Zero-energy quasiparticles in an interacting nanowire containing a topological Josephson junction

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(Received 9 March 2020; revised 21 May 2020; accepted 2 July 2020; published 13 July 2020)

We study a Josephson junction in a Kitaev chain with particle-hole symmetric nearest-neighbor interactions. When the phase difference across the junction is π, we show analytically that the full spectrum is fourfold degenerate up to corrections that vanish exponentially in the system size. The Majorana bound states at the ends of the chain are known to survive interactions. Our result proves that the same is true for the zero-energy quasiparticle localized at the junction. We further study finite-size corrections numerically, and show how repulsive interactions lead to stronger end-to-end correlations than in a noninteracting system with the same bulk gap.

DOI: 10.1103/PhysRevB.102.024513

I. INTRODUCTION

In many superconducting systems, the fundamental degrees of freedom are noninteracting Bogoliubov quasiparticles. However, sometimes interactions cannot be ignored. For instance, when a nanowire is placed on top of a bulk superconductor, superconductivity is induced in the wire via the proximity effect, but the superconductor does not necessarily screen the short-range part of Coulomb interactions intrinsic to the wire [1–5]. Due to the 1D nature of the wire, it may not be possible to treat these interactions by the same mean-field techniques used to account for Cooper pairing in the bulk superconductor. In this case, the occupation numbers of single-quasiparticle orbitals are not conserved, due to scattering between quasiparticles, and we are dealing with a nontrivial interacting many-body system.

For the past decade, systems consisting of a nanowire with strong spin-orbit coupling on top of an *s*-wave superconductor have been studied intensively [6]. For a suitable choice of system parameters, the wire effectively becomes a spinless 1D *p*-wave superconductor. The system hosts a zero-energy quasiparticle orbital comprised of two spatially separated Majorana bound states localized at the ends of the wire [7–10]. Because filling this orbital costs no energy, each energy level of the many-body system is at least twofold degenerate. The ground-state degeneracy is topologically protected [11–14] and may allow for fault-free processing of quantum information in systems consisting of multiple chains.

However, if the system is not in the ground state, i.e., if it contains finite-energy quasiparticles, and scattering between quasiparticles cannot be neglected, the occupation number of the zero-energy quasiparticle orbital can change, thus destroying the information being processed. This is referred to as quasiparticle poisoning [15,16]. It is important to ask whether electron-electron interactions in the wire are a significant source of quasiparticle poisoning. Fendley [17] analized a Kitaev chain, [7] a minimal model for a spinless *p*-wave wire, and constructed Majorana end-state operators for the interacting system at the particle-hole symmetric point. The existence of these operators implies that the full spectrum of the interacting system is still twofold degenerate. Associated with the degeneracy is a dressed zero-energy quasiparticle mode whose occupation number remains conserved, also when the system is not in the ground-state manifold. In particular, it cannot be changed by electron-electron scattering when a finite energy quasiparticle comes near an edge. In a system containing multiple chains, quantum information can be stored in these zero-energy quasiparticle modes, and electron-electron interactions will not lead to quasiparticle poisoning. (Of course, we still expect other mechanisms, such as the electron-phonon interaction, to induce poisoning.)

When the superconducting phase difference across a Josephson junction in a *noninteracting* spinless *p*-wave wire equals π, there is a single Andreev bound state localized around the junction [18]. In this case, the bound-state orbital is not associated with spatially separated Majorana operators. Nonetheless, it has zero energy and constitutes an essential ingredient for the 4π Josephson effect that may serve to detect Majorana bound states [15,19,20]. Its presence in the noninteracting system produces a further twofold degeneracy, resulting in an overall fourfold degeneracy of the spectrum. Here we ask whether this degeneracy also survives particle-hole symmetric electron-electron interactions. In other words, do interactions cause quasiparticle poisoning of the 4π Josephson effect in a Kitaev chain? We analytically prove that fourfold degeneracy of the full spectrum survives, implying no quasiparticle poisoning by electron-electron interactions, until strong interactions induce a quantum phase transition.

There is, however, another way in which interactions do degrade the 4π-Josephson signal, namely, by enhancing finite-size effects. A perfect 4π signal requires the quasiparticle at the junction to have precisely zero energy and an infinite lifetime, a condition that is only met in the limit of an infinite wire. In a finite system, there are corrections that manifest as a lifting of the fourfold degeneracy. These

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decay exponentially as a function of the system size, but may still be non-negligible in realistic systems [21,22]. We investigate finite-size effects numerically. Not surprisingly, repulsive electron-electron interactions reduce the bulk gap and enhance finite size corrections. However, if we compare two wires with the same gap, one with repulsive interactions and the other without, we still find a significantly slower decay of finite-size corrections in the interacting case. We conclude that repulsive interactions frustrate superconductivity in a way that cannot be accounted for by a noninteracting model with a reduced gap.

II. MODEL

The model we study is based on a Kitaev chain with zero chemical potential, and describes a 1D lattice hosting spinless fermions with hopping $W > 0$ and superconducting pairing $|\Delta|$ between nearest-neighbor sites [7]. We add short-range electron-electron interactions [2,4,5,23] of strength U and readjust the external potential to maintain particle-hole symmetry in the presence of interactions. We model a Josephson junction [8,19] by setting superconducting pairing to zero between a pair of neighboring sites. For sites to the left of the junction, pairing terms are of the form $-\Delta(aa_< + a_>a + \text{H.c.})/2$, where a is the fermion annihilation operator associated with a given site while $a_<$ and $a_>$ are, respectively, associated with its left and right neighbors. We take $0 < \Delta < W$. For a π superconducting phase difference across the junction, pairing terms to the right of the junction are of the form $-\Delta(a_<a + aa_> + \text{H.c.})/2$. We introduce an index $\alpha \in \{1, 2\}$ to distinguish sites on the left and right of the junction, and an index $j = 1, \dots, N$ to label sites by their distance from the junction. With these conventions, the Hamiltonian that we consider is

$$H = H_0 + H_{\text{int}},\ H_0 = \sum_{\alpha=1}^{2} H_\alpha + H_{\text{J}}, \tag{1}$$

where

$$\begin{aligned} H_\alpha = &-\frac{W}{2}\sum_{j=1}^{N-1}(a_{\alpha,j}^\dagger a_{\alpha,j+1} + a_{\alpha,j+1}^\dagger a_{\alpha,j}) \\ &-\frac{\Delta}{2}\sum_{j=1}^{N-1}(a_{\alpha,j} a_{\alpha,j+1} + a_{\alpha,j+1}^\dagger a_{\alpha,j}^\dagger) \end{aligned} \tag{2}$$

describes the chain to the left ($\alpha = 1$) and right ($\alpha = 2$) of the Josephson junction. The coupling between the two sites comprising the junction is given by

$$H_{\text{J}} = -\frac{W}{2}(a_{1,1}^\dagger a_{2,1} + a_{2,1}^\dagger a_{1,1}). \tag{3}$$

The interaction term reads

$$\begin{aligned} H_{\text{int}} = &U\sum_{\alpha=1}^{2}\sum_{j=1}^{N-1}\left(n_{\alpha,j} - \frac{1}{2}\right)\left(n_{\alpha,j+1} - \frac{1}{2}\right) \\ &+ U\left(n_{1,1} - \frac{1}{2}\right)\left(n_{2,1} - \frac{1}{2}\right), \end{aligned} \tag{4}$$

where $n_{\alpha,j} = a_{\alpha,j}^\dagger a_{\alpha,j}$. The following should be noted. Our proof relies on particle-hole conjugation symmetry together with properties of the system within a finite distance from the ends, N being assumed sufficiently larger than this distance. Apart from symmetry constraints, the proof therefore does not depend on the details near the junction. For instance, it does not matter whether the junction is atomically sharp, whether it sits exactly in the middle of the chain, or how the junction affects the values of W, U, and Δ in its vicinity. We adopt a minimal model here purely so we may later numerically calculate finite-size corrections to our analytical results.

Important for our purposes is that H possesses a particle-hole symmetry, i.e., $PHP^\dagger = H$ where the unitary operator P is defined as

$$\begin{aligned} P = &\gamma_{2,N}^{\text{Re}}\gamma_{1,N}^{\text{Re}}\gamma_{2,N-1}^{\text{Im}}\gamma_{1,N-1}^{\text{Im}} \times \dots \\ &\dots \times \begin{cases} \gamma_{2,2}^{\text{Re}}\gamma_{1,2}^{\text{Re}}\gamma_{2,1}^{\text{Im}}\gamma_{1,1}^{\text{Im}} & \text{if } N \text{ is even,} \\ \gamma_{2,2}^{\text{Im}}\gamma_{1,2}^{\text{Im}}\gamma_{2,1}^{\text{Re}}\gamma_{1,1}^{\text{Re}} & \text{if } N \text{ is odd,} \end{cases} \end{aligned} \tag{5}$$

with

$$\gamma_{\alpha,j}^{\text{Re}} = a_{\alpha,j} + a_{\alpha,j}^\dagger,\ \gamma_{\alpha,j}^{\text{Im}} = i(a_{\alpha,j} - a_{\alpha,j}^\dagger), \tag{6}$$

so

$$Pa_{\alpha,j}P^\dagger = (-)^{N+\alpha-j-1}a_{\alpha,j}^\dagger. \tag{7}$$

Furthermore, we rely on the fact that while H does not conserve fermions, it does conserve fermion parity $\Pi = \exp i\pi\sum_{\alpha,j} a_{\alpha,j}^\dagger a_{\alpha,j}$, with $+1 =$ even and $-1 =$ odd. We note that P also preserves fermion parity.

III. PROOF OF FOURFOLD DEGENERACY

Fourfold degeneracy will follow if an operator c_1 exists such that

$$\Pi c_1 \Pi^\dagger = -c_1, \tag{8}$$

$$\{c_1, c_1^\dagger\} = 1,\ c_1^2 = [H, c_1] = 0, \tag{9}$$

$$Pc_1P^\dagger = c_1^\dagger. \tag{10}$$

The first condition Eq. (8) implies that c_1 flips the a-fermion parity. The second set of conditions Eq. (9) implies that c_1 can be viewed as a fermionic annihilation operator for a zero-energy quasiparticle mode with occupation number $c_1^\dagger c_1$. The spectrum of H is thus twofold degenerate: for every eigenstate in which the zero-energy mode is empty, there is an eigenstate with the same energy in which the zero-energy mode is filled, and vice versa. The two states have opposite fermion parity. Now consider one such pair $|E, 0, \pm\rangle$ and $|E, 1, \mp\rangle \equiv c_1^\dagger|E, 0, \pm\rangle$. Here the first quantum number E is energy, the second quantum number (0 or 1) is the occupation number of the zero-energy mode, and the third $\pm$ is the fermion parity. The third condition Eq. (10) implies that there are two more states orthogonal to the above two that have the same energy. The first is $|E, 1, \pm\rangle \equiv P|E, 0, \pm\rangle$. Because the Hamiltonian has particle-hole symmetry, it is an eigenstate of H with energy E. It has parity $\pm$ and hence is orthogonal to $|E, 1, \mp\rangle$. The occupation number of the zero-energy mode is one because $c_1^\dagger c_1 P|E, 0, \pm\rangle =$

$Pc_1P^\dagger Pc_1^\dagger P^\dagger P|E,0,\pm\rangle = Pc_1c_1^\dagger|E,0,\pm\rangle = P|E,0,\pm\rangle$. Thus it is orthogonal to $|E,0,\pm\rangle$. The fourth state is straightforwardly seen to be $|E,0,\mp\rangle = c_1P|E,0,\pm\rangle$.

In the noninteracting system ($U=0$), the operator c_1 is built from Majorana end-state operators that are zero-energy Bogoliubov quasiparticles (linear combinations of $a_{\alpha,j}$ and $a_{\alpha,j}^\dagger$). Explicitly,

$$\gamma_\alpha^{\rm end} = A_{\rm end}\sum_{n=0}^{\lfloor\frac{N}{2}\rfloor}\left(-\frac{W-\Delta}{W+\Delta}\right)^n\gamma_{\alpha,N-2n}^{\rm Re}, \tag{11}$$

with $\alpha\in\{1,2\}$. $A_{\rm end}$ is a normalization constant that ensures $(\gamma_\alpha^{\rm end})^2=1$. We see that the Majorana operators are exponentially localized, with decay constant

$$\nu = \ln\sqrt{\frac{W+\Delta}{W-\Delta}}. \tag{12}$$

It is straightforward to check that $\{\gamma_\alpha^{\rm end},\gamma_\alpha^{\rm end}\}=2\delta_{\alpha,\beta}$ and

$$[H_0,\gamma_\alpha^{\rm end}]=0, \tag{13}$$

up to finite-size corrections that vanish like $[(W-\Delta)/(W+\Delta)]^N$. We disregard these and similar corrections that vanish exponentially as a function of system size throughout our analytical work. In view of Eq. (13), we can define a fermion annihilation operator,

$$c_1 = \gamma_1^{\rm end} + i\gamma_2^{\rm end}, \tag{14}$$

that satisfies conditions Eqs. (8) and (9). From the particle-hole conjugation relation Eq. (7) for a fermions, it then follows that

$$P\gamma_\alpha^{\rm end}P^\dagger = (-)^{\alpha-1}\gamma_\alpha^{\rm end}, \tag{15}$$

so Eq. (10) is indeed satisfied. Thus we have rederived the known result that the noninteracting system has fourfold degeneracy, up to corrections that decay exponentially as a function of system size [18]. It was not necessary for our proof to find the Andreev bound state localized at the junction explicitly. It is, however, straightforward to do so by solving the Bogoliubov de Gennes equation associated with H_0. One finds $c_2=(\gamma_+^{\rm J}+i\gamma_-^{\rm J})/2$, where

$$\gamma_\pm^{\rm J} = A_{\rm J}\sum_{n=1}^N e^{-\nu(n-1)}\left(\sin\frac{\pi n}{2}\pm\cosh\nu\cos\frac{\pi n}{2}\right)\times\frac{1}{\sqrt{2}}\left(\gamma_{1,n}^{\rm Im}\pm\gamma_{2,n}^{\rm Im}\right). \tag{16}$$

The normalization constant $A_{\rm J}$ is chosen such that $(\gamma_\pm^{\rm J})^2=1$ and ν is defined as in Eq. (12).

It is instructive to ask where the above proof breaks down if there is a zero rather than π phase difference across the junction. In that case, pairing on both sides of the junction is of the form $-\Delta(aa_< + a_>a + {\rm H.c.})/2$. In H_0, the sign of the pairing term to the right of the junction must then be reversed, i.e., $\Delta\to-\Delta$ for $\alpha=2$. This has the effect of replacing $\gamma_{2,N-2n}^{\rm Re}$ with $\gamma_{2,N-2n}^{\rm Im}$ in the expression Eq. (11) for $\gamma_2^{\rm end}$. As a result, the particle-hole conjugation properties of this right end-state operator changes from Eq. (15) to $P\gamma_2^{\rm end}P^\dagger=\gamma_2^{\rm end}$. This brings about the key modification. Now instead of Eq. (10), c_1 becomes invariant under particle-hole conjugation, i.e., $Pc_1P^\dagger=c_1$. As a consequence, $|E,0,\pm\rangle$ and $P|E,0,\pm\rangle$ have the same $c_1^\dagger c_1$ eigenvalue, and we can no longer conclude that they are orthogonal.

Now we consider the interacting case $H=H_0+H_{\rm int}$. It is convenient to define Wigner-Jordan spin operators:

$$\sigma_{\alpha,j}^x = \gamma_{\alpha,j}^{\rm Re}\prod_{k=j+1}^N\sigma_{\alpha,k}^z,\quad \sigma_{\alpha,j}^y = \gamma_{\alpha,j}^{\rm Im}\prod_{k=j+1}^N\sigma_{\alpha,k}^z, \tag{17}$$

$$\sigma_{\alpha,j}^z = 2a_{\alpha j}^\dagger a_{\alpha j}-1. \tag{18}$$

Operators with the same α index obey angular momentum commutation relations, i.e., $[\sigma_{\alpha,j}^\mu,\sigma_{\alpha,k}^\nu]=2i\delta_{jk}\varepsilon^{\mu\nu\rho}\sigma_{\alpha,k}^\rho$. In our construction, the commutation relations between operators with different α are less canonical. For $\mu\in\{x,y,z\}$, we still have $[\sigma_{1j}^\mu,\sigma_{2k}^z]=[\sigma_{1j}^z,\sigma_{2k}^\mu]=0$. However, for $l,m\in\{x,y\}$, we have $\{\sigma_{1j}^l,\sigma_{2k}^m\}=0$ instead of the more usual commuting operators. While it is obviously possible to alter the definition of the spin operators in such a way that all commutators are canonical, it turns out that the above construction is more useful for our purposes.

Away from the junction ($j>1$), the above Wigner Jordan transformation maps the Kitaev chain onto two spin-1/2 Heisenberg chains with nearest-neighbor exchange couplings:

$$J_x=(W+\Delta)/4,\ J_y=(W-\Delta)/4,\ J_z=U/4. \tag{19}$$

From the known properties of the XYZ model [24], it can then be inferred that either attractive or repulsive interactions close the superconducting gap when $|U|=W+\Delta$. For interactions stronger than this critical value, the system is again in a gapped phase. We collectively refer to the $|U|>W+\Delta$ regimes as the intrinsically gapped phase, because the gap is driven by interactions inside the wire $\propto U$ rather than by extrinsic pairing $\propto\Delta$. We refer to the $|U|<W+\Delta$ regime as the extrinsically gapped phase.

To prove fourfold degeneracy along the same lines as in the noninteracting case, we need to consider the Majorana end-state operators of the interacting system. They have in common with the noninteracting case that they are Hermitian, decay into the bulk, commute with the Hamiltonian up to terms that vanish exponentially in system size, and square to unity. To translate the explicit expressions that Fendley [17] derived into our notation, we define

$$\Psi_{\alpha,xyz} = A_{xyz}\sum_{b=1}^N\left(\frac{J_yJ_z}{J_x^2}\right)^{b-1}\sigma_{\alpha,N+1-b}^x\times\sum_{s=0}^{\lfloor\frac{b-1}{2}\rfloor}\sum_{(l_1,\dots,l_{2s})}\prod_{t=1}^s Q_{xyz}(\alpha,l_{2t-1},l_{2t}). \tag{20}$$

Here the sum $\sum_{(l_1,\dots,l_{2s})}$ is over all distinct sets of integers $l_1,\ \dots,\ l_{2s}$ such that

$$N+1-b<l_1<\dots<l_{2s}<N+1 \tag{21}$$

and

$$Q_{xyz}(\alpha, j, k) = \left(\frac{J_z}{J_x}\right)^{j-k}\left(1-\frac{J_x^2}{J_y^2}\right)\sigma^y_{\alpha,j}\sigma^y_{\alpha,k} + \left(\frac{J_y}{J_x}\right)^{j-k}\left(1-\frac{J_x^2}{J_z^2}\right)\sigma^z_{\alpha,j}\sigma^z_{\alpha,k}. \quad (22)$$

The normalization constant is

$$A_{xyz} = \sqrt{\left(1-\frac{J_y^2}{J_x^2}\right)\left(1-\frac{J_z^2}{J_x^2}\right)}. \quad (23)$$

In the extrinsically gapped phase ($|U| < W + \Delta$), the Majorana end-state operators are $\gamma^{\rm end}_{\alpha,\,\rm ext} = \Psi_{\alpha,xyz}$. In the intrinsically gapped phase ($|U| > W + \Delta$), the dominant exchange term switches from x to z, with the result that the end-state operators are obtained from those of the extrinsically gapped phase by exchanging the roles of x and z, i.e., $\gamma^{\rm end}_{\alpha,\,\rm int} = \Psi_{\alpha,zyx}$. A crucial difference between the end-state operators in the two phases is immediately apparent: Each term in $\gamma^{\rm end}_{\alpha,\,\rm ext}$ contains an unpaired $\sigma^x_{\alpha,j}$ operator and therefore flips fermion number parity, while the corresponding unpaired operator in the intrinsically gapped phase is $\sigma^z_{\alpha,j}$, which preserve fermion number parity. A related property is that while $\gamma^{\rm end}_{1,\,\rm ext}$ and $\gamma^{\rm end}_{2,\,\rm ext}$ anticommute, $\gamma^{\rm end}_{1,\,\rm int}$ and $\gamma^{\rm end}_{2,\,\rm int}$ *commute*. The operators also have different behavior under particle-hole conjugation. From the definitions of the spin operators, it follows that $P\sigma^x_{\alpha,j}P^\dagger = (-)^{\alpha-1}\sigma^x_{\alpha,j}$, $P\sigma^y_{\alpha,j}P^\dagger = (-)^{\alpha}\sigma^y_{\alpha,j}$, and $P\sigma^z_{\alpha,j}P^\dagger = -\sigma^z_{\alpha,j}$. As a result,

$$P\gamma^{\rm end}_{\alpha,\,\rm ext}P^\dagger = (-)^{\alpha-1}\gamma^{\rm end}_{\alpha,\,\rm ext}, \quad (24)$$

as in the noninteracting case while, on the other hand,

$$P\gamma^{\rm end}_{\alpha,\,\rm int}P^\dagger = -\gamma^{\rm end}_{\alpha,\,\rm int}. \quad (25)$$

In the extrinsically gapped phase, we have all the ingredients required to derive fourfold degeneracy. Since $\{\gamma^{\rm end}_{\alpha,\,\rm ext}, \gamma^{\rm end}_{\beta,\,\rm ext}\} = 2\delta_{\alpha,\beta}$, we can define a fermionic operator $c_{1,\rm ext} = \gamma^{\rm end}_{1,\rm ext} + i\gamma^{\rm end}_{2,\rm ext}$ in analogy to the noninteracting case Eq. (14). The annihilation operator $c_{1,\rm ext}$ is no longer a simple linear combination of $a_{\alpha,j}$ and $a^\dagger_{\alpha,j}$. The interaction dresses the Bogoliubov quasiparticle with particle-hole excitations. Nonetheless, $c_{1,\rm ext}$ meets all three conditions Eqs. (8)–(10). Thus, in the extrinsically gapped phase, the full spectrum is fourfold degenerate up to corrections that vanish exponentially as a function of the length of the wire. Adiabatic continuity with the noninteracting system implies that just as there is a dressed zero-energy fermionic quasiparticle divided between the ends of the system whose occupation number is conserved, there is one localized around the junction. Here we also note that when the phase difference across the junction is 0 rather than π, the proof of fourfold degeneracy breaks down in exactly the same way as in the noninteracting case. In the expression for the right end-state operator, the unpaired $\gamma^{\rm Re}_{2,j}$ operator is replaced by a $\gamma^{\rm Im}_{2,j}$, with the result that the dressed $c_{1,\rm ext}$ operator becomes invariant under particle-hole conjugation, rather than transforming into its conjugate, as is required for fourfold degeneracy.

In the intrinsically gapped phase, the above construction does not work: because $\gamma^{\rm end}_{1,\,\rm int}$ and $\gamma^{\rm end}_{2,\,\rm int}$ commute rather than anticommute, and furthermore preserve fermion number parity, we can no longer construct a fermionic operator in analogy to c_1 of the noninteracting case Eq. (14). It seems the best we can do is to diagonalize H, $\gamma^{\rm end}_{1,\,\rm int}$, and $\gamma^{\rm end}_{2,\,\rm int}$ simultaneously. The eigenvalues λ_1 and λ_2 of the latter two operators are ± 1. If $|E, \lambda_1, \lambda_2\rangle$ is a simultaneous eigenstate with energy E, then $P|E, \lambda_1, \lambda_2\rangle$ is a different simultaneous eigenstate, that, due to Eq. (25), has $\gamma^{\rm end}_{1,\,\rm int}$ and $\gamma^{\rm end}_{2,\,\rm int}$ eigenvalues $-\lambda_1$ and $-\lambda_2$. Thus we have only proved twofold degeneracy. In the limit $|U| \gg W$, Δ, the system becomes equivalent to a spin-1/2 Ising chain, in which case the degeneracy of the ground state as well as the most excited state is indeed only twofold. Below we report a numerical calculation confirming that in the intrinsically gapped phase, the degeneracy remains twofold as $|U|$ is reduced toward the critical point.

IV. FINITE-SIZE CORRECTIONS

The finite-size corrections that we have ignored throughout the above analysis cause lifting of degeneracies. While the corrections vanish exponentially in system size, there may be practical considerations limiting how long a real device can be made. For instance, the longer the wire, the more likely it is to include (rare) regions where the disorder potential is large enough to destroy topological order. Indeed, in Ref. [22], it was estimated that the maximum exponential suppression factor achievable in an InAs wire realization is of order 10^{-2}. This estimate did not take interactions into account. It is therefore worth asking how interactions affect finite-size corrections. To address this question, we have performed Density Matrix Renormalization Group (DMRG) calculations on a finite system. Our results are accurate to at least seven significant digits. We have calculated the ground state and the first few excited state energies as a function of U, for a system with $\Delta/W = 0.2$ and $N = 41$ sites on each side of the junction. (We confined our attention to a few low-lying energy levels because a separate and increasingly expensive numerical calculation is required for each higher excited state.)

In Fig. 1 we show the low-energy spectrum at a few representative interaction strengths. For attractive interactions $U < 0$, finite-size corrections are small and the predicted degeneracy of the infinite system's spectrum is clearly seen: above the critical point $U_{\rm c,-} = -(W + \Delta)/2$ (extrinsically gapped phase) there is fourfold degeneracy while below the critical point (intrinsically gapped phase), some levels, such as the ground state, are only twofold degenerate.

For repulsive interactions in the window $(0.3\,U_{\rm c,+}, 1.5\,U_{\rm c+})$, on the other hand, where $U_{\rm c+} = (W + \Delta)/2$ is the repulsive critical point, degeneracy is obscured by finite-size effects. (See main panel of Fig. 2.) Whereas on the attractive side, the gap closes linearly in $|U - U_{\rm c,-}|$, on the repulsive side the gap closes quadratically in $U - U_{\rm c,+}$ [25]. As a result, there is a larger region around the repulsive critical point compared to the attractive critical point where the bulk gap is small. It is not surprising that degeneracy lifting in this region should be more noticeable than in regions where the bulk gap is larger. However, closer inspection of the finite-size corrections reveals a nontrivial interaction effect, namely, significantly slower decay than

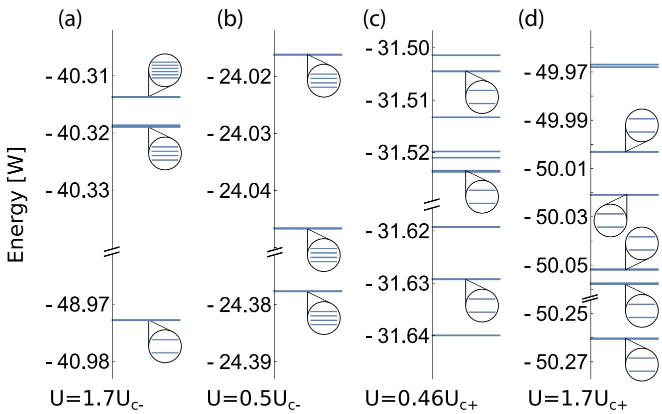

FIG. 1. Low-lying spectrum for a chain with 41 sites on each side of the Josephson junction and π-phase difference across the junction, for several values of the interaction strength U. We used $\Delta/W = 0.2$ throughout. Where near degeneracies cannot be resolved with the naked eye, the degeneracy is schematically indicated. In panels (a) and (d), the system is in the intrinsically gapped phase, while in panels (b) and (c) the system is in the extrinsically gapped phase. For attractive interactions [(a) and (b)], finite-size errors are too small to be resolved at the scale of the figure, and the degeneracy structure of the infinite system is clearly seen–twofold at worst in panel (a) and fourfold in panel (b). For the repulsive interactions in panel (c), finite-size errors are significant and the degeneracy structure of the infinite system's spectrum is obscured.

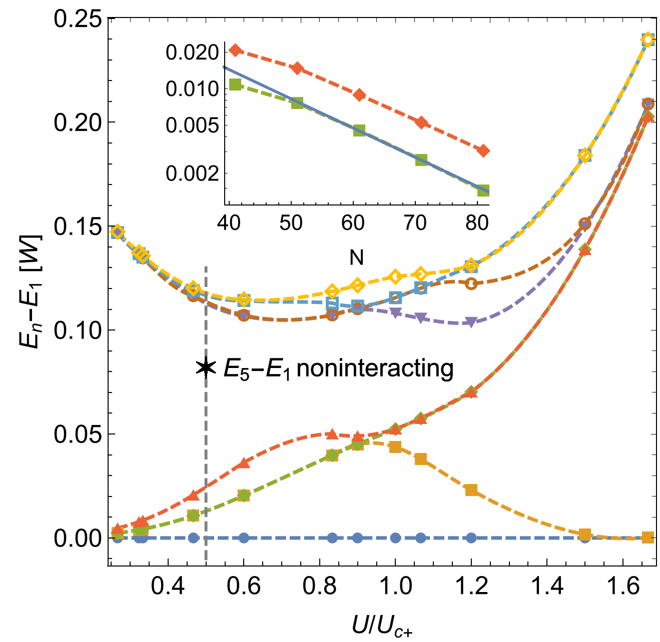

FIG. 2. Finite-size corrections in the case of repulsive interactions. Main panel: Excitation energies $E_n - E_1$, $n = 1, \ldots 8$, for a chain with 41 sites on each side of the Josephson junction and π-phase difference across the junction, as a function of $U \in (0.3\,U_{c+}, 1.7\,U_{c+})$. $\Delta/W = 0.2$ throughout. (Some of this data is also plotted in Fig. 1). Lines are there to guide the eye. The vertical line at $U = 0.5U_{c+}$ indicates the point for which we performed a finite-size scaling analysis of $E_n - E_1, n = 2, \ \ldots 4$. The black star indicates the excitation energy $E_5 - E_1$, i.e., from the ground state to the first bulk excitation, in a noninteracting system with $\Delta = 0.054\,W$. This is the value of Δ that produces the same decay of finite-size corrections as observed in the interacting system with $\Delta = 0.2\,W$ and $U = 0.5U_{c+}$. Inset: Finite-size scaling. The excitation energies $E_n - E_1$, $n = 2, \ \ldots 4$, as a function of N, for $U = 0.5U_{c+}$ and $\Delta/W = 0.2$. (The $n = 2$ and $n = 3$ data lie on top of each other.) The solid curve is the fit $E_2 - E_1 = 0.134 \times (0.946)^N$.

would be the case in a noninteracting system with the same excitation gap $E_5 - E_1$ and bare hopping amplitude W.

To see this, we fixed $U = U_{c+}/2$ (the vertical line in the main panel of Fig. 2) and calculated the low-lying excitation energies as a function of system size. As seen in the inset of Fig. 2, exponential decay $\sim(0.946)^N$ is observed. We then ask, What value of Δ would produce the same decay in a noninteracting system? From Eq. (12), we get $\Delta = 0.054\,W$. We also confirmed this numerically (not shown). In the infinite noninteracting system, this would be the excitation energy between the ground state E_1 and the first state above the gap E_5, but to compare with our finite interacting system, we need to take finite-size corrections in the noninteracting system into account as well. In a noninteracting system with 41 sites on each side of the junction and $\Delta = 0.054\,W$, the excitation energy from the ground state to the first state above the gap is $E_5 - E1 = 0.082\,W$. The same excitation energy in the interacting system is $0.116\,W$, which is significantly larger. The conclusion is that a given rate of decay of finite-size corrections corresponds to a larger value of the bulk gap in the system with repulsive interactions than in the noninteracting system with the same hopping amplitude. Because finite size corrections decay faster for a larger gap, this also means that for a given bulk gap, finite-size corrections decay more slowly in a chain with repulsive interactions than in the noninteracting chain.

The finite-size corrections that we analyzed result from the nonzero (but exponentially small) anticommutator between Majorana operators localized at the ends of the system and fermion operators at the junction. This means that finite-size corrections are present to the same extent that the outcome of measurements at the junction are correlated with the state of degrees of freedom at the ends of the system. Thus, finite-size corrections diagnose end-to-end (or more strictly end-to-junction) correlations and our results reveal that repulsive interactions enhance end-to-end correlations as compared to a noninteracting system with the same excitation gap and hopping amplitude. We have also analyzed the case of attractive interactions (not shown) and curiously here we find the opposite, i.e., finite-size corrections and end-to-end correlations are smaller in the interacting system than in a noninteracting system with the same excitation gap and hopping amplitude.

V. PARTICLE-HOLE SYMMETRY BREAKING

Our analytical results assume particle-hole symmetry and apply to half filling. However, in the noninteracting system, the fourfold degeneracy survives for any chemical potential in the band, i.e., $|\mu| < W$. To shed light on the extent of the robustness of fourfold degeneracy in the interacting system, we add a chemical potential term $-\mu \sum_{\alpha,j} n_{\alpha,j}$ to the Hamiltonian, and using DMRG, calculate the low-lying

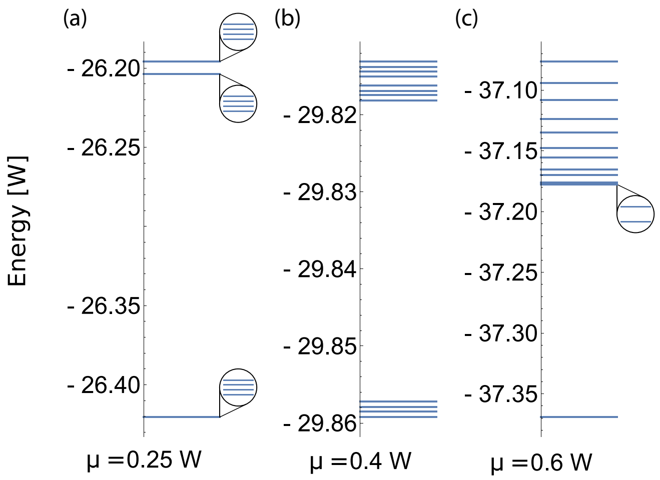

FIG. 3. Particle-hole symmetry breaking: Low-lying spectrum for a chain with 41 sites on each side of the Josephson junction and π-phase difference across the junction, for several values of the chemical potential μ. We used $\Delta/W = 0.2$ and $U = 0.5U_{c,-}$ throughout. Where near degeneracies cannot be resolved with the naked eye, the degeneracy is schematically indicated.

spectrum of a system with 41 sites on either side of the junction. We use $\Delta = 0.2\,W$ as before. To minimize finite-size corrections, we take $U = 0.5\,U_{c,-}$ where we saw the smallest finite-size corrections at $\mu = 0$ in Fig. 1. Results are shown in Fig. 3. At $\mu = 0.25\,W$, the lowest three energies are fourfold degenerate to within the numerical accuracy of our algorithm. At $\mu = 0.4$, the excitation gap $E_5 - E_1$ is only $0.04\,W$, and degeneracy is lifted. However, the clear grouping of energies into similar clusters of four suggests that here too there will be fourfold degeneracy in the $N \to \infty$ limit, and that the observed lifting is due to enhanced finite-size corrections resulting from the small gap. Finally, at $\mu = 0.6\,W$ we observe the unambiguous lifting of fourfold degeneracy for the ground as well as excited states. It is interesting to note that the critical chemical potential is lower than in the noninteracting case where it equals the hopping amplitude W. These results support the hypothesis that the full spectrum remains fourfold degenerate at a finite but subcritical chemical potentials. However, proving this analytically remains an open problem.

VI. CONCLUSION

In conclusion, we have shown that the full spectrum of a particle-hole symmetric *interacting* Kitaev chain is fourfold degenerate when the system contains a Josephson junction with a π-phase difference, up to corrections that vanish exponentially as a function of system size. This proves that the occupation number of the dressed Andreev bound state localized around the junction is a constant of motion. Thus, in a sufficiently long chain, intrinsic electron-electron interactions are not a source quasiparticle poisoning where the 4π Josephson effect is concerned. We have numerically studied finite-size corrections, and found that for a fixed bare-hopping amplitude, they decay more slowly in a chain with repulsive interactions than in a noninteracting chain with the same measured excitation gap and hopping amplitude. This shows that repulsive interactions frustrate superconductivity in a way that cannot be accounted for by an effective noninteracting model.

ACKNOWLEDGMENT

This work is based on research supported by the National Research Foundation of South Africa (Grant No. 90657).

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