AND MODES IN THE SPINLESS FERMI GAS

Relaxation of the one-dimensional spinless Fermi gas was studied in Ref. [5]. In the case of short-range interaction the
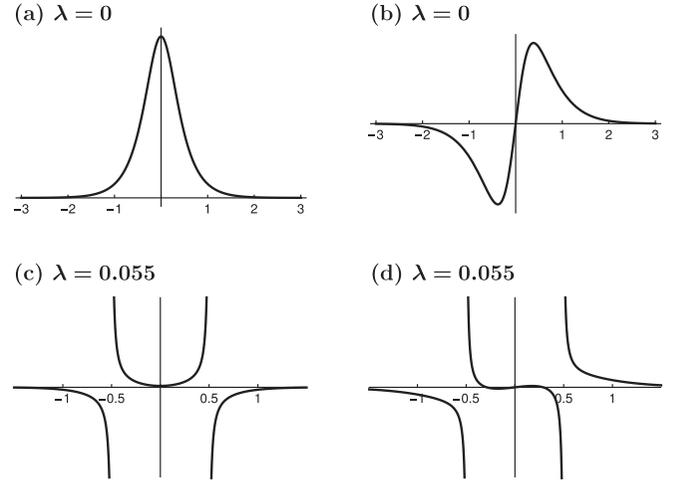

FIG. 3. Numerically obtained eigenfunctions $\Phi_l(\xi)$ of the operator $\widehat{M}$. (a, b) The two modes with zero eigenvalues. (c, d) The even and odd modes with nonzero eigenvalues. The position of the singularity is $\xi_0 = 0.5$.

relaxation rates are given by

$$\frac{1}{\tau_l} = \frac{2\pi^3 \Lambda^2 T^7}{\hbar^5 v_F^8} \lambda_l. \tag{B1}$$

Here the parameter Λ is quadratic in $V(q)$ but different from our earlier expression (11); it is given by Eq. (80) of Ref. [5]. The parameters λ_l are obtained by solving the eigenvalue problem

$$\widehat{M}\Phi_l(\xi) = \lambda_l \Phi_l(\xi). \tag{B2}$$

Here the operator $\widehat{M}$ is defined by

$$\widehat{M}\Phi(\xi) = A(\xi)\Phi(\xi) + \int d\xi' [B_1(\xi,\xi') + B_2(\xi,\xi')]\Phi(\xi'), \tag{B3}$$

where the integration is from $-\infty$ to $+\infty$ and

$$A(\xi) = \frac{(1+4\xi^2)(9+4\xi^2)(5+44\xi^2)}{5760}, \tag{B4}$$

$$B_1(\xi,\xi') = \frac{1}{6}(\xi-\xi')^2 \frac{(\xi+\xi')[1+(\xi+\xi')^2]}{\sinh(\pi(\xi+\xi'))}, \tag{B5}$$

$$B_2(\xi,\xi') = -\frac{\xi-\xi'}{240\sinh(\pi(\xi-\xi'))}$$
$$\times(7 + 120\xi\xi' + 128\xi^4 - 752\xi^3\xi'$$
$$+ 1488\xi^2\xi'^2 - 752\xi\xi'^3 + 128\xi'^4). \tag{B6}$$

Similarly to the relaxation problem (27) with $\widehat{W}$ defined by Eq. (31), only the right-moving particles are accounted for by the operator $\widehat{M}$. The relaxation of the left-moving particles can be obtained by using the inversion symmetry of the system.

The integral equation (B2) can be solved numerically by replacing the infinite limits of integration with finite but large ones and discretizing the function $\Phi(\xi)$. The resulting spectrum of eigenvalues is shown in Fig. 2. The two lowest

eigenvalues vanish. A gap $\Delta = A(0) = 1/128$ separates $\lambda_0 = \lambda_1 = 0$ and $\lambda_2 \approx 0.00781$. The dense set of eigenvalues above the gap represents a continuous spectrum of relaxation rates and extends to $+\infty$.

The two modes with zero eigenvalues are plotted in Figs. 3(a) and 3(b). Up to a numerical prefactor they are given by

$$\Phi_0(\xi) = \frac{1}{\cosh(\pi\xi)}, \quad \Phi_1(\xi) = \frac{\xi}{\cosh(\pi\xi)}. \quad \text{(B7)}$$

These two modes account for the conservation of the number of particles and energy and are fully analogous to the modes (39) for fermions with spin.

The modes corresponding to nonvanishing eigenvalues are qualitatively different. Each mode is either an even or an odd function of ξ and has two singularities $(\xi \pm \xi_0)^{-1}$ for some value of ξ_0. The corresponding eigenvalue is related to the positions $\pm\xi_0$ of the singularities by $\lambda = A(\xi_0)$. Pairs of even and odd modes are present for all real ξ_0. Typical modes with $\xi_0 = 0.5$ are shown in Figs. 3(c) and 3(d).

[1] E. M. Lifshitz and L. P. Pitaevskii, *Statistical Physics, Part 2* (Butterworth-Heinemann, Oxford, 1980).

[2] A. Imambekov, T. L. Schmidt, and L. I. Glazman, One-dimensional quantum liquids: Beyond the Luttinger liquid paradigm, Rev. Mod. Phys. **84**, 1253 (2012).

[3] M. Arzamasovs, F. Bovo, and D. M. Gangardt, Kinetics of Mobile Impurities and Correlation Functions in One-Dimensional Superfluids at Finite Temperature, Phys. Rev. Lett. **112**, 170602 (2014).

[4] I. V. Protopopov, D. B. Gutman, and A. D. Mirlin, Relaxation in Luttinger liquids: Bose-Fermi duality, Phys. Rev. B **90**, 125113 (2014).

[5] K. A. Matveev and Z. Ristivojevic, Thermal conductivity of the degenerate one-dimensional Fermi gas, Phys. Rev. B **99**, 155428 (2019).

[6] K. A. Matveev and A. Furusaki, Decay of Fermionic Quasiparticles in One-Dimensional Quantum Liquids, Phys. Rev. Lett. **111**, 256401 (2013).

[7] T. Karzig, L. I. Glazman, and F. von Oppen, Energy Relaxation and Thermalization of Hot Electrons in Quantum Wires, Phys. Rev. Lett. **105**, 226407 (2010).

[8] K. A. Matveev and A. V. Andreev, Propagation and attenuation of sound in one-dimensional quantum liquids, Phys. Rev. B **98**, 155441 (2018).

[9] R. Samanta, I. V. Protopopov, A. D. Mirlin, and D. B. Gutman, Thermal Transport in One-Dimensional Electronic Fluids, Phys. Rev. Lett. **122**, 206801 (2019).

[10] A. M. Lunde, K. Flensberg, and L. I. Glazman, Three-particle collisions in quantum wires: Corrections to thermopower and conductance, Phys. Rev. B **75**, 245418 (2007).

[11] M. Gaudin, Un systeme a une dimension de fermions en interaction, Phys. Lett. A **24**, 55 (1967).

[12] C. N. Yang, Some Exact Results for the Many-Body Problem in One Dimension with Repulsive Delta-Function Interaction, Phys. Rev. Lett. **19**, 1312 (1967).

[13] B. Sutherland, *Beautiful Models: 70 Years of Exactly Solved Quantum Many-Body Problems* (World Scientific, Singapore, 2004).

[14] A similar linearization of the excitation spectrum was performed in the problem of scattering of phonons in a one-dimensional Wigner crystal in Ref. [19].

[15] H. Bateman, Some properties of a certain set of polynomials, Tohoku Math. J. **37**, 23 (1933).

[16] H. Bateman, The polynomial $F_n(x)$, Ann. Math. **35**, 767 (1934).

[17] Polynomials $B_l(u)$ are related to the Bateman polynomials $F_l(u)$ via $B_l(u) = i^l F_l(iu)$.

[18] W. DeGottardi and K. A. Matveev, Equilibration of quasi-one-dimensional Fermi gases, Phys. Rev. B **99**, 245121 (2019).

[19] J. Lin, K. A. Matveev, and M. Pustilnik, Thermalization of Acoustic Excitations in a Strongly Interacting One-Dimensional Quantum Liquid, Phys. Rev. Lett. **110**, 016401 (2013).

[20] We have multiplied the result (41) of Ref. [5] by 2 to account for the spins of the fermions.

[21] For spinless systems, it was recently predicted [9] that well below the quasiparticle relaxation rate the frequency-independent result κ_{ex} for the thermal conductivity of the system is replaced by $\kappa \propto \omega^{-1/3}$. We leave the study of this effect for systems of fermions with spin for future work.

[22] K. A. Matveev and M. Pustilnik, Viscous Dissipation in One-Dimensional Quantum Liquids, Phys. Rev. Lett. **119**, 036801 (2017).