Particle number fluctuations, Rényi entropy, and symmetry-resolved entanglement entropy in a two-dimensional Fermi gas from multidimensional bosonization

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We revisit the computation of particle number fluctuations and the Rényi entanglement entropy of a two-dimensional Fermi gas using multidimensional bosonization. In particular, we compute these quantities for a circular Fermi surface and a circular entangling surface. Both quantities display a logarithmic violation of the area law, and the Rényi entropy agrees with the Widom conjecture. Lastly, we compute the symmetry-resolved entanglement entropy for the two-dimensional circular Fermi surface and find that, while the total entanglement entropy scales as $R\ln R$, the symmetry-resolved entanglement scales as $\sqrt{R\ln R}$, where R is the radius of the subregion of our interest.

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I. INTRODUCTION

In recent years, there has been a surge of interest in quantum entanglement and its various measures, in the condensed-matter as well as the high-energy physics communities [1–9]. One of the most profound results pertaining to entanglement entropy in many-body systems is the area law for ground states of gapped systems, where the entanglement entropy is known to be proportional to the area of a subregion [1]. This area law underlies the simulatability of gapped ground states by matrix product states [10]. Intuitively, degrees of freedom in a system with local interactions are entangled only with their neighbors, so the entanglement entropy receives contributions primarily from the degrees of freedom situated close to the boundary.

Even though the appearance of the area-law behavior in ground states is ubiquitous, there are known exceptions where the area law is violated, typically with a logarithmic correction. Some well-known examples are conformal field theories in one spatial dimension, which describe quantum critical points, and fermionic systems in higher spatial dimensions with a Fermi surface [11,12].

While the von Neumann entropy and related measures are important physical quantities, they are difficult to compute analytically for generic many-body systems. Conformal field theories in one spatial dimension are among the most analytically tractable systems since the replica technique can be applied there. In these cases, the computation of the entanglement entropy boils down to the evaluation of the correlation functions of twist operators. Another approach for one-dimensional systems would be to use the Fisher-Hartwig formula for free systems. There are, however, fewer analytical calculations done in spatial dimensions greater than one. Calculations for the entanglement entropy of a higher dimensional Fermi surface can either be done by applying the Widom conjecture or bosonization, or by simply dividing up the multidimensional Fermi surface into many one-dimensional pieces where one can use the known one-dimensional results [5,13–15]. Another example where the entanglement entropy and particle number cumulants can be computed in two spatial dimensions is a case of noninteracting fermions trapped in a harmonic potential [16].

In this paper, we apply the multidimensional bosonization technique developed in Refs. [17,18] to calculate the entanglement entropy and related quantities of a Fermi gas analytically and nonperturbatively. First, we compute the particle number cumulants' generating function. This quantity is then used to carefully derive the entanglement entropy for an isotropic Fermi gas, which is found to be in agreement with Widom's conjecture. This implies that the leading term in the entanglement entropy of a Fermi gas comes primarily from the modes near the Fermi surface. The calculations presented in this paper can potentially be generalized to higher dimensions or to systems with Fermi liquid interactions. Next, the particle number cumulants' generating function is also used to compute the symmetry-resolved entanglement of a two-dimensional Fermi gas [19]. We find that each particle number sector contributes an entanglement of $\sqrt{R\ln R}$, while the total entanglement entropy scales as $R\ln R$, where R is the radius of the subregion of our interest.

II. REVIEW OF MULTIDIMENSIONAL BOSONIZATION

Before proceeding with the calculations, we review the scheme of multidimensional bosonization developed in Refs. [17,18]. Alternate formulations of multidimensional bosonization can be found in Refs. [20,21]. Given a filled Fermi sea, we can create and annihilate particle-hole pairs with the following operators:

$$n_{\vec{q}}(\vec{k}) = c^{\dagger}_{\vec{k}-\frac{\vec{q}}{2}} c_{\vec{k}+\frac{\vec{q}}{2}}, \tag{1}$$

where $c_{\vec{k}}/c^{\dagger}_{\vec{k}}$ are the electron annihilation/creation operators with momentum $\vec{k}$. Because they are quadratic in the fermion operators, their commutators are almost bosonic. However, they do not annihilate the Fermi sea. We need to normal order the particle-hole operators relative to the Fermi sea, so

Ref. [18] defined the following operators:

$$a_{\vec{q}}(\vec{k}_F) = \sum_{\vec{k}} \Phi_\Lambda(|\vec{k}-\vec{k}_F|)\,[n_{\vec{q}}(\vec{k})\Theta(\vec{v}_{\vec{k}_F}\cdot\vec{q}) + n_{-\vec{q}}(\vec{k})\Theta(-\vec{v}_{\vec{k}_F}\cdot\vec{q})],$$
$$a^\dagger_{\vec{q}}(\vec{k}_F) = \sum_{\vec{k}} \Phi_\Lambda(|\vec{k}-\vec{k}_F|)\,[n_{-\vec{q}}(\vec{k})\Theta(\vec{v}_{\vec{k}_F}\cdot\vec{q}) + n_{\vec{q}}(\vec{k})\Theta(-\vec{v}_{\vec{k}_F}\cdot\vec{q})], \tag{2}$$

where $\Theta(x) = 1(-1)$ if $x > 0(< 0)$ and $\Phi_\Lambda(|\vec{k}-\vec{k}_F|)$ is some dimensionless smearing function that keeps the vectors $\vec{k}$ close to the Fermi momentum $\vec{k}_F$. More precisely, it is defined as

$$\lim_{\Lambda\to 0}\Phi_\Lambda(|\vec{k}-\vec{k}_F|) = \delta_{\vec{k},\vec{k}_F}, \tag{3}$$

where Λ is a momentum space cutoff. We have also defined the velocity of the particles as $\vec{v}_{\vec{k}} = \vec{\nabla}\epsilon_k$, with $\epsilon_{\vec{k}}$ being the spectrum of the one-particle states. The idea is to divide the Fermi surface into patches of radius Λ centered about $\vec{k}_F$, and $\vec{q}$ is constrained to lie within the patch, so $q \ll \Lambda \ll k_F$. By construction, $a_{\vec{q}}(\vec{k}_F)$ annihilates the Fermi sea $|\mathrm{FS}\rangle$, i.e., $a_{\vec{q}}(\vec{k}_F)|\mathrm{FS}\rangle = 0$.

For each patch, the local density of states is

$$N_\Lambda(\vec{k}_F) = \frac{1}{V}\sum_{\vec{k}} |\Phi_\Lambda(|\vec{k}-\vec{k}_F|)|^2\delta(\mu-\epsilon_{\vec{k}}), \tag{4}$$

where the chemical potential is $\mu = \epsilon_{\vec{k}_F}$ and the total system size is V. The total density of states is

$$N(0) = \frac{1}{V}\sum_{\vec{k}}\delta(\mu-\epsilon_{\vec{k}}) \tag{5}$$

and they are related by $N_\Lambda(\vec{k}_F) = \frac{N(0)}{S_d}$ for an isotropic Fermi surface, where S_d is the d-dimensional solid angle. For convenience, rescale the bosonic operators Eq. (2) as

$$b_{\vec{q}}(\vec{k}_F) = [N_\Lambda(\vec{k}_F)V|\vec{q}\cdot\vec{v}_{\vec{k}_F}|]^{-1/2}a_{\vec{q}}(\vec{k}_F). \tag{6}$$

These operators obey the usual bosonic algebra:

$$[b_{\vec{q}}(\vec{k}_F), b^\dagger_{\vec{q}'}(\vec{k}'_F)] = \delta_{\vec{k}_F,\vec{k}'_F}\left(\delta_{\vec{q},\vec{q}'} + \delta_{\vec{q},-\vec{q}'}\right). \tag{7}$$

For the restricted Hilbert space that contains excitations close to the Fermi surface, the noninteracting Hamiltonian is effectively given by

$$H_0 = \sum_{\vec{k}_F}\sum_{\vec{q}}^{\vec{q}\cdot\vec{k}_F>0} |\vec{q}\cdot\vec{v}_{\vec{k}_F}|\,b^\dagger_{\vec{q}}(\vec{k}_F)b_{\vec{q}}(\vec{k}_F). \tag{8}$$

We see that these bosons diagonalize the noninteracting low-energy Hamiltonian. The electronic density is related to the bosons as follows:

$$\rho(\vec{q}) = \sum_{\vec{k}_F}^{\vec{v}_{\vec{k}_F}\cdot\vec{q}>0} [N_\Lambda(\vec{k}_F)|\vec{q}\cdot\vec{v}_{\vec{k}_F}|]^{1/2}[b^\dagger_{\vec{q}}(-\vec{k}_F) + b_{\vec{q}}(\vec{k}_F)]. \tag{9}$$

This is the multidimensional bosonization identity which relates the fermionic density with the bosonic modes.

For the rest of the paper, we will restrict ourselves to two spatial dimensions.

III. FREE FERMION PARTICLE NUMBER CUMULANT GENERATING FUNCTIONAL

For a given subregion A, we define the generating function of particle number cumulants to be [22,23]

$$\left\langle e^{i\lambda\hat{N}_A}\right\rangle, \quad \lambda\in\mathbb{C}, \tag{10}$$

where $\hat{N}_A$ is the number operator of subregion A. The generating function produces the cumulants of the particle number distribution in subregion A via

$$V_A^{(m)} = (-i\partial_\lambda)^m \ln\langle e^{i\lambda\hat{N}_A}\rangle|_{\lambda=0}. \tag{11}$$

In particular, the second cumulant (variance) is

$$V_A^{(2)} = \langle(\hat{N}_A - \langle\hat{N}_A\rangle)^2\rangle. \tag{12}$$

Without interactions, the Hamiltonian is given by Eq. (8) for low-lying states, so the ground state is annihilated by $b_{\vec{q}}(\vec{k}_F)$. Defining $f(\vec{r}) = i\lambda\Theta(\vec{r}\in A)$, where $\Theta(\vec{r})$ is the two-dimensional step function, the generating function Eq. (10) can be written as

$$\langle e^{i\lambda\hat{N}_A}\rangle = \left\langle \exp\left[\int d^d r\,\rho(\vec{r})f(\vec{r})\right]\right\rangle = \left\langle \exp\left[\sum_{\vec{k}}\rho(\vec{k})f(-\vec{k})\right]\right\rangle, \tag{13}$$

where the momentum-space density operator can be related to the bosonic modes via Eq. (9). Since the expectation value is computed in the ground state of Eq. (8), it can be simplified by the Baker-Campbell-Hausdorff formula. Let us further restrict ourselves to a circular Fermi surface for the rest of the paper. The generating function then simplifies to

$$\langle e^{i\lambda\hat{N}_A}\rangle = \exp\Bigg[-\frac{1}{2}\frac{\lambda^2}{V}\sum_{\vec{k}_F}\int_{\vec{r}\in A}d^d r\int_{\vec{r}'\in A}d^d r' \times \sum_{\vec{q}}^{\vec{v}_{\vec{k}_F}\cdot\vec{q}>0} N_\Lambda(\vec{k}_F)(\vec{q}\cdot\vec{v}_{\vec{k}_F})\,e^{i\vec{q}\cdot(\vec{r}-\vec{r}')}e^{-\frac{\alpha|\vec{q}\cdot\vec{v}_{\vec{k}_F}|}{|\vec{v}_{\vec{k}_F}|}}\Bigg]. \tag{14}$$

Here, a momentum regulator α has been introduced so only states with small excitation momenta normal to the Fermi surface are kept.

It remains to perform the momentum sums and the spatial integrals. We first integrate over the excitation momentum $\vec{q}$. This can be done by decomposing $\vec{q}$ into two parts, $\vec{q} = \vec{q}_N + \vec{q}_T$, where $\vec{q}_{N/T}$ are the components normal/tangent to the Fermi surface. The subsequent spatial integral leads to

$$\langle e^{i\lambda\hat{N}_A}\rangle = \exp\left[-\frac{\lambda^2}{2\pi}NN_\Lambda(\vec{k}_F)|\vec{v}_{\vec{k}_F}|I(R)\right], \tag{15}$$

where $N = \sum_{\vec{k}_F}$ is the number of patches, and

$$I(R) = \int_{\vec{r}\in A}d^d r\,\frac{\sqrt{R^2-y^2}}{R^2-y^2-(x+i\alpha)^2}, \tag{16}$$

with $\vec{r} = (x, y)$. The coordinate system was chosen such that $\vec{q}_N$ is parallel to $\hat{x}$ and $\vec{q}_T$ is parallel to $\hat{y}$. The final result is independent of the direction of the Fermi momentum $\vec{k}_F$ due to the isotropy of the Fermi surface, so the sum over the Fermi momentum of the patches of the Fermi surface simply becomes the number of such patches. Evaluating the final one-dimensional spatial integral numerically, we find that $I(R) \simeq 2R \ln \frac{R}{\alpha} + 0.776R$. The generating function for the particle number cumulants is thus

$$\langle e^{i\lambda \hat{N}_A} \rangle = \exp\left[-\frac{\lambda^2 k_F}{(2\pi)^2}\left(2R \ln \frac{R}{\alpha} + 0.776R\right)\right]. \tag{17}$$

This equation is independent of the number of patches of the Fermi surface, as it should be. The coefficient of the logarithmic term converged unambiguously to 2 in the numerical integration. With the generating function at hand, one can easily obtain the variance Eq. (12):

$$V_A^{(2)} = \frac{k_F}{2\pi^2}\left(2R \ln \frac{R}{\alpha} + 0.776R\right). \tag{18}$$

The leading term is simply the two-dimensional area law with a logarithmic violation, and is thus proportional to the entanglement entropy, as it should be [12,24]. Intuitively, if the number of particles in a subregion has significant fluctuations, then there must be a large number of particles moving between the subregion and its complement, giving rise to a large amount of entanglement, so these two quantities should be proportional.

The variance Eq. (18) can be checked numerically with the following formula, valid for free fermions systems, in terms of the overlap matrix [22,25,26],

$$V_A^{(2)} = \mathrm{Tr}\,[\mathbb{A}(1 - \mathbb{A})], \tag{19}$$

where $\mathbb{A}$ is the overlap matrix given in terms of the single-particle eigenfunctions $\phi_n(\vec{r})$ as

$$\mathbb{A}_{nm} = \int_{\vec{r}\in A} d^d r\, \phi_n^*(\vec{r})\phi_m(\vec{r}). \tag{20}$$

The overlap matrix $\mathbb{A}$ is the continuum limit of the more familiar correlation function $\mathbb{C}$ used in computing entanglement entropy for free fermions on a lattice [22], $\mathrm{Tr}\,\mathbb{C}^n = \mathrm{Tr}\,\mathbb{A}^n$. (Working in the continuum allows us to select a perfectly circular subregion, which is not possible in a lattice model). For a total system of size $L \times L$ with periodic boundary conditions,

$$\begin{aligned}\mathbb{A}_{nm} &= \frac{1}{L^2}\int_{\vec{r}\in A} d^2 r\, e^{i(\vec{k}_m - \vec{k}_n)\cdot\vec{r}} \\ &= \frac{2\pi R}{|\vec{k}_m - \vec{k}_n| L^2} J_1(|\vec{k}_m - \vec{k}_n| R),\end{aligned} \tag{21}$$

where $\vec{k}$ lies within a circular Fermi surface of radius k_F and $J_1(x)$ is a Bessel function of the first kind. The subregion A is a disk of radius R as before. The numerically obtained variance Eq. (19) with $L = 20$, $k_F = \pi$ is compared with the analytical result Eq. (18) in Fig. 1. Here, we use the regulator α as a fitting parameter with $\alpha = 0.0603$.

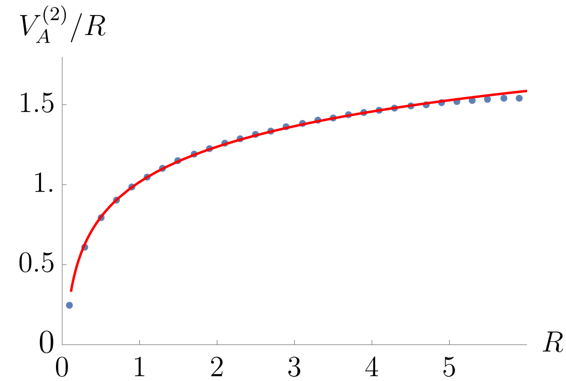

FIG. 1. The ratio of the variance to the subregion radius $V_A^{(2)}/R$ as a function of the subregion radius R obtained from the numerical evaluation of Eq. (19) (dots) fitted against the analytical result Eq. (18) (curve). The fit gives $\alpha = 0.0603$. The system size is $L = 20$ and $k_F = \pi$.

IV. ENTANGLEMENT ENTROPY

In this section, we compute the Rényi entanglement entropy of a free Fermi gas in a circular subregion A with a two-dimensional isotropic Fermi surface:

$$S_A^{(n)} = \frac{1}{1-n} \ln \mathrm{Tr}\left(\rho_A^n\right). \tag{22}$$

As mentioned earlier, for a free Fermi gas, the quantum entanglement and the particle number variance for a given subregion are proportional to each other [22–24],

$$\frac{S_A^{(n)}}{V_A^{(2)}} = \frac{(1 + n^{-1})\pi^2}{6} + \mathcal{O}(1), \tag{23}$$

where $V_A^{(2)}$ is the second cumulant of the generating function we found earlier. Using our previously obtained analytical result for the particle number variance, we readily obtain

$$S_A^{(n)} = (1 + n^{-1})\frac{k_F}{6} R \ln \frac{R}{\alpha} + \cdots. \tag{24}$$

The fact that we are able to obtain the leading term in the Rényi entropy by multidimensional bosonization implies that the leading contribution comes from the modes near the Fermi surface.

Let us also mention that Eq. (23) is not the only way to relate the Rényi entropy with the particle number variance. The Rényi entropy can be written in terms of expectation values of the form Eq. (10) with particular choices of λ [27]. This approach yields the same result as in Eq. (23).

V. SYMMETRY-RESOLVED ENTANGLEMENT

Having computed the Rényi entropy, we turn our attention to the charged entanglement:

$$S_A^{(n)}(c) = \mathrm{Tr}\left(\rho_A^n\, e^{ic\hat{N}_A}\right), \quad c \in \mathbb{R}. \tag{25}$$

This quantity has a nice interpretation of a replicated path integral with flux insertion [19], as will be demonstrated later. The charged entanglement entropy and its variants have been used to detect and distinguish symmetry-protected topological phases [28,29]. It has also been studied holographically

[30–33]. A related quantity known as accessible entanglement entropy has been studied in Refs. [34,35]. Performing the Fourier transform, we obtain the symmetry-resolved entanglement [19,36],

$$S_A^{(n)}(N_A) = \int_{-\pi}^{\pi} \frac{dc}{2\pi}\, S_A^{(n)}(c)\, e^{-icN_A} = \mathrm{Tr}\, \left(\rho_A^n\, \mathcal{P}_{N_A}\right), \tag{26}$$

where $\mathcal{P}_{N_A}$ is the projector onto the subspace with N_A particles in region A. In other words, the symmetry-resolved entanglement is the contribution to the nth Rényi entropy from states with N_A particles in region A.

We begin by computing the charged entanglement. The following derivation generalizes the computation of the entanglement entropy in Ref. [27] to compute the charged entanglement entropy. Let us consider the partial $U(1)$ rotation restricted to region A:

$$M = e^{ic\hat{N}_A}. \tag{27}$$

In the basis of fermionic coherent states,

$$M = \int d\psi d\bar{\psi} d\chi d\bar{\chi}\, e^{-(\bar{\psi}\psi+\bar{\chi}\chi)} M(\bar{\psi},\chi)|\psi\rangle\langle\bar{\chi}|, \tag{28}$$

where $\psi, \bar{\psi}, \chi, \bar{\chi}$ are Grassmann numbers. (These coherent states are constructed in the basis that diagonalizes the entanglement Hamiltonian.) We have suppressed the indices of the Grassmann variables for notational simplicity. All normalization constants have also been absorbed into the integration measure. Here,

$$M(\bar{\psi},\chi) = e^{\phi\bar{\psi}\chi}, \quad \phi = e^{ic}. \tag{29}$$

Performing the Grassmann integrals [37], we obtain the following simple form for the charged Rényi entropy:

$$\mathrm{Tr}\, \left(\rho_A^n\, e^{ic\hat{N}_A}\right) = \int \prod_i d\alpha_i d\bar{\alpha}_i\, \rho_A(\bar{\alpha}_i,\alpha_i)\, e^{\sum_{i,j}\bar{\alpha}_i T_{ij}\alpha_j}, \tag{30}$$

where $\alpha_i, \bar{\alpha}_i$ are Grassmann variables, and the T matrix in the replica space is given by

$$T = \begin{bmatrix} & & & & e^{-ic} \\ -1 & & & & \\ & -1 & & & \\ & & \ddots & & \\ & & & -1 & \end{bmatrix} \tag{31}$$

with eigenvalues

$$\lambda_k = e^{i\left(\frac{2\pi k - c}{n}\right)}, \quad k = -\frac{n-1}{2}, \cdots, \frac{n-1}{2}. \tag{32}$$

This matrix connects the fermions in each sheet of the replica path integral to the next and the phase factor in the upper right-hand corner of the matrix corresponds to the Aharanov-Bohm phase that the fermions acquire if they pass through all the sheets of the replicated space-time and go back to the original sheet. This is the reason why we can interpret the charged Rényi entropy as a replicated path integral with flux insertion. One can then factorize $S_A^{(n)}(c)$ as

$$S_A^{(n)}(c) = \mathrm{Tr}\, \left(\rho_A^n e^{ic\hat{N}_A}\right) = \prod_k Z_k, \tag{33}$$

where Z_k is a ground-state expectation value,

$$\begin{aligned} Z_k &= \langle\Psi|T_k|\Psi\rangle = \mathrm{Tr}\left(\rho_A T_k\right), \\ T_k &= \exp\left(i\Lambda_k \sum_j c_j^\dagger c_j\right) = \exp\left(i\Lambda_k \sum_\mu f_\mu^\dagger f_\mu\right). \end{aligned} \tag{34}$$

Here, c_j are the real-space fermions while f_μ are the fermions that diagonalize the entanglement Hamiltonian and they are related by a unitary transformation. Λ_k is related to λ_k as $\Lambda_k = \frac{c-2\pi k}{n}$. We can now utilize our previous multidimensional bosonization result for the generating function of particle number cumulants Eq. (17) with $\lambda = \Lambda_k$. We thus arrive at

$$\begin{aligned} S_A^{(n)}(c) &= \exp\left[-\frac{k_F I(R)}{(2\pi)^2}\sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \Lambda_k^2\right] \\ &= e^{-\frac{n^2-1}{12n}k_F I(R)} e^{-\frac{c^2}{4n\pi^2}k_F I(R)}. \end{aligned} \tag{35}$$

Since $I(R) = \frac{2\pi^2}{k_F}V_A^{(2)}$, the larger the variance, the smaller the charged Rényi entropy.

The Fourier transform of charged entanglement $S_A^{(n)}(c)$ gives the symmetry-resolved Rényi entropy for particle number N_A:

$$S_A^{(n)}(N_A) = S_A^{(n)}(c=0)\int_{-\pi}^{\pi}\frac{dc}{2\pi}\, e^{-\frac{k_F I(R)}{n}\left(\frac{c}{2\pi}\right)^2 - icN_A}. \tag{36}$$

Assuming $R\ln R \gg 1$, the integrand is negligible for large c, so we might as well extend the integration region to $\mathbb{R}$ and get a Gaussian integral, leading to

$$S_A^{(n)}(N_A) = \sqrt{\frac{\pi n}{k_F I(R)}}\, e^{-\frac{k_F I(R)}{12}\frac{n^2-1}{n} - \frac{n\pi^2 N_A^2}{k_F I(R)}}. \tag{37}$$

Finally, the entanglement entropy for a given particle number N_A is

$$S_A(N_A) = \frac{1}{6}\sqrt{2\pi k_F R\ln\frac{R}{\alpha}}\, e^{-\frac{\pi^2 N_A^2}{2k_F R\ln\frac{R}{\alpha}}}, \tag{38}$$

Dividing Eq. (37) by $\mathrm{Tr}(\rho_A^n)$ gives a normalized symmetry-resolved Rényi entropy which can be used to compute the accessible entanglement entropy discussed in Ref. [34]. where we kept only the leading order term in the final expression. Summing up the contributions from all the particle number sectors, we recover the von Neumann entropy for subsystem A:

$$\int dN_A S_A(N_A) = \frac{k_F}{3}R\ln\frac{R}{\alpha} = S_A. \tag{39}$$

VI. CONCLUSION

In conclusion, we applied multidimensional bosonization to compute the generating function of the particle number cumulants of a circular subregion A for a two-dimensional Fermi gas. This generating function is then used to compute the Rényi entropy of the Fermi gas, which agrees with known results. These quantities show a logarithmic violation of the area law. We then proceed to compute the symmetry-resolved entanglement of the two-dimensional Fermi gas, extending the results in Ref. [19] to two spatial dimensions. Each charge sector is observed to give a $\sqrt{R \ln R}$ contribution to the total von Neumann entanglement entropy, which scales as $R \ln R$. The success of multidimensional bosonization in computing these quantities suggests that one could try to apply multidimensional bosonization to compute other quantities in a nonperturbative *ab initio* approach.

It should, in principle, be possible to extend the analysis to higher dimensions, although the integrals in Eq. (14) become a lot more complicated. To make progress in this direction, a more efficient approach to evaluating the integrals is required.

While we focused on a noninteracting Fermi gas, the power of bosonization lies in its ability to treat Fermi liquid interactions. Before closing, we give a brief comment on the effects of Fermi liquid interactions on the particle number cumulants. Utilizing the formalism of Ref. [18], we computed the particle number cumulants' generating function with an isotropic contact interaction (with a spherical Fermi surface and a spherical entangling surface, as in the case of the free Fermi gas). In this computation, the effects of interactions can be incorporated by a Bogoliubov transformation, which relates the modes that diagonalize the interacting Hamiltonian to the noninteracting modes, thereby realizing Landau's adiabatic principle. Unfortunately, we found that the calculation is plagued by an infrared (IR) divergence. The best way to deal with the IR divergence is at this moment unclear, and left for future investigation. We suspect that the IR divergence might be cured with an improved treatment of collective modes within the bosonization framework. Heuristically dropping the IR divergence (appearing in a subleading contribution) by hand, we found that the coefficient of the leading logarithmic term decreases as we turn on interactions,

$$\langle e^{i\lambda \hat{N}_A} \rangle = \exp\left[-\frac{\lambda^2 k_F}{2\pi^2} \left(1 - \frac{1}{2} \left(\frac{gN}{1+gN} \right)^2 \right) R \ln \frac{R}{\alpha} + \cdots \right], \tag{40}$$

where g is the dimensionless coupling constant. In particular, it approaches half of the noninteracting value in the limit of infinite coupling. This decrease is consistent with the known result in one-dimensional Tomonaga-Luttinger liquids [23]. It would be interesting to calculate the particle number cumulants numerically and compare them with the above findings.

Note added. Recently, Ref. [38] appeared on arXiv, where the symmetry-resolved entanglement for higher-dimensional Fermi gases was computed using Widom's conjecture.

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[1] J. Eisert, M. Cramer, and M. B. Plenio, Colloquium: Area laws for the entanglement entropy, Rev. Mod. Phys. **82**, 277 (2010).

[2] N. Laflorencie, Quantum entanglement in condensed matter systems, Phys. Rep. **646**, 1 (2016).

[3] J. I. Latorre and A. Riera, A short review on entanglement in quantum spin systems, J. Phys. A: Math. Theor. **42**, 504002 (2009).

[4] D. A. Abanin, E. Altman, I. Bloch, and M. Serbyn, Colloquium: Many-body localization, thermalization, and entanglement, Rev. Mod. Phys. **91**, 021001 (2019).

[5] S. Ryu and T. Takayanagi, Aspects of holographic entanglement entropy, J. High Energy Phys. 08 (2006) 045.

[6] T. Nishioka, S. Ryu, and T. Takayanagi, Holographic entanglement entropy: An overview, J. Phys. A: Math. Theor. **42**, 504008 (2009).

[7] M. Rangamani and T. Takayanagi, *Holographic Entanglement Entropy*, Lecture Notes in Physics (Springer International Publishing, Cham, Switzerland, 2017).

[8] M. Headrick, Lectures on entanglement entropy in field theory and holography, arXiv:1907.08126.

[9] T. Nishioka, Entanglement entropy: Holography and renormalization group, Rev. Mod. Phys. **90**, 035007 (2018).

[10] R. Orús, A practical introduction to tensor networks: Matrix product states and projected entangled pair states, Ann. Phys. **349**, 117 (2014).

[11] D. Gioev and I. Klich, Entanglement Entropy of Fermions in Any Dimension and the Widom Conjecture, Phys. Rev. Lett. **96**, 100503 (2006).

[12] M. M. Wolf, Violation of the Entropic Area Law for Fermions, Phys. Rev. Lett. **96**, 010404 (2006).

[13] B. Swingle, Entanglement Entropy and the Fermi Surface, Phys. Rev. Lett. **105**, 050502 (2010).

[14] B. Swingle, Rényi entropy, mutual information, and fluctuation properties of Fermi liquids, Phys. Rev. B **86**, 045109 (2012).

[15] W. Ding, A. Seidel, and K. Yang, Entanglement Entropy of Fermi Liquids Via Multidimensional Bosonization, Phys. Rev. X **2**, 011012 (2012).

[16] B. Lacroix-A-Chez-Toine, S. N. Majumdar, and G. Schehr, Rotating trapped fermions in two dimensions and the complex Ginibre ensemble: Exact results for the entanglement entropy and number variance, Phys. Rev. A **99**, 021602(R) (2019).

[17] A. H. Castro Neto and E. Fradkin, Bosonization of Fermi liquids, Phys. Rev. B **49**, 10877 (1994).

[18] A. H. Castro Neto and E. H. Fradkin, Exact solution of the Landau fixed point via bosonization, Phys. Rev. B **51**, 4084 (1995).

[19] M. Goldstein and E. Sela, Symmetry-Resolved Entanglement in Many-Body Systems, Phys. Rev. Lett. **120**, 200602 (2018).

[20] A. Houghton, H.-J. Kwon, and J. B. Marston, Multidimensional bosonization, Adv. Phys. **49**, 141 (2000).

[21] P. Kopietz, *Bosonization of Interacting Fermions in Arbitrary Dimensions*, Lecture Notes in Physics Monographs (Springer-Verlag, Berlin, 1997).

[22] P. Calabrese, M. Mintchev, and E. Vicari, Exact relations between particle fluctuations and entanglement in Fermi gases, Europhys. Lett. **98**, 20003 (2012).

[23] H. F. Song, S. Rachel, C. Flindt, I. Klich, N. Laflorencie, and K. Le Hur, Bipartite fluctuations as a probe of many-body entanglement, Phys. Rev. B **85**, 035409 (2012).
[24] I. Klich, Lower entropy bounds and particle number fluctuations in a Fermi sea, J. Phys. A: Math. Gen. **39**, L85 (2006).
[25] P. Calabrese, M. Mintchev, and E. Vicari, Entanglement Entropy of One-Dimensional Gases, Phys. Rev. Lett. **107**, 020601 (2011).
[26] P. Calabrese, M. Mintchev, and E. Vicari, Entanglement entropies in free-fermion gases for arbitrary dimension, Europhys. Lett. **97**, 20009 (2012).
[27] H. Shapourian, K. Shiozaki, and S. Ryu, Partial time-reversal transformation and entanglement negativity in fermionic systems, Phys. Rev. B **95**, 165101 (2017).
[28] S. Matsuura, X. Wen, L.-Y. Hung, and S. Ryu, Charged topological entanglement entropy, Phys. Rev. B **93**, 195113 (2016).
[29] I. Marvian, Symmetry-protected topological entanglement, Phys. Rev. B **95**, 045111 (2017).
[30] A. Belin, L.-Y. Hung, A. Maloney, S. Matsuura, R. C. Myers, and T. Sierens, Holographic charged Rényi entropies, J. High Energy Phys. 12 (2013) 059.
[31] A. Belin, L.-Y. Hung, A. Maloney, and S. Matsuura, Charged Rényi entropies and holographic superconductors, J. High Energy Phys. 01 (2015) 059.
[32] G. Pastras and D. Manolopoulos, Charged Rényi entropies in CFTs with Einstein-Gauss-Bonnet holographic duals, J. High Energy Phys. 11 (2014) 007.
[33] P. Caputa, M. Nozaki, and T. Numasawa, Charged entanglement entropy of local operators, Phys. Rev. D **93**, 105032 (2016).
[34] H. Barghathi, C. M. Herdman, and A. Del Maestro, Rényi Generalization of the Accessible Entanglement Entropy, Phys. Rev. Lett. **121**, 150501 (2018).
[35] H. Barghathi, E. Casiano-Diaz, and A. Del Maestro, Operationally accessible entanglement of one-dimensional spinless fermions, Phys. Rev. A **100**, 022324 (2019).
[36] S. Murciano, G. Di Giulio, and P. Calabrese, Symmetry resolved entanglement in gapped integrable systems: A corner transfer matrix approach, SciPost Phys. **8**, 046 (2020).
[37] M. Salmhofer, *Renormalization: An Introduction*, Theoretical and Mathematical Physics (Springer, Berlin, 2007).
[38] S. Fraenkel and M. Goldstein, Symmetry resolved entanglement: Exact results in 1D and beyond, J. Stat. Mech. (2020) 033106.