Stability of Weyl semimetals with quasiperiodic disorder

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Weyl semimetals are phases of matter with excitations effectively described by massless Dirac fermions. Their critical nature makes unclear the persistence of such a phase in the presence of disorder. We present a theorem ensuring the stability of the semimetallic phase in the presence of weak quasiperiodic disorder. The proof relies on the subtle interplay of the relativistic quantum field theory description combined with number-theoretical properties used in Kolmogorov-Arnold-Moser theory.

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I. INTRODUCTION

Conduction electrons in metals are well described by the Schrödinger equation but in certain cases the interaction with the lattice produces an effective relativistic description in terms of massless Dirac particles; this happens, in particular, in Weyl semimetals [1], which have been recently experimentally discovered [2–5]. This offers the possibility of observing the counterpart of high-energy phenomena at a much lower-energy scale, and to have materials with unusual physical properties. The critical nature of excitations has the effect that in several cases predictions are ambiguous and sensitive to approximations. Indeed, while there is agreement that at weak coupling many-body interactions do not destroy the semimetalllic phase [6–9], it is still a subject of debate the effect of disorder. Field-theoretical approaches find that a weak random disorder does not destroy the semimetallic phase [10,11] while other studies [12] based on the inclusion of rare region effects lead to the opposite conclusion, namely that even an arbitrary weak random potential destabilizes the system. Numerical investigations were done for random [13–19] or quasiperiodic disorder [20,21], but conclusions are subjected to finite-size effects [22].

Rigorous results in this context are useful as they can act as a benchmark to check approximations or conjectures. In this paper we rigorously analyze Weyl semimetals on a lattice in the presence of a weak quasiperiodic disorder. Such disorder is the one realized in cold atoms experiments [23,24]; in addition, the quasiperiodic potential can effectively describe coupled Dirac systems like Moire' superlattices [25]. The effect of quasiperiodic potentials for quantum particles has been deeply studied in one dimension; in the noninteracting case a very detailed mathematical knowledge was reached [26,27], and recently great progress in understanding the effect of the interaction has been obtained [28–37]. In contrast, very little is known for higher-dimensional Dirac systems, with the exception of [20,21], where numerical evidence of stability of the Weyl semimetallic phase was found. The main difficulty of quasiperiodic disorder is the presence of infinitely many processes involving a large exchange of momentum, which, due to Umklapp and the incommensurability of frequencies, connect fermions with momenta close to the Weyl points. Such processes are dimensionally relevant in the renormalization group (RG) sense and the effect of disorder in principle increases at each RG iteration and could destroy the Weyl semimetallic phase. This phenomenon manifests in the presence in the series expansion of small divisors which could break convergence.

A similar situation is encountered in classical mechanics and in particular in Kolmogorov-Arnold-Moser (KAM) theory, where quasiperiodic solutions are written as Lindestedt series see, e.g., [38]. Such series are plagued by small divisors but their convergence is ensured by subtle cancellations due to the number-theoretical properties of irrational numbers, see, e.g., [39]. In this paper we show that a similar phenomenon allows to prove the stability of the semimetallic phase in Weyl semimetals; number theoretical properties allow to prove that the relevant terms almost connecting Weyl points are indeed ineffective. Physical quantities are written as convergent series so that nonperturbative effects due to small divisors are excluded.

The paper is organized in the following way. In Sec. II the model is presented, in Sec. III we describe the effect of Umklapp terms, in Sec. IV we recall number theoretical properties of irrationals, and in Sec. V the main result is presented. Finally, in Sec. VI the RG analysis is presented and Sec. VII is devoted to conclusions.

II. WEYL SEMIMETALS WITH QUASIPERIODIC DISORDER

A basic model for Weyl semimetals, see [1], is obtained assuming a pair of orbitals on each site of a lattice, preserving inversion but with broken time reversal symmetry; if $x = (x_1, x_2, x_3)$ are points in a cubic three-dimensional lattice Λ, $a^{\pm}_{x,1}, a^{\pm}_{x,2}$ fermionic creation or annihilation operators, the

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hopping Hamiltonian is $H_0 =$

$$\sum_{x\in\Lambda}\Bigg\{\sum_{j=1}^{2}(-1)^{j-1}\Big[(\zeta-1)a^\dagger_{x,j}a_{x,j}+\frac{1}{2}a^\dagger_{x,j}(-\Delta a)_{x,j}\Big] +\frac{it_1}{2}\big[a^\dagger_{x,1}\big(a_{x+e_1,2}-a_{x-e_1,2}\big)+a^\dagger_{x,2}\big(a_{x+e_1,1}-a_{x-e_1,1}\big)\big] +\frac{t_2}{2}\big[a^\dagger_{x,1}\big(a_{x+e_2,2}-a_{x-e_2,2}\big)-a^\dagger_{x,2}\big(a_{x+e_2,1}-a_{x-e_2,1}\big)\big]\Bigg\}, \tag{1}$$

where in the first line Δ is the standard lattice Laplacian: $\Delta f(x)=\sum_{l=1}^{3}[f(x+e_l)+f(x-e_l)-2f(x)]$. The Hamiltonian H_0 in Fourier space can be written as $H_0=\int\frac{dk}{(2\pi)^3}\,\widehat{a}^\dagger_k h(k)\widehat{a}_k$ with

$$h(k)=\begin{pmatrix}\alpha(k) & \beta(k)\\ \beta^*(k) & -\alpha(k)\end{pmatrix}, \tag{2}$$

where $k\in(0,2\pi]^3$, $\alpha(k)=2+\zeta-\cos k_1-\cos k_2-\cos k_3$, and $\beta(k)=t_1\sin k_1-it_2\sin k_2$. We assume that $\zeta\in[0,1)$, in which case $\widehat{h}(k)$ is singular at $k=\pm p_F$, with $p_F=(0,0,\arccos\zeta)$ called the Weyl point. In the vicinity of $\pm p_F$, $k=q\pm p_F$,

$$\widehat{H}^0(q\pm p_F)=t_1\sigma_1 q_1+t_2\sigma_2 q_2\pm\sin p_F\sigma_3 q_3+O(q^2). \tag{3}$$

We include now a many-body interaction and quasiperiodic disorder writing

$$H=H_0+\varepsilon\sum_x\phi_x\big(a^+_{x,1}a^-_{x,1}-a^+_{x,2}a^-_{x,2}\big)+\lambda\sum_{x,y}v(x-y)\rho_x\rho_y, \tag{4}$$

where $v(x-y)$ is a short-range potential, $\rho_x=a^+_{x,1}a^-_{x,1}+a^+_{x,2}a^-_{x,2}$, and

$$\phi_x=\sum_n\widehat{\phi}_n e^{i2\pi(\omega_1 n_1x_1+\omega_2 n_2 x_2+\omega_3 n_3 x_3)}, \tag{5}$$

with $n\in\mathbb{Z}^3$, $\widehat{\phi}_n=\widehat{\phi}_{-n}$, and $|\widehat{\phi}_n|\leqslant Ce^{-\xi(|n_1|+|n_2|+|n_3|)}$. We assume the periodicity of the potential incommensurate with the lattice periodicity, by taking ω_i as *irrational*. The above potential includes the basic example $\phi_x=\sum_i\cos(\omega_i x_i)$ and respects the inversion symmetry; it corresponds to a quasiperiodic staggered chemical potential.

If $\psi^\pm_{\mathbf{x}}=e^{Hx_0}\psi^\pm_x e^{-Hx_0}$, $\mathbf{x}=(x_0,x)$, x_0 the imaginary time, the 2-point function is given by $S(\mathbf{x},\mathbf{y})=\frac{\mathrm{Tr}e^{-\beta H}T\psi^-_{\mathbf{x}}\psi^+_{\mathbf{y}}}{\mathrm{Tr}e^{-\beta H}}$ and $\widehat{S}(\mathbf{k})$ is the Fourier transform, $\mathbf{k}=(k_0,k)$. In the noninteracting case $\lambda=\varepsilon=0$ one has $S(\mathbf{x},\mathbf{y})|_0=g(\mathbf{x}-\mathbf{y})$ with

$$g(\mathbf{x})=\frac{1}{L^3\beta}\sum_{\mathbf{k}}e^{i\mathbf{k}\mathbf{x}}[-ik_0I+h(k)]^{-1} \tag{6}$$

and $\widehat{g}(\mathbf{k})=[-ik_0I+h(k)]^{-1}$ is its Fourier transform. From (3) we see that, close to the Weyl momenta, the propagator $\widehat{g}(\mathbf{q}\pm\mathbf{p}_F)$ is equal to the massless Dirac propagator up to corrections. By this, one can easily deduce the physical properties; for instance, the real part of the zero-temperature optical conductivity vanishes linearly with the frequency $\sigma(\omega)\sim\omega$.

To investigate the stability of the Weyl semimetallic phase in the presence of incommensurate potential, it is convenient to write the interacting correlations as $S(\mathbf{x},\mathbf{y})=\frac{\partial^2 W}{\partial\phi^-_{\mathbf{x}}\partial\phi^+_{\mathbf{y}}}$, where $W(\phi)$ is the Grassmann integral defined in the following way:

$$e^{W(\phi)}=\int P(d\psi)e^V, \tag{7}$$

where ϕ is an external field, $\psi^\pm_{\mathbf{x},i}$ are Grassmann variables, $P(d\psi)$ is the Grassman integration with propagator $g(\mathbf{x})$, and

$$V=\lambda\int d\mathbf{p}\widehat{v}(\mathbf{p})\widehat{\rho}_{\mathbf{p}}\widehat{\rho}_{-\mathbf{p}}+\int d\mathbf{x}(\psi^+_{\mathbf{x}}\phi^-_{\mathbf{x}}+\psi^-_{\mathbf{x}}\phi^+_{\mathbf{x}}) +\varepsilon\sum_{n,i}\widehat{\phi}_n\int d\mathbf{k}(-1)^i\widehat{\psi}^+_{i,\mathbf{k}_1}\widehat{\psi}_{i,\mathbf{k}_2} \times\delta_p(\mathbf{k}_1-\mathbf{k}_2+\bar{\omega}_n 2\pi), \tag{8}$$

where $\bar{\omega}_n=(0,\omega_n)$, $\omega_n=(\omega_1 n_1,\omega_2 n_2,\omega_3 n_3)$, $\int d\mathbf{x}=\int_{-\beta/2}^{\beta/2}dx_0\sum_x$, and

$$\widehat{\rho}_{\mathbf{p}}=\int d\mathbf{k}(\widehat{\psi}^+_{\mathbf{k},1}\widehat{\psi}^-_{\mathbf{k}+\mathbf{p},1}+\widehat{\psi}_{\mathbf{k},2}\widehat{\psi}^+_{\mathbf{k}+\mathbf{p},2}). \tag{9}$$

Finally,

$$\delta_p(\mathbf{k})=\delta_p(k_0)\prod_{i=1}^{3}\delta_p(k_i),\qquad \delta_p(k_i)=L\sum_n\delta_{k_i,2n\pi}. \tag{10}$$

Note that momentum is conserved to momenta $2\pi n$ due to the presence of the lattice.

III. RELEVANT PROCESSES AND UMKLAPP TERMS

A natural way to understand the effect of the interaction and disorder is to use the RG. The physical information are encoded in the marginal or relevant processes, that is, the terms with vanishing or positive scaling dimension. The linear divergence at the Weyl points of the propagator (6) says that the scaling dimension of the interactions with n ψ fields is $D=4-3n/2$, so that the only relevant terms are the bilinear ones. In the absence of quasiperiodic potential $\varepsilon=0$, there is only one relevant term corresponding to a shift in the position of the Weyl points. The irrelevance of the quartic terms has the effect that, in the weak coupling regime, the semimetallic behavior persists and the only effects of the interaction are the finite renormalization of the velocities and wave function, see [6].

The presence of the quasiperiodic potential produces infinitely many relevant terms quadratic in the fields, with momenta $\mathbf{k}_1,\mathbf{k}_2$ such that $k_{1,i}-k_{2,i}+2\omega_i n_i\pi+2l_i\pi=0$ with l_i,n_i positive or negative integers. The factor $2\omega_i n_i\pi$ is the momentum exchanged with the quasiperiodic disorder while the factor $2l_i\pi$ is exchanged with the lattice (Umklapp). Only the terms connecting fermions with momenta close to the Weyl points are really important and, due to Umklapp, this can happen also in correspondence of a nonvanishing transfer of momentum produced by the disorder. The important processes involve fermions with momenta close to the same Weyl point $\sigma=0$ or to opposite ones $\sigma=1/\pi$ ones; if $p_F=(0,0,p_{F,3})$

this requires

$$n_1\omega_1 - l_1 \sim 0, \quad n_2\omega_2 - l_2 \sim 0, \quad n_3\omega_3 - l_3 \pm \sigma p_{F,3} \sim 0. \tag{11}$$

Note the basic difference between periodic or quasiperiodic potentials. In the first case ω_i is rational $\omega_i = p/q$ so that the differences in (11) either are exactly vanishing or are $O(1/q)$: there are no processes (for $p_{F,3} \neq n\pi\omega_3$) connecting momenta arbitrarily close to the Weyl points except the one with $n_i = 0$, a process corresponding to the shift of the chemical potential. Therefore a periodic potential is not expected to modify the physical behavior for generic values of p_F, at least for small ε (except opening of gaps at $p_{F,3} = n\pi\omega_3$).

In contrast, in the quasiperiodic case (11) can be arbitrarily close to zero, for the basic properties of irrational numbers. This means that there are infinitely many relevant processes connecting the Weyl points. Such a feature makes the case of quasiperiodic potentials very close to the random case, where the difference of momenta of relevant terms is $k_1 - k_2 = p$ with p the momentum carried by a random field $\widehat{\phi}_p$ which can be arbitrarily small.

IV. KAM THEOREM AND DIOPHANTINE CONDITIONS

In the case of random potential the issue of stability is related to the probability that certain dangerous configurations happen. In the quasiperiodic case, the problem is deterministic and related to the irrationality properties of the frequencies. Therefore, quantitative estimates saying how much an irrational is close to a rational one are necessary. For instance, the golden number $\omega = \frac{\sqrt{5}-1}{2}$ verifies $|q\omega - p| \geqslant \frac{1}{(3+\sqrt{5})}\frac{1}{2\pi q}$. If such ω is the frequency of the quasiperiodic potential, this says that, looking at (11), *only* the processes involving a *large* transfer of momentum can involve fermions *close* to the Weyl points. Such a property is indeed generic. There is a class of irrationals called *Diophantine*, such that, for $q \neq 0$, $p, q \in \mathbb{Z}^2/(0,0)$

$$|q\omega - p| \geqslant \frac{C_0}{2\pi q^\tau}. \tag{12}$$

The irrationals not verifying (12) in the unit segment have measure $O(C_0)$; as C_0 can be taken arbitrarily small, the set of Diophantine numbers is full, see, e.g., [38]. Indeed the set of ω in the unit cube verifying $|q\omega - p| < \frac{C_0}{q^\tau}$ for a certain q, p is smaller than $2C_0/q^{\tau+1}$ hence summing over p (a sum bounded by $C|q|$) and q we get a set with measure bounded by $CC_0 \sum_q \frac{1}{q^\tau}$ which is $O(C_0)$ for $\tau > 1$.

It is therefore not restrictive to assume the following conditions on the frequencies, $i = 1, 2, 3$

$$|2\pi\omega_i n|_T \geqslant \frac{C_0}{|n|^\tau} \quad |2\pi\omega_3 n \pm 2p_{F,3}|_T \geqslant \frac{C_0}{|n|^\tau} \quad n \in \mathbb{Z}/0, \tag{13}$$

where by $|.|_T$ we mean the average on the torus, that is $|2\pi\omega n|_T = \inf_p |2\pi\omega n - 2\pi p|$; the first condition is (12) and the second is a requirement of incommensurability for $p_{F,3}$. As we will see, Diophantine conditions are crucial to prove the stability of the Weyl semimetallic phase.

Another point to stress is that, to impose a periodic boundary condition, we have to choose a sequence of ω rational converging to an irrational in the infinite volume limit. To do that we start from the continued fraction representation of a number ω

$$\omega = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}. \tag{14}$$

We approximate ω by a sequence of rational numbers (*convergents*) $\frac{p_1}{q_1} = a_0 + \frac{1}{a_1}$, $\frac{p_2}{q_2} = a_0 + \frac{1}{a_1 + \frac{1}{a_2}}$, and so on. The properties of the convergents imply that if ω verifies the Diophantine condition then $|\pi(n\frac{p_i}{q_i} - k)| \geqslant \frac{C}{2|n|^\tau}$ if $q_1 \leqslant n \leqslant \frac{q_i}{2}$ and any k. Therefore, we can impose periodic boundary conditions by considering a sequence of frequencies $\omega_i = \frac{p_i}{q_i}$ and $L_i = q_i$.

Finally, it is worth recalling that the number-theoretical conditions are unusual in condensed matter, but rather common in other branches of physics. For instance, planets around the sun neglecting the mutual attraction have an integrable Hamilltonian dynamics which is quasiperiodic, and according to KAM theory only quasiperiodic motions with Diophantine frequencies survive in the presence of perturbation-breaking integrability [38]. Indeed quasiperiodic solutions are written as a series in the perturbation, called the Lindstedt series, whose convergence follows by subtle cancellations due to Diophantine conditions, see, e.g., [39].

V. MAIN RESULT

As the interaction in general moves the location of the Weyl mometum, we write $\xi = \cos p_F + \nu$ in (4) and we choose ν so that p_F is the just the interacting Weyl momentum.

Theorem. For λ, ε small enough, and assuming that the frequencies ω_i in (5) verify (13), there exists ν such the 2-point function $\widehat{S}(\mathbf{k})$ behaves as, if $\mathbf{p}_F = (0, p_F)$

$$S(\mathbf{q} \pm \mathbf{p}_F) = \frac{1}{Z}\begin{pmatrix} -iq_0 \pm v_3 q_3 & v_1 q_1 - i v_2 q_2 \\ v_1 q_1 + i v_2 q_2 & -iq_0 \mp v_3 q_3 \end{pmatrix}^{-1} [1 + O(\mathbf{q})], \tag{15}$$

with $Z = 1 + O(\lambda, \varepsilon)$, $v_1 = t_1 + O(\lambda, \varepsilon)$, $v_2 = t_2 + O(\lambda, \varepsilon)$, $v_3 = \sin p_F + O(\lambda, \varepsilon)$.

This result proves the stability of the Weyl semimetallic phase, as quasiperiodic disorder does not modify qualitatively the 2-point function but produces only a finite renormalization of the parameters; no phase transition is present at small disorder. As a consequence the real part of optical conductivity vanishes as $O(\omega)$ as in the noninteracting case. Even if there are infinitely many relevant terms due to quasiperiodic disorder, they do not modify the physical behavior. The result is in agreement with the numerical evidence in [20,21].

VI. RENORMALIZATION GROUP

To prove (15) we need to evaluate the generating function $\int P(d\psi)e^{\mathcal{V}}$ with $\mathcal{V} = V + \nu \int \widehat{\psi}^+ \sigma_3 \widehat{\psi}^-$ with V given by (8) and the propagator given by $g(\mathbf{x})$. We introduce two smooth cutoff functions $\chi_\pm(\mathbf{k} \mp \mathbf{p}_F)$ nonvanishing in a region $|\mathbf{k} \mp \mathbf{p}_F| \leqslant \gamma$ and nonoverlapping, $\gamma > 1$ a suitable constant: we define $\widehat{g}_\rho^{(\leqslant 0)}(\mathbf{k}) = \chi_\rho(\mathbf{k} - \rho\mathbf{p}_F)\widehat{g}(\mathbf{k})$ and

$$g(\mathbf{x}) = g^{(1)}(\mathbf{x}) + \sum_{\rho=\pm} g_\rho^{(\leqslant 0)}(\mathbf{x}) \tag{16}$$

with $\widehat{g}^{(1)}(\mathbf{k}) = (1 - \sum_\rho \chi_\rho)\widehat{g}(\mathbf{k})$; this induces the Grassmann variable decomposition $\psi_{\mathbf{x}} = \psi^{(1)}_{\mathbf{x}} + \sum_{\rho=\pm}\psi^{(\leqslant 0)}_\rho$ with propagators given by $g^{(1)}(\mathbf{x})$ and $g^{(\leqslant 0)}_\rho(\mathbf{x})$, respectively. Note that $\psi^{(1)}$ corresponds to fermions with momenta far from the Weyl points, while $\psi^{(\leqslant 0)}_\pm$ with momenta around $\pm\mathbf{p}_F$.

We can further decompose $\chi_\rho(\mathbf{k} - \rho\mathbf{p}_F) = \sum_{h=-\infty}^0 f_{h,\rho}(\mathbf{k})$ with $f_{h,\rho}(\mathbf{k}) = \chi_\rho(\gamma^{-h}(\mathbf{k} - \rho\mathbf{p}_F)) - \chi_\rho(\gamma^{-h+1}(\mathbf{k} - \rho\mathbf{p}_F))$ nonvanishing in $\gamma^{h-1} \leqslant |\mathbf{k} - \rho\mathbf{p}_F| \leqslant \gamma^{h+1}$; therefore, setting $\widehat{g}^{(h)}_\rho(\mathbf{k}) = f_{h,\rho}(\mathbf{k})\widehat{g}(\mathbf{k})$ we get $\widehat{g}_\rho(\mathbf{k}) = \sum_{h=-\infty}^0 \widehat{g}^{(h)}_\rho(\mathbf{k})$, $\rho = \pm$; note that $\widehat{g}^{(h)}_\rho(\mathbf{k})$ is the "single scale propagator," corresponding to fermions with momenta measured from $\mathbf{p}_F$ which are $O(\gamma^h)$ ($\gamma > 1$ and $h < 0$). This decomposition of the propagator corresponds to a decomposition of the Grasmann variables $\psi^{(\leqslant 0)}_\rho = \sum_{h=-\infty}^0 \psi^{(h)}_\rho$, where $\psi^{(h)}_\rho$ has propagator $g^{(h)}_\rho$. In the following we distinguish between the momentum $\mathbf{k}$ and the momentum measured from the Fermi points, $\mathbf{q} = \mathbf{k} - \rho\mathbf{p}_F$; in the support of f_h we have that $\mathbf{q} \sim \gamma^h$.

After the integration of $\psi^{(1)}, \psi^{(0)}, \dots, \psi^{(h+1)}$ the generating function has the form

$$e^{W(\phi)} = \int P(d\psi^{(\leqslant h)})e^{\mathcal{V}^{(h)}(\psi^{(\leqslant h)},\phi)}, \tag{17}$$

where $P(d\psi^{(\leqslant h)})$ has propagator

$$\widehat{g}^{(\leqslant h)}_\pm(\mathbf{q}) = \frac{1}{Z_h}\chi_{h,\rho}(\mathbf{q})\begin{pmatrix} -iq_0 \pm v_{3,h}q_3 & v_{1,h}q_1 - iv_{2,h}q_2 \\ v_{1,h}q_1 + iv_{2,h}q_2 & -iq_0 \mp v_{3,h}q_3 \end{pmatrix}^{-1} \tag{18}$$

with $\chi_{h,\rho}(\mathbf{q}) = \chi_\rho(\gamma^{-h}\mathbf{q})$ and $\mathcal{V}^{(h)}(\psi, 0) =$

$$\sum_{m,n,\underline{\rho}} \int d\mathbf{q}_1 \dots d\mathbf{q}_m W^{(h)}_{n,m}(\underline{\mathbf{q}})\psi^{\varepsilon_1(\leqslant h)}_{\rho_1,\mathbf{q}_1} \dots \psi^{\varepsilon_m(\leqslant h)}_{\rho_m,\mathbf{q}_m}\delta_{n,m}(\underline{\mathbf{q}}), \tag{19}$$

where $\varepsilon = \pm$, $\delta_{n,m}(\underline{\mathbf{q}})$ is $L\beta$ times a periodic Kronecker delta nonvanishing for $\sum_{i=1}^m \varepsilon_i q_{0,i} = 0$ and $p_F = (0, 0, p_{F,3})$

$$\sum_{i=1}^m \varepsilon_i q_i = -\sum_{i=1}^m \varepsilon_i\rho_i p_F + 2\pi\omega_n + 2l\pi, \tag{20}$$

with $l = (l_1, l_2, l_3)$ and $\omega_n = (\omega_1 n_1, \omega_2 n_2, \omega_3 n_3)$. $\mathcal{V}^{(h)}(\psi, \phi)$ has a similar expression with some ψ field replaced by an external field ϕ.

Our aim is to prove the following bound:

$$\left|W^{(h)}_{n,m}(\underline{\mathbf{q}})\right| \leqslant C[\max(|\varepsilon|, |\lambda|)]^{m-1}\gamma^{Dh}e^{-\frac{\xi}{2}|n|}, \tag{21}$$

with $D = 4 - \frac{3}{2}m$ the scaling dimension, from which (15) easily follows. The main point of the above statement is that, up to the dimensional factor γ^{Dh}, the kernels $W^{(h)}_{n,m}$ are small for ε, λ small uniformly in $-h$. The validity of (21) is nontrivial at all for the presence of infinitely many relevant terms. We recall that, in the usual terminology, the relevant terms are the ones with $D > 0$, actually the bilinear terms $m = 2$ and any n in (19). This could produce an infinite number of running coupling constants which could possibly produce an instability. However, we can distinguish among the relevant bilinear terms in (19) the ones such that the left-hand side (l.h.s.) of (20) is vanishing, which we call resonant, from the others, which we call nonresonant. The resonant terms with $m = 2$ are possible only for $n = 0$ and $\rho_1 = \rho_2$; the case $\rho_1 = -\rho_2$ would be possible if $p_{F,3} = n_3\pi\omega_3$, a case excluded by (13). In the RG approach one needs to renormalize the relevant terms introducing an operator $\mathcal{R} = 1 - \mathcal{L}$. The key point is that there is no need to renormalize the nonresonant terms, even if they are dimensionally relevant, as their size is controlled by the Diophantine condition. We define therefore

$$\mathcal{L}W^{(h)}_{0,2}(\mathbf{q}) = W^{(h)}_{0,2}(0) + \mathbf{q}\partial W^{(h)}_{0,2}(0) \tag{22}$$

and $\mathcal{L}W^{(h)}_{n,m}(\underline{\mathbf{q}}) = 0$ otherwise; this says that the renormalization $\mathcal{R} = 1 - \mathcal{L}$ makes the scaling dimension, which is originally 1, negative. Note also that $\mathcal{R}$ is trivial on the relevant nonresonant terms. By symmetry, no new relevant and marginal terms are produced by the $\mathcal{L}$ operation. Indeed the nondiagonal part of $W^{(h)}_{0,2}$ is the sum of terms with an odd number of nondiagonal propagators, hence it is vanishing while the diagonal part has opposite signs. In addition, the derivative with respect to 0,3 of the terms contributing to the nondiagonal part is zero, as they contain an odd number of nondiagonal propagators, and the derivative with respect to 1,2 of the terms contributing to the diagonal part is zero, as it contain an even number of nondiagonal propagators. We can write therefore

$$\mathcal{V}^h = \mathcal{L}\mathcal{V}^h + \mathcal{R}\mathcal{V}^h \tag{23}$$

with $\mathcal{R} = 1 - \mathcal{L}$; moreover, we can insert the marginal terms in $\mathcal{L}\mathcal{V}^h$ in the fermionic integrations so that a renormalization of the velocities v_h and of the wave-function renormalization Z_h is produced. In conclusion, we can rewrite (17) as

$$\int \widetilde{P}(d\psi^{(\leqslant h)})e^{\gamma^h v_h F^{(h)} + \mathcal{R}\mathcal{V}^{(h)}(\psi^{(\leqslant h)},\phi)} \tag{24}$$

with $F^{(h)} = \int d\mathbf{x}(\psi^+_{\mathbf{x},1}\psi^-_{\mathbf{x},1} - \psi^+_{\mathbf{x},2}\psi^-_{\mathbf{x},2})$ and $\widetilde{P}(d\psi^{(\leqslant h)})$ has a propagator similar to (18) with $v_{i,h-1}$ replacing $v_{i,h}$ and Z_{h-1} replacing Z_h, with $v_{h-1,i} = v_{h,i} + \partial_i W^h_{0,2}$, $i = 1, 2, 3$, and $Z_{h-1} = Z_h + \partial_0 W^h_{0,2}$.

We can use the additional property of Gaussian Grassmann integrations $P(d\psi^{(\leqslant h)}) = P(d\psi^{(\leqslant h-1)})P(d\psi^{(h)})$, with $P(d\psi^{(h)})$ with propagator $g^{(h)}$ and $P(d\psi^{(\leqslant h-1)})$ with propagator $g^{(\leqslant h-1)}$ ("high- and low-energy fields"), where $g^{(h)}$ coincides with $g^{(\leqslant h-1)}$ (18) with $f_{h,\rho}$ replacing $\chi_{h-1,\rho}$.

We integrate the single scale variable $\psi^{(h)}$

$$e^{\mathcal{V}^{(h-1)}(\psi^{(\leqslant h-1)},\phi)} = \int P\big(d\psi^{(h)}\big)e^{\gamma^h v_h F^{(h)} + \mathcal{R}\mathcal{V}^{(h)}(\psi^{(\leqslant h)},\phi)}, \tag{25}$$

obtaining for $\mathcal{V}^{(h-1)}$ an expression similar to (17) with $h - 1$ replacing h, and the procedure can be iterated. By definition

$$\mathcal{V}^{(h-1)} = \sum_{n=1}^\infty \frac{1}{n!}\mathcal{E}^T_n(\gamma^h v_h F^{(h)} + \mathcal{R}\mathcal{V}^{(h)}; n), \tag{26}$$

where $\mathcal{E}^T_n$ are truncated expectations (also called cumulants), expressed in terms of connected Feynman graphs. The procedure can be therefore iterated.

The result of the above RG analysis is an expansion of $W^{(h)}_{n,m}(\underline{\mathbf{q}})$ in powers of $\lambda, \varepsilon, v_h$ with coefficients expressed in terms of the Feynman graphs defined, as usual, contracting bilinear ε or v_h vertices or quartic λ vertices in oriented lines ℓ, so that to each line ℓ is associated with a propagator $\widehat{g}^{(h_\ell)}_{\rho_\ell}(\mathbf{q}_\ell)$ ($\mathbf{q} = \mathbf{k} + \rho\mathbf{p}_F$); the difference of momenta $\mathbf{k}$ coming

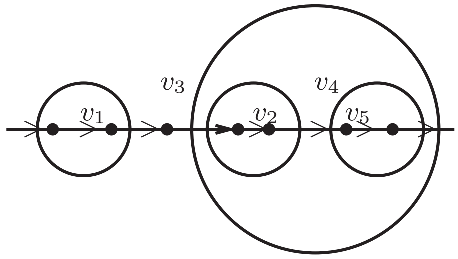

FIG. 1. The graph with value (27).

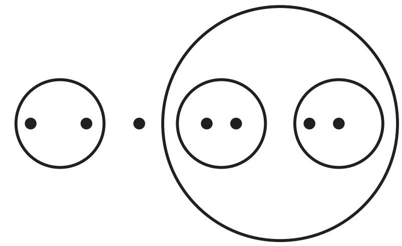

FIG. 2. The cluster structure of the graph in Fig. 1.

in and out an $\varepsilon\widehat{\phi}_n$ vertex is $2\pi\omega_n = 2\pi(\omega_1 n_1, \omega_2 n_2, \omega_3 n_3)$, while the sum of momenta coming in and out a λ-vertex or ν vertex is zero. Note that the propagators carry now a scale index h_ℓ such that $h_\ell < h$. Given a graph we can consider a maximally connected subset of propagators with scale h_ℓ such that $h_\ell \geqslant h_v$ and there is at least a scale h_v, and such that the external lines have scale $< h_v$; the points connected by such lines define a cluster labeled by v, and to each Feynman diagram is associated a hierarchy of clusters. We can reorganize the expansion in the opposite way; one can choose a hierarchy of clusters and find the diagrams associated with it. The momenta $\mathbf{q}$ of the propagators in a cluster measured from the Weyl points are larger than the ones of the external lines; this induces a decomposition in subgraphs avoiding overlapping divergences, see, e.g., [39,40] for more details. Note finally that the $\mathcal{R} = 1 - \mathcal{L}$ operation acts on the clusters with two external lines and $n = 0$, that is on the resonant clusters; this means that $W_{0,2}^{(h_v)}(\mathbf{q})$ is replaced by $W_{0,2}^{h_v}(\mathbf{q}) - W_{0,2}^{(h)}(0) - \mathbf{q}\partial W_{0,2}^{(h)}(0)$.

An example of the Feynman graph defined according to the above rules is in Fig. 1. The graph is a contribution of order 7 in ε; the vertices are represented by dots and the clusters, labeled by v_i, by circles enclosing the subsets of vertices. The clusters $v_1, \dots, v_6$ are such that, by definition, $h_{v_3} < h_{v_1}$, $h_{v_3} < h_{v_4}$, $h_{v_4} < h_{v_2}$, $h_{v_4} < h_{v_5}$. We will use the notation that v' is the first cluster enclosing v, that is, $v_2' = v_4$ and so on, and $h_{v'} < h_v$. As only ε vertices are present, the graph is linear with value

$$\varepsilon^7 \int d\mathbf{q}\, \psi^{+(\leqslant h)}_{\mathbf{q},\rho} \psi^{-(\leqslant h)}_{\mathbf{q}+2\pi\omega_n,\rho'} \prod_{i=1}^{7} \widehat{\phi}_{n_i}$$
$$\times \widehat{g}^{h_{v_1}}_{\rho_1}(\mathbf{q}_1) \widehat{g}^{h_{v_3}}_{\rho_2}(\mathbf{q}_2) \widehat{g}^{h_{v_3}}_{\rho_3}(\mathbf{q}_3) \widehat{g}^{h_{v_2}}_{\rho_4}(\mathbf{q}_4) \widehat{g}^{h_{v_4}}_{\rho_5}(\mathbf{q}_5) \widehat{g}^{h_{v_5}}_{\rho_6}(\mathbf{q}_6), \quad (27)$$

where $\mathbf{k} = \mathbf{q} + \rho\mathbf{p}_F$, $\mathbf{k}_1 = \mathbf{k} + 2\pi\omega_{n_1}$ mod. 2π, $\mathbf{k}_2 = \mathbf{k} + 2\pi\omega_{n_1+n_2}$, $\mathbf{k}_3 = \mathbf{k} + 2\pi\omega_{n_1+n_2+n_3}$ and so on; finally, $\sum_i n_i = n$ and $\omega_n = (\omega_1 n_1, \omega_2 n_2, \omega_3 n_3)$. The bounds leading to (21) are largely independent from the details of the graphs; what really matters is the structure of clusters with scales and the momentum conservation rule. This fact is also essential in proving the convergence of the series expansion: we bound a class of graphs with the same cluster structure, and in this way we avoid the dangerous combinatorial $n!$ which would be obtained by bounding each graph separately. In Fig. 2 it is represented in the cluster structure and the vertices corresponding to the graph in Fig. 1. In general, graphs obtained by contractions with λ vertices are not linear; in Fig. 3. it represents a graph contributing to $W_{n,4}^{(h)}$ of order $\lambda^3\varepsilon^3$ and the associated cluster structure; the propagator in the circle, representing a cluster, has a scale lower than the four propagators external to it.

We want now to bound the Feynman graph associated to a hierarchy of clusters. We call v' the minimal cluster containing v so that $h_v - h_{v'} > 0$, and S_v is the number of clusters or vertices contained in v and not in any smaller cluster. We also call n_v^e the number of lines external to a cluster v and m_v^4, m_v^2 the vertices in a cluster v of kind λ or ε, ν. Using that the propagator $g^{(h)}(\mathbf{x})$ is bounded by γ^{3h} and the integral of the propagator over coordinates by $\gamma^{-4h}\gamma^{3h}$, a graph of order s is bounded by $C^s[\max(|\varepsilon|, |\lambda|, |\nu_h|)^s]$ times

$$\prod_v \gamma^{-4h_v(S_v-1)} \prod_v \gamma^{3h_v n_v} \prod_v \gamma^{z_v(h_{v'}-h_v)}, \quad (28)$$

where n_v is the number of propagators in the cluster v but not in any smaller one, and $z_v = 2$ where v is a resonant cluster with $n_v^e = 2$ and zero otherwise; the last term in the above expression is produced by the renormalization $\mathcal{R}$. Note that the same bound is valid for the sum over all Feynman graphs with a fixed hierarchy of clusters from cancellations due to the Pauli principle, see [40].

By using the relations

$$\sum_v (h_v - h)(S_v - 1) = \sum_v (h_v - h_{v'})\big(m_v^4 + m_v^2 - 1\big),$$
$$\sum_v (h_v - h) n_v = \sum_v (h_v - h_{v'})\big(2m_v^4 + m_v^2 - n_v^e/2\big),$$

one gets

$$\gamma^{Dh} \prod_v \gamma^{(h_v - h_{v'})(D_v - z_v)} \prod_v \gamma^{2h_v \bar{m}_v^4} \prod_v \gamma^{-h_v \bar{m}_v^2} \left[\prod_i e^{-\xi|n_i|}\right], \quad (29)$$

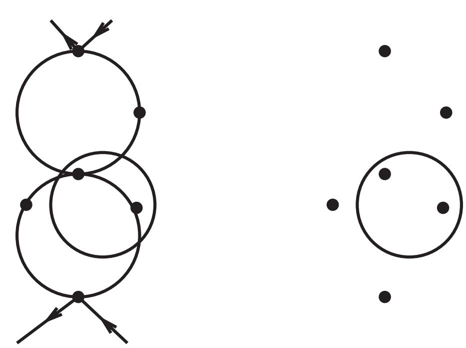

FIG. 3. An example of graph with λ and ε vertices and the associated cluster structure.

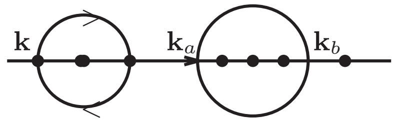

FIG. 4. A cluster with the gain (33).

where $\bar{m}_v^4$ is the number of vertices λ contained in v and not in any smaller cluster, $\bar{m}_v^2$ is the number of vertices ε contained in v and not in any smaller cluster [such a factor is absent for ν_h vertices as canceled by the γ^h in (25)], s is the order, $D_v = 4 - 3n_v^e/2$; the last term comes from the factors $\widehat{\phi}_n$.

One needs to sum over all the choices of scales $\{h\}$. By looking to (29) we see indeed that if for all v one has $D_v - z_v < 0$ then one can sum over all the choices of scales, that is, $\sum_{\{h\}} \prod_v \gamma^{(h_v - h_{v'})(D_v - z_v)}$ is bounded by C^s (remember that $h_v - h_{v'} > 0$). There are, however, clusters with $D_v - z_v = 1$, actually the nonresonant clusters with $n_v^e = 2$ which are dimensionally relevant, and this produces a divergent bound γ^{-hs}. Such a divergence may suggests that the Weyl semimetallic behavior is unstable.

We need, however, to take into account the number-theoretical properties of the frequencies. Let us consider a cluster with two external lines (see, e.g., Fig. 4), associated to propagators with momenta $\mathbf{k}_a, \mathbf{k}_b$. If q_a, q_b are the momenta measured from the Weyl points, $k = q \pm p_F$, external to a cluster v, one has $|q_a| \leqslant \gamma^{h_{v'}}$, $|q_b| \leqslant \gamma^{h_{v'}}$ for the compact support properties of the propagator. We call $N = (N_1, N_2, N_3)$, $N = \sum_i n_i$ where n_i is the momentum associated with each ε vertex in the cluster; therefore, $k_a - k_b = 2\pi(N_1\omega_1, N_2\omega_2, N_3\omega_3)$ so that, if $|q|_T = \sqrt{|q_1|_T^2 + |q_2|_T^2 + |q_3|_T^2}$

$$2\gamma^{h_{v'}} \geqslant |q_a|_T + |q_b|_T \geqslant |q_a - q_b|_T, \tag{30}$$

where we used the triangular inequality on the torus. Now we use the Diophantine property (13), $\varepsilon = 0, \pm$

$$2\gamma^{h_{v'}} \geqslant \sqrt{|2\pi\omega_1 N_1|_T^2 + |2\pi\omega_2 N_2|_T^2 + |2\pi\omega_3 N_3 + \varepsilon 2p_{F,3}|_T^2} \geqslant \frac{3C_0}{\bar{N}^\tau}, \tag{31}$$

so that, if $\bar{N} = \max(N_1, N_2, N_3)$ then

$$\bar{N} \geqslant C\gamma^{-h_{v'}/\tau}. \tag{32}$$

This inequality says that if the momenta external to a nonresonant cluster are very small, than the momentum transferred is very large. On the other hand, $N = \sum_i n_i$ and

$$\prod_i e^{-\xi|n_i|} \leqslant e^{-\xi\bar{N}} \leqslant e^{-C\gamma^{-h_{v'}/\tau}}, \tag{33}$$

as $\sum_i |n_i| \geqslant |\sum_i n_i| \geqslant \bar{N}$. An example is in Fig. 4; the first circle represents the propagators of the two λ vertices while the second circle represents a cluster v with scale h_v containing three ε vertices with momenta n_1, n_2, n_3; if k_a, k_b are the external momenta if $k_a - k_b = 2\pi\omega_N$ and $|q_a| \sim \gamma^{-h_{v'}}$, $|q_b| \sim \gamma^{-h_{v'}}$ then $|\bar{N}| \geqslant \gamma^{-h_{v'}/\tau}$, see (31), $N = n_1 + n_2 + n_3$. The small factor (33) is sufficient to compensate the factor $\gamma^{-h_{v'}}$ for dimensional reasons, that is the dimensionally relevant terms are indeed irrelevant.

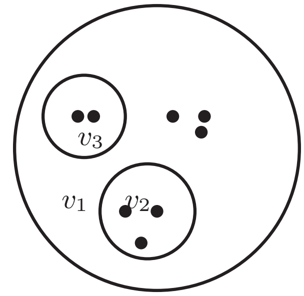

FIG. 5. An example of hierarchy clusters leading to (37).

In the above argument we have not taken into account that there is, in general, a sequence of clusters enclosed into clusters, as in Fig. 2, and not all the gain coming from the factors $e^{-\xi|n|}$ contained in a cluster can be consumed. To get a decay factor for each cluster we can write each factor $e^{-\xi|n|/2}$ associated to $\widehat{\phi}_n$ as

$$e^{-\xi|n|/2} = \prod_{h=-\infty}^{-1} e^{-\xi 2^h |n|/2}, \tag{34}$$

so that, if N_v is the sum of the n_i of the ε-vertices in v

$$\prod_i e^{-\xi|n_i|/2} \leqslant \prod_v e^{-\xi\bar{N}_v 2^{h_{v'}}}, \tag{35}$$

where the produce in the l.h.s. is over all the ε vertices. An example of the above argument is in Fig. 5; if the vertices represent $\widehat{\phi}_{n_i}$, assuming 1,2,3 in v_2, 3,4 in v_3 and 4,5,6 in v_1, then $\prod_{i=1}^8 e^{-|n_i|}$ is bounded by $e^{-(|n_1|+|n_2|+|n_3|)2^{h_{v_2}}} e^{-(|n_4|+|n_5|)2^{h_{v_3}}} e^{-(|n_1|+\cdots|n_8|)2^{h_{v_1}}}$.

We get, using (32),

$$e^{-\xi\bar{N}_v 2^{h_{v'}}} \leqslant e^{-\xi 2^{h_{v'}} \gamma^{-h_{v'}/\tau}}. \tag{36}$$

If we choose $\gamma^{1/\tau} = 4$ then $e^{-\xi 2^{-h_{v'}}} \leqslant (N/e\xi)^N 2^{N 2 h_{v'}}$ by using $e^{-\alpha x} x^N \leqslant (\frac{N}{e\alpha})^N$. Therefore, choosing N so that $2^N = \gamma$ ($N = 2\tau$)

$$\left[\prod_i e^{-\xi|\bar{n}_i|/2}\right] \leqslant C^s \prod_v \gamma^{h_v 2 S_v^{\rm NR}}, \tag{37}$$

where $S_v^{\rm NR}$ is the number of nonresonant clusters with two external lines or ε vertices in v and not in any smaller cluster and s is the order.

Using that

$$\prod_v^* \gamma^{-2(h_{v'} - h_v)} \prod_v \gamma^{-h_v 2\bar{m}_v^2} \leqslant \prod_v \gamma^{-h_v 2 S_v^{\rm NR}}, \tag{38}$$

where the first product is over the nonresonant relevant v we get that (29) is replaced by, using (37),

$$\gamma^{Dh} \prod_v \gamma^{(h_v - h_{v'})(D_v - \bar{z}_v)} \prod_v \gamma^{h_v \bar{m}_v^4} \left[\prod_i e^{-\xi|\bar{n}_i|/2}\right] \tag{39}$$

with $\bar{z}_v = 2$ for $n_v^e = 2$ and zero otherwise. Note that the Diophantine equation has been used to derive (37), from which one can associate a gain γ^{2h_v} to each nonresonant relevant cluster, making their scaling dimension negative, that is, replacing z_v with $\bar{z}_v$ in (39). As $D_v - \bar{z}_v \leqslant -1$ we can sum over all the scale choices getting a bound $O[C^s \max(|\lambda|, |\varepsilon|, |\nu_h|)^s]$ from which the convergence of the series expansions follows, provided that ν is chosen so that ν_h vanishes as $h \to -\infty$.

Finally we note that the velocities verify a recursive relation $v_{h-1} = v_h + \beta_v^h$ and the wave-function renormalization verifies $Z_{h-1} = Z_h + \beta_z^h$; note that the Feynman graphs contributing to β_h have at least a λ vertex so that $\beta^h = O(\lambda\gamma^h)$ (the quartic terms are irrelevant) and $v_{-\infty} = v_0 + O(\lambda)$, $Z_{-\infty} = 1 + O(\lambda)$. One can therefore sum over all the scales h_v in (39) obtaining (21).

Note that the crucial role of the exponential decay of the harmonics $\widehat{\phi}_n$ to prove the irrelevance of the nonresonant terms, see (33). While such a condition is probably nonoptimal, some fast decay is likely to be necessary. Indeed this is what happens in the case of interacting one-dimensional fermions: nonresonant terms are irrelevant at weak coupling with some fast decay [29] while if the decay is slow $O(1/n)$ as in the Fibonacci potential, there is evidence that they are instead relevant and produce instability [41].

VII. CONCLUSION

We rigorously established the stability of the Weyl semimetallic phase in the presence of weak interaction and quasiperiodic disorder. Even if the infinitely many relevant terms produced by the disorder could possibly destabilize the semimetallic phase, this is avoided by subtle cancellations due to the number of theoretical properties. The physical properties appear to be determined by the interplay of relativistic quantum field theory with classical mechanics and KAM theory. There are no phase transitions for weak quasiperiodic disorder, where rare region effects are absent. If a similar rigorous RG analysis can be performed in the case of random disorder is a very interesting open question.

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