Rapid Communications

Thermoelectric response and entropy of fractional quantum Hall systems

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We study the thermoelectric transport properties of fractional quantum Hall systems based on an exact diagonalization calculation. Based on the relation between the thermoelectric response and thermal entropy, we demonstrate that thermoelectric Hall conductivity α_{xy} has power-law scaling $\alpha_{xy} \propto T^{\eta}$ for gapless composite Fermi-liquid states at filling numbers $\nu = 1/2$ and $1/4$ at low temperatures (T), with an exponent $\eta \sim 0.5$ distinctly different from Fermi liquids. The power-law scaling remains unchanged for different forms of interaction including Coulomb and short-range ones, demonstrating the robustness of non-Fermi-liquid behavior of these interacting systems at low T. In contrast, for the $1/3$ fractional quantum Hall state, α_{xy} vanishes at low T with an activation gap associated with neutral collective modes rather than charged quasiparticles. Our results establish another manifestation of the non-Fermi-liquid nature of quantum Hall fluids at a finite temperature.

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Introduction. Thermoelectric phenomena that provide a direct conversion between heat and electricity are interesting and useful. Decades of research have been devoted to finding materials and methods to increase thermoelectric energy conversion efficiency [1,2]. Recent theoretical works suggested the possibility of record-high thermoelectric conversion efficiency in semiconductors and semimetals under a quantizing magnetic field [3,4], where the thermoelectric response is directly related to entropy [5–8]. Based on this relation, it is found that the thermopower of three-dimensional (3D) Dirac and Weyl materials in the quantum limit increases unboundedly with magnetic field [3,9–12]. Very recently, it is shown that two-dimensional (2D) quantum Hall systems can reach a thermoelectric figure of merit on the order of unity down to low temperature (T), as a consequence of the thermal entropy from the massive Landau level (LL) degeneracy [4].

The degeneracy of a partially filled Landau level is lifted by an electron-electron interaction and disorder. Therefore, the thermoelectric response of quantum Hall systems is expected to depart from the noninteracting and clean limit when the thermal energy $k_B T$ is smaller than a characteristic energy scale proportional to the electron interaction strength or a disorder-induced Landau level broadening Γ. Previous works [6,13] have shown that disorder leads to a T-linear thermoelectric Hall conductivity $\alpha_{xy} \propto T$ for $k_B T \ll \Gamma$, in accordance with the thermal entropy of disorder broadened Landau levels.

On the other hand, in clean systems an electron-electron interaction lifts the massive Landau level degeneracy and forms a many-body ground state at fractional filling. These include gapped fractional quantum Hall (FQH) states [14] and gapless composite Fermi liquids [15,16], which provide a fertile ground for exotic quantum states of matter. After nearly four decades of theoretical and experimental studies, ground state properties and low-energy excitations at various fractional fillings are largely understood. In contrast, much less is known about FQH systems at finite temperature. Based on the relation between the entropy and the thermoelectric transport coefficient, a linear T scaling behavior for thermopower S_{xx} has been conjectured [17] for composite Fermi-liquid states. While there are a few measurements [18–25], theoretical or numerical calculations on finite T thermoelectric transport coefficients for fractional quantum Hall states are lacking.

In this Rapid Communication, we investigate the thermoelectric Hall response for interacting quantum Hall systems through exact diagonalization calculations of the entropy of finite-size systems at finite temperature. For even denominator filling numbers $\nu = 1/2$ and $1/4$, we identify the robust power-law scaling behavior of the thermoelectric Hall conductivity $\alpha_{xy} \propto T^{\eta}$ in a wide temperature range, with the exponent $\eta \sim 0.5$ distinctly different from the linear T behavior of Fermi liquids. We further show that the scaling behavior is robust against weak disorder, and the exponent η gradually increases with disorder strength. In contrast, for a $1/3$ FQH system, we observe a vanishing α_{xy} at low T below the excitation gap. Our prediction of an anomalous power-law temperature dependence of the thermoelectric response establishes another fundamental property of $\nu = 1/2$ and $\nu = 1/4$ quantum Hall fluids, which can be measured in future experiments.

Model and method. We consider a two-dimensional electron system subject to a perpendicular strong magnetic field, whose energy spectrum is composed of discrete LLs. Throughout this work we assume the cyclotron energy is much larger than other energy scales set by interaction, disorder scattering, or temperature, so that it suffices to work with the restricted Hilbert space of a partially filled LL.

The many-body Hamiltonian can be written as

$$H = \sum_{i<j} \sum_{\mathbf{q}} e^{-q^2/2} V(q) e^{i\mathbf{q}\cdot(\mathbf{R}_i - \mathbf{R}_j)} + \sum_{i} \sum_{\mathbf{q}} e^{-q^2/4} U_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{R}_i}, \quad (1)$$

where $\mathbf{R}_i$ is the guiding center coordinate of the ith electron, $V(q) = 2\pi e^2/\epsilon q$ is the Coulomb potential, and $U_{\mathbf{q}}$ is the impurity potential with the wave vector $\mathbf{q}$. We set the magnetic length $\ell = 1$ and $e^2/\epsilon\ell = 1$ for convenience. The Gaussian white noise potential we use is generated according to the correlation relation in q space, $\langle U_{\mathbf{q}} U_{\mathbf{q}'}\rangle = (W^2/A)\delta_{\mathbf{q},-\mathbf{q}'}$, which corresponds to $\langle U(\mathbf{r})U(\mathbf{r}')\rangle = W^2\delta(\mathbf{r}-\mathbf{r}')$ [26] in real space, where W is the strength of the disorder and A is the area of the system. The filling fraction is then defined as $\nu = N_e/N_s$, where N_e and N_s are the number of electrons in the partially filled LL and the number of the flux quanta, respectively.

Thermoelectric conductivity α_{ij} is defined by the electrical current generated by a temperature gradient in the absence of any voltage (short-circuit condition), or via an Onsager relation, by the heat current generated by a voltage difference at a uniform temperature. Since heat current is carried by thermal excitations, thermoelectric conductivity is purely a property of the partially occupied LL. In our case, its value depends on temperature k_BT and disorder strength W (both in units of $e^2/\epsilon\ell$). While thermoelectric conductivity is conceptually convenient for a theoretical analysis, experiments usually measure the thermopower S_{xx} and Nernst signal S_{xy} directly. These are given by the product of α_{ij} and resistivity ρ_{jk}, $S_{ik} = \alpha_{ij}\rho_{jk}$.

We first consider the clean limit $W = 0$. As shown explicitly for both noninteracting systems [4,6] and a generic interacting electron fluid [17,27], in the absence of disorder, thermoelectric Hall conductivity α_{xy} is directly proportional to entropy density s, $\alpha_{xy} = s/B$. This remarkable formula enables us to obtain α_{xy} by numerically calculating entropy—a thermodynamic property—without invoking the Kubo formula for transport coefficients. The validity of the relation can be justified in the weak disorder scattering limit ($W \ll e^2/\epsilon\ell$), where the electric Hall conductivity is proportional to the electron filling number [17], consistent with a semiclassical approximation.

We perform thermal entropy calculations based on the exact calculation of the energy spectrum of the Hamiltonian, and obtain $\alpha_{xy} = S/N_s$ (in units of k_Be/h), where $\boldsymbol{S = sA}$ is the thermal entropy of the system. We consider systems with a subdimension of Hilbert space up to $N_h = 102\,348$ (2 119 036) for full (partial) diagonalizations, which is slightly smaller than the largest accessible sizes used in ground state simulations and gives reliable results for all temperature regimes we considered. Further details on numerical calculations are discussed in the Supplemental Material [28] (see also Refs. [29–31] therein).

Thermoelectric Hall response of composite Fermi liquids. We first consider interacting quantum Hall systems without disorder scattering. Two even denominator filling numbers $\nu = 1/2$ and $1/4$ will be considered first, where the low-temperature behavior of such systems is controlled by the physics of the composite Fermi liquid [16]. By exact diagonalization, we can study systems with up to the number of flux quanta $N_s = 28$ for electrons at $1/2$ filling using magnetic translational symmetry. By obtaining all energy eigenvalues of the system, we determine α_{xy} from the entropy per flux. As shown in Figs. 1(a) and 1(b), we show α_{xy} for different system sizes with $N_s = 12$–28 for both $n = 0$ and 1 LLs. The α_{xy} grows with T monotonically, and saturates towards a universal

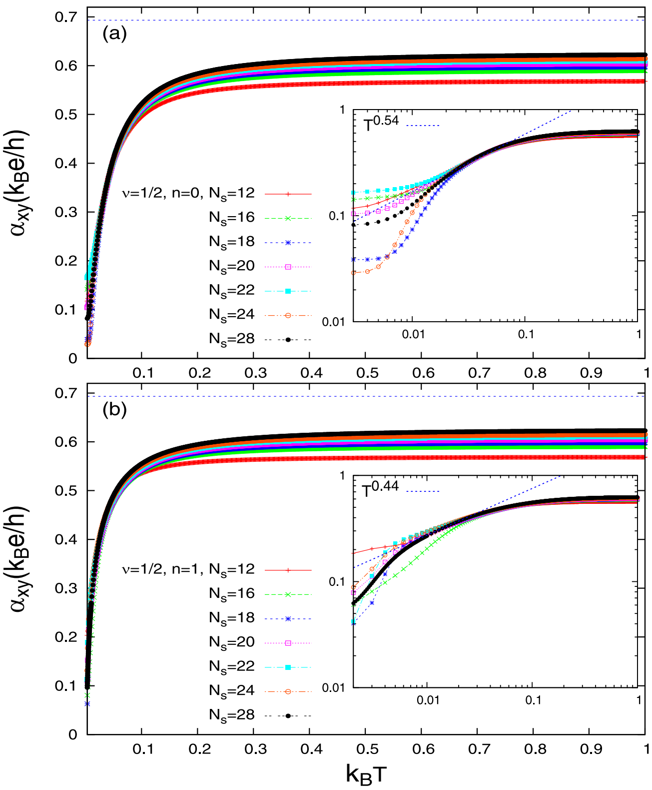

FIG. 1. Thermoelectric Hall coefficient α_{xy} (in units of k_Be/h) for electrons at filling number $\nu = 1/2$ of the nth LL with the number of flux quanta varying between $N_s = 12$ and 28. The thermal energy k_BT is in units of energy $e^2/\epsilon\ell = 1$ in all figures. α_{xy} increases with T, which saturates towards $\ln 2$ at the thermodynamic and large T limit according to their saturated entropy per LL orbital. In the insets of (a) and (b) we show the same data in the logarithmic plot, where a power-law scaling is found $\alpha_{xy} \propto T^\eta$ for a range of T. (a) For $n = 0$ LL with a Coulomb interaction, $\eta \sim 0.54 \pm 0.03$. (b) For $n = 1$ LL with a Coulomb interaction, $\eta \sim 0.44 \pm 0.03$. The error bar is estimated by fitting low T data from different N_s.

value $\ln(2) \times k_Be/h$ determined by the entropy per flux for LL at half filling. As shown in the insets of Figs. 1(a) and 1(b), we identify a power-law behavior at low temperature $\alpha_{xy} \propto T^\eta$ as a straight line fitting to the data in the algorithmic plots, and find the exponent $\eta \sim 0.54 \pm 0.03$ and $\sim 0.44 \pm 0.03$ for $n = 0$ and 1, respectively. We remark that although the nature of the ground states for the $n = 0$ and $n = 1$ are different at the $T = 0$ limit corresponding to the composite-Fermi liquid and Moore-Read non-Abelian FQH state [32], respectively, α_{xy} of both systems in the small to intermediate T can demonstrate similar scaling behavior. This is because α_{xy} is controlled by the gapless excitations once k_BT is larger than the excitation gap of the FQH of $n = 1$ LL. Furthermore, we compare α_{xy} of systems with different types of electron-electron interactions including the Haldane short-range pseudopotential, and find quantitative similar results. Our results indicate a composite Fermi liquid has non-Fermi-liquid behavior, consistent with the entanglement probe [33]. Due to the strong interaction effect, the many-body density of states are shown to increase very fast with the excitation energy in these systems in contrast to the constant density of states

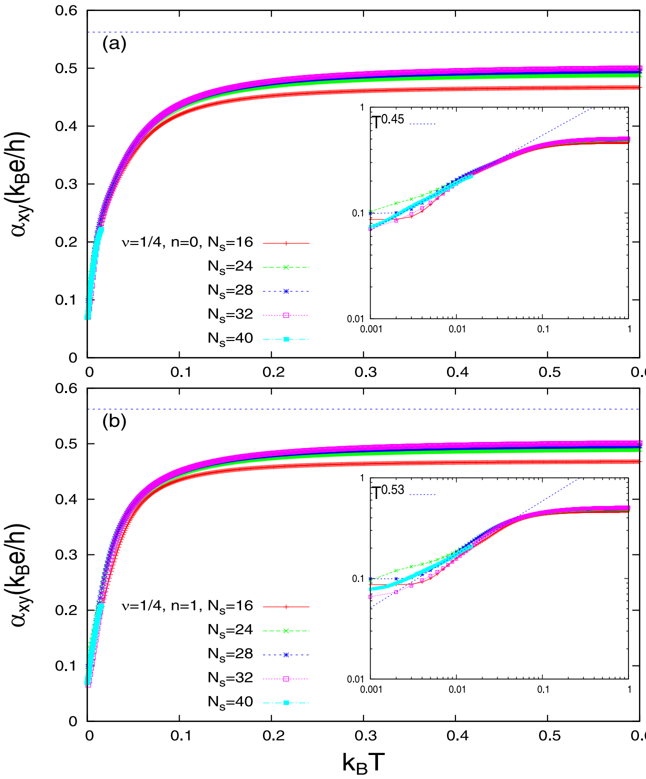

FIG. 2. α_{xy} for electrons at $\nu = 1/4$ filling of the nth LL with $N_s = 16$–40. The dotted line above the data curve indicates the expected α_{xy} at the infinite T and N_s limit according to their saturated entropy. The fittings shown in the insets of (a) and (b) illustrate the power-law scaling $\alpha_{xy} \propto T^{\eta}$. (a) For $n = 0$ LL with a Coulomb interaction, $\eta \sim 0.45 \pm 0.03$. (b) For $n = 1$ LL with a Coulomb interaction, $\eta \sim 0.53 \pm 0.03$.

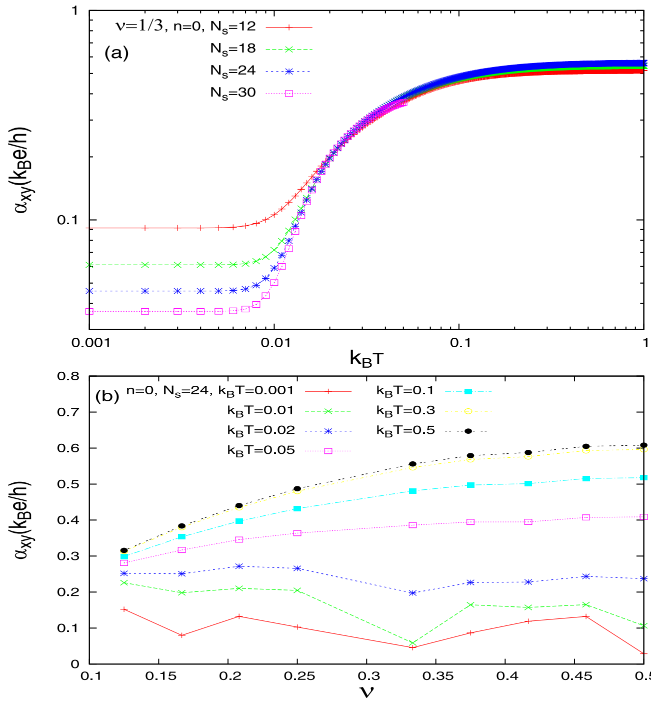

FIG. 3. α_{xy} for electrons at different filling numbers of the lowest LL for pure interacting systems. (a) α_{xy} vs k_BT for $N_s = 12$–30 at fixed $1/3$ filling number. In the inset we show $\alpha_{xy} - \ln 3/N_s$ as a function of $1/k_BT$ for larger systems $N_s = 18$–30. (b) α_{xy} vs ν for $N_s = 24$ for different $k_BT = 0.001$–0.5.

of a 2D Fermi liquid, which induces nonlinear scaling for α_{xy} [28].

Now we move on to the $\nu = 1/4$ filling number, where larger systems can be accessed with N_s up to 32 (40) for full (partial) diagonalization. As shown in Figs. 2(a) and 2(b) for systems with $N_s = 16$–40 for both LLs $n = 0$ and $n = 1$, α_{xy} increases with T rapidly and it saturates to a universal value (0.562) determined by the entropy per flux for the $1/4$ partially filled LL at a high T limit. As shown in the insets of Figs. 2(a) and 2(b), we demonstrate a power-law behavior at low temperature $\alpha_{xy} \propto T^{\eta}$, where almost all data points from the low T regime can be well fitted by such a scaling behavior up to $T \sim 0.07$, beyond which α_{xy} starts to saturate. Clearly, the finite-size effect is reduced comparing to the $\nu = 1/2$ case as we can access larger systems with larger N_s at $\nu = 1/4$ filling. The exponent is identified to be $\eta \sim 0.45 \pm 0.03$ and $\sim 0.53 \pm 0.03$ for $n = 0$ and 1, respectively. The average exponent for the power-law behavior is consistent with $\eta \sim 0.50$ for both $\nu = 1/4$ and $1/2$ filling numbers, indicating a possible universal scaling behavior for α_{xy} at low T.

Thermoelectric Hall response for different electron filling numbers. At low temperatures, different correlated states emerge for interacting quantum Hall systems. As we have shown before, the even denominator state is either gapless (for the $n = 0$ case) or having a small excitation gap for the quantum Hall effect (QHE) state ($1/2$ filling of $n = 1$ LL), which exhibits power-law scaling for α_{xy} down to very low temperatures. To explore the possible distinct physics from the thermoelectric Hall effect of a gapped state, we present results of $1/3$ FQH with a Coulomb interaction for a pure system at $W = 0$. As shown in Fig. 3(a) with flux quanta $N_s = 12$–30, we find that α_{xy} at low $T < 0.01$ decreases with an increase of N_s, while $N_s\alpha_{xy} = \ln 3$, in accordance with the threefold degeneracy of the system. The data for the largest system $N_s = 30$ are obtained through Lanczos for the lowest 600 states in each momentum sector, which allows us to obtain a lower temperature α_{xy} accurately. The temperature regime to see such a vanishing α_{xy} behavior is $T \leqslant 0.01$, indicating a finite gap for such a fractionalized topological state. From the energy excitation spectrum and many-body density of states, we identify a roton gap of size 0.062 [28], consistent with earlier numerical results [30]. Beyond $T > 0.01$, α_{xy} increases with T very sharply, and saturates to a universal value at the large T limit. To compare these data with the activation behavior of a gapped system, we use the following form, $\alpha_{xy} - \ln 3/N_s \propto \exp(-\frac{E_g}{k_BT})$, to fit the low T data. The constant term $\ln 3/N_s$ is the size-dependent contribution from topological degeneracy. As shown in the inset of Fig. 3(a), we identify the collective excitation gap $E_g = 0.06$, consistent with the direct measurement in the excitation spectrum [28]. The obtained roton gap should be distinguished from the quasiparticle and quasihole charge gap [28]. Therefore the thermoelectric response provides another experimental way to detect collective excitations.

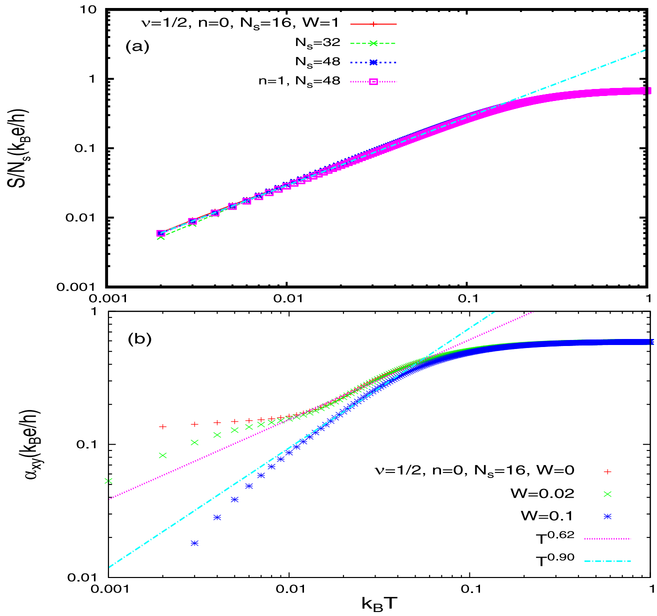

FIG. 4. (a) S/N_s for *noninteracting* electrons at half filling of the $n = 0$ and 1 LLs with $N_s = 16$–48 and the disorder strength $W = 1$. The dashed line shows the linear fit of the data. (b) α_{xy} for electrons at half filling of the lowest LL for disordered interacting systems with $N_s = 16$.

We compare α_{xy} at other filling numbers with the behavior of composite Fermi-liquid systems at $\nu = 1/2$ and $1/4$. In Fig. 3(b), we show α_{xy} vs ν for filling numbers $\nu = 1/8$–$1/2$ at fixed $N_s = 24$ (the result is symmetric about $\nu = 1/2$ due to particle-hole symmetry). At low temperatures, the gapped 1/3 FQH state has suppressed α_{xy}, as discussed above. Other systems are either gapless, or have tiny gaps compared to the 1/3 FQH system. We find a clearly local minimum at $\nu = 1/3$ for $k_BT \leqslant 0.02$. For lower temperatures at $T = 0.001$, we suspect that the finite-size effect dominates due to the finite-energy splitting for these finite-size systems [28]. As T is further increased, we identify α_{xy} as a smooth increasing function of ν, which reaches a broad maximum value around $\nu = 1/2$.

Disorder effect. For experimentally realized quantum Hall systems, the impurity scattering effect is always present in addition to electron-electron Coulomb interactions. We first show the results of entropy for disordered quantum Hall systems without considering the Coulomb interaction. The obtained S/N_s as a function of k_BT is shown in Fig. 4(a). For system flux numbers $N_s = 16$–48, we find that all the data collapse into one universal curve, which can be well fitted by a linear dependence $S/N_s \propto T$ shown as the dashed line fitting in the log-log plot in Fig. 4(a).

Now we turn to the effect of weak random disorder for interacting quantum Hall systems. In this case, the relation between thermoelectric Hall conductivity and entropy is approximately valid, when $W \ll e^2/\epsilon\ell = 1$. For such systems, the magnetic translational symmetry is broken due to momentum nonconserving scattering present in the Hamiltonian. So we have to diagonalize the whole Hilbert space, which limits us to smaller systems with $N_s = 16$ and 18. As shown in Fig. 4(b) for $N_s = 16$ at filling number $\nu = 1/2$, thermal entropy is always an increasing function of T for different W, which saturates to the universal value determined by the maximum entropy per orbital in such systems. In a smaller and intermediate T regime, α_{xy} appears to follow the power-law scaling behavior $\alpha_{xy} \propto T^{\eta}$, with the exponent η increasing from around 0.62 for $W = 0.02$, to $\eta \sim 0.90$ for $W = 0.1$. Very similar results are obtained for $N_s = 16$ and 18, indicating these behaviors are robust.

Summary and discussion. We now discuss the importance of our work for understanding the thermoelectric Hall effect of different quantum Hall systems at low temperature. Since most of the quantum Hall systems realized in experiments have high mobility, the interaction effect plays an essential role in lifting the LL degeneracy. Our work focuses on finite temperature, where these interacting systems are either in gapless composite Fermi-liquid states, or thermally excited QHE states. The nonlinear power-law scaling behavior we established for $\nu = 1/2$ and $1/4$ filling numbers strongly proves that these systems at finite temperature are strongly correlated electron fluids, distinctly different from Fermi liquids for which the linear scaling law of α_{xy} is expected from the low-energy excitations around the Fermi level. The scaling behavior of $\alpha_{xy} \propto T^{\eta}$ (with $\eta \sim 0.5$) appears to be very general for quantum Hall systems at different filling numbers. In particular, the $\nu = 1/2$ and $1/4$ quantum Hall states have much enhanced α_{xy} at the lowest temperature, and therefore are promising candidates for thermoelectric energy conversion with high efficiency [4]. Our calculations can be naturally extended to other quantum Hall systems [34–39] with different LL degeneracies or multicomponent interactions such as graphene. The intriguing strange metal transport we discovered calls for a theory of fractional quantum Hall liquids at finite temperature, which we will provide in forthcoming works.

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