# Kosterlitz-Thouless phase and $Z_d$ topological quantum phase

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It has been known that encoding Boltzmann weights of a classical spin model in amplitudes of a many-body wave function can provide quantum models whose phase structure is characterized by using classical phase transitions. In particular, such correspondence can lead to finding new quantum phases corresponding to well-known classical phases. Here, we investigate this problem for the Kosterlitz-Thouless (KT) phase in the $d$-state clock model, where we find a corresponding quantum model constructed by applying a local invertible transformation on a $d$-level version of Kitaev's toric code. In particular, we show the ground-state fidelity in such a quantum model is mapped to the heat capacity of the clock model. Accordingly, we identify an extended topological phase transition in our model in the sense that, for $d \geqslant 5$, a KT-like quantum phase emerges between a $Z_d$ topological phase and a trivial phase. Then, using a mapping to the correlation function in the clock model, we introduce a nonlocal (string) observable for the quantum model which exponentially decays in terms of distance between two end points of the corresponding string in the $Z_d$ topological phase, while it shows a power law behavior in the KT-like phase. Finally, using well-known transition temperatures for the $d$-state clock model, we give evidence to show that while the stability of both the $Z_d$ topological phase and the KT-like phase increases by increasing $d$, the KT-like phase is even more stable than the $Z_d$ topological phase for large $d$.

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## I. INTRODUCTION

Characterizing different phases of matter is a central problem in condensed-matter physics [1]. While this problem seems well established in classical physics, it is specifically challenging in quantum physics, where a completely different property, namely, entanglement, plays a very specific role [2]. This problem has led to cross-fertilization between condensed-matter physics and quantum information theory, where using concepts from quantum information theory, one is able to characterize a quantum phase transition [3–5]. In particular, besides different measures provided by quantum information theory, the ground-state fidelity has attracted much attention in recent decades, where a quantum phase transition can be well characterized by a singularity in the ground-state fidelity [6–8].

Among different quantum phases, topological quantum phases have attracted much attention because of their applications in fault-tolerant quantum computation. A well-known example is Kitaev's toric code [9], which shows a robust topological degeneracy against any local perturbation [10–12]. The Abelian $d$-level version of this model, which we call a $Z_d$ Kitaev model, is also interesting because it has been used as a string-net model with a $Z_d$ topological phase [13–16] which is even more robust than the $Z_2$ one [17,18]. In spite of the above important applications, since topological phases have a nonlocal order and do not follow the symmetry-breaking paradigm of Landau [19], their characterization is a challenging task. Different approaches for characterizing topological phases [20–23] are based on the fact that the nonlocal nature of a topological order should lead to stability against any local transformation. In particular, considering local stochastic transformations [i.e., local invertible (LI) transformations] on entangled states was recently shown to be a very important approach in which topological phases are stable against small LI transformations [24].

On the other hand, recently, it has been common to consider some quantum phase transitions by mapping to classical phase transitions [25–35]. An interesting idea behind such mappings is that the thermal fluctuations are mapped to quantum fluctuations by encoding Boltzmann weights of a classical model in amplitudes of a quantum entangled state. In particular, it has been shown that the ground-state fidelity in such entangled states is mapped to the heat capacity of the corresponding classical models [36,37]. Therefore, any singularity in the heat capacity corresponds to a singularity in the ground-state fidelity, and the corresponding quantum and classical phase transitions will be related to each other.

Motivated by the above classical-quantum mappings, one can ask whether it is possible to find new quantum phases by considering such mappings for various classical spin models with well-known classical phases. In particular, one of the most interesting classical phases is the Kosterlitz-Thouless (KT) phase, which was originally seen in the classical $X$-$Y$ model [38]. Such a phase has also been seen in the classical $d$-state clock model, where, for $d \geqslant 5$, a KT phase emerges between the ferromagnetic and paramagnetic phases [39–42]. In this respect, in order to find a quantum phase corresponding to the KT phase, it is enough to find a quantum entangled

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state corresponding to the classical clock model. Interestingly, we show that such a quantum state is a simple deformation of the $Z_d$ Kitaev state, where it is constructed by applying an LI transformation in the $Z_d$ Kitaev state, and can also be considered the ground state of a quasi-Hermitian Hamiltonian [43–46]. We explicitly prove that the ground-state fidelity of such a quantum model is mapped to the heat capacity of the clock model. Therefore, we identify an extended topological phase transition in the deformed Kitaev model where a KT-like quantum phase emerges between the $Z_d$ topological phase and a trivial phase.

We also provide a mapping from the correlation function in the clock model to a nonlocal (string) observable [47] in the quantum model. Accordingly, we show that such a string parameter can characterize the KT-like phase in the sense that it exponentially decays in terms of distance between two end points of the corresponding string in the $Z_d$ topological phase, while it shows a power law behavior in the KT-like quantum phase. Furthermore, using the fact that for $d \leqslant 4$ the clock model shows a single ferromagnetic-paramagnetic phase transition, we conclude that in this case the KT-like quantum phase disappears in the quantum model. Finally, we also use the well-known transition temperatures for the clock model for different values of $d$ [39–41] to derive transition points of our quantum model. In particular, using an interpretation of transition points as a measure of stability against small LI transformation, we give evidence which shows that the stability of both the $Z_d$ topological phase and KT-like phase increases by increasing $d$. Furthermore, we show that the KT-like phase is even more stable than the $Z_d$ topological phase for large $d$.

This paper is structured as follows: In Sec. II, we give a brief review of the $Z_d$ Kitaev state and its corresponding Hamiltonian. In Sec. III, we introduce the deformed Kitaev model constructed by an LI transformation on the $Z_d$ Kitaev model. Then in Sec. IV, we prove the correspondence between the ground-state fidelity and the heat capacity, and finally, in Sec. V, we give the main results of the paper, where we use well-known facts about the clock model to identify the KT-like quantum phase.

## II. $Z_d$ KITAEV STATE

The $Z_d$ Kitaev state is defined as a simple generalization of Kitaev's toric code state where qubits are replaced by $d$-level quantum systems called qudits [48–51]. Therefore, consider a two-dimensional $L \times L$ square lattice with qudits living on the edges of the lattice. Furthermore, we also label each edge by a direction [for example, see Fig. 1(a)]. Like the qubit case, the $Z_d$ Kitaev state is also a stabilizer state which is stabilized by a generalized stabilizer group on $N$ qudits. A generalized stabilizer group is a subgroup of generalized Pauli group $P_N$ which is constructed by the product of $d$-level Pauli operators that commute with each other. The $d$-level Pauli operator of $Z$ is a diagonal matrix with eigenvalues in the form $1, \omega, \omega^2, \dots, \omega^{(d-1)}$, where $\omega = \exp\{2\pi i/d\}$ in the sense that if we denote eigenstates of $Z$ by $|m\rangle$, we will have $Z = \sum_{m=0}^{d-1} \omega^m |m\rangle\langle m|$. The Pauli operator of $X$ is also a ladder operator in the form of $X = \sum_m |m+1\rangle\langle m|$

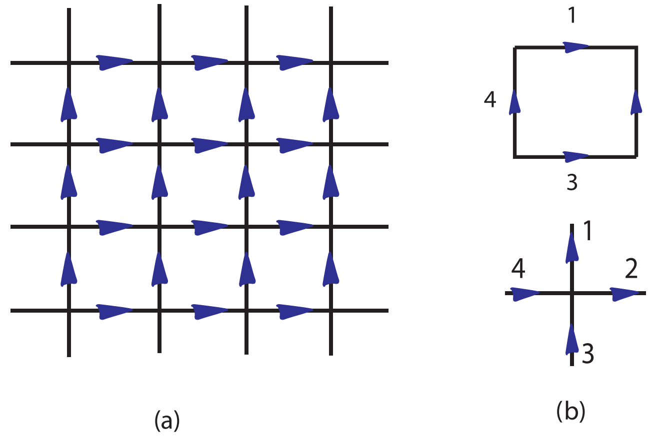


FIG. 1. (a) A square lattice with qudits living on the edges. Each edge is labeled by a direction. (b) A plaquette operator of $Z_1Z_2^{-1}Z_3^{-1}Z_4$ and a vertex operator of $X_1X_2X_3^{-1}X_4^{-1}$ are defined, corresponding to each plaquette and vertex of the lattice, respectively.

and $X|m\rangle = |m+1\rangle$. It is clear that these generalized Pauli operators are not Hermitian, but they are unitary in the sense that $XX^{-1} = 1$ and $ZZ^{-1} = 1$. Furthermore, since $\omega^d = 1$, one can conclude that $X^d = Z^d = 1$. Finally, one can check that there is a commutation relation between the $X$ and $Z$ operators in the form of $ZX = \omega XZ$.

Now we are ready to introduce stabilizers of the $Z_d$ Kitaev state. To this end, we consider a plaquette of the lattice as shown in Fig. 1(b). Corresponding to each plaquette of the lattice, a plaquette operator is defined in the following form:

$$B_p = Z_1Z_2^{-1}Z_3^{-1}Z_4. \tag{1}$$

If we turn the plaquette around in a clockwise direction, the above stabilizer can be written in a general form as $\prod_{e\in\partial p} Z_e^{\sigma_e}$, where $e \in \partial p$ refers to edges around the plaquette $p$ and $\sigma_e$ is equal to 1 if the direction of that edge is matched with the clockwise direction and is equal to $-1$ otherwise. The advantage of such a general definition is that it is independent of directions that we had considered for edges in Fig. 1(a), and it works for any given direction for edges. Then, corresponding to each vertex of the lattice [see Fig. 1(b)], a vertex operator is defined in the following form:

$$A_v = X_1X_2X_3^{-1}X_4^{-1}. \tag{2}$$

This operator can also be written in a compact form as $\prod_{e\in v} X_e^{\gamma_e}$, where $e \in v$ refers to edges connected to the vertex $v$ and $\gamma_e$ is equal to $-1$ if the direction of the edge $e$ is incoming to the vertex $v$ and is equal to $+1$ otherwise. According to the commutation relation of generalized Pauli operators and since each vertex operator has zero or two qudits in common with plaquette operators, it is simple to check that vertex and plaquette operators commute with each other. Therefore, vertex and plaquette operators generate a stabilizer group.

Then, the $Z_d$ Kitaev state, denoted by $|K_d\rangle$, is defined as a quantum state which is stabilized by vertex and plaquette operators where $B_p|K_d\rangle = |K_d\rangle$ and $A_v|K_d\rangle = |K_d\rangle$. Up to a normalization factor, such a state can be written in the following form:

$$|K_d\rangle = \prod_v \left(1 + A_v + A_v^2 + \cdots + A_v^{d-1}\right)|0\rangle^{\otimes N} \quad (3)$$

where $N = 2L^2$ is total number of qudits. In order to show that the above state is stabilized by $A_v$'s and $B_p$'s, it is enough to note that $A_v^d = 1$ and therefore we have $A_v(1 + A_v + A_v^2 + \cdots + A_v^{d-1}) = (1 + A_v + A_v^2 + \cdots + A_v^{d-1})$. Furthermore, since $B_p$ commutes by $A_v$'s and $B_p|0\rangle^{\otimes N} = |0\rangle^{\otimes N}$, it is simply concluded that $B_p|K_d\rangle = |K_d\rangle$. Furthermore, in the same way, one can prove that the $Z_d$ Kitaev state can also be written in the following form, up to a normalization factor:

$$|K_d\rangle = \prod_p \left(1 + B_p + B_p^2 + \cdots + B_p^{d-1}\right)|+\rangle^{\otimes N}, \quad (4)$$

where $|+\rangle = \frac{1}{\sqrt{d}}(|0\rangle + |1\rangle + \cdots + |d-1\rangle)$ is the eigenstate of Pauli operator $X$.

On the other hand, the $Z_d$ Kitaev state can also be considered a ground state of a Hermitian Hamiltonian in the following form:

$$H_0 = -\sum_p \left(B_P + B_P^{-1}\right) - \sum_v \left(A_v + A_v^{-1}\right). \quad (5)$$

Interestingly, it has been shown that when the square lattice has a periodic boundary condition on a torus, the above Hamiltonian shows a topological degeneracy where $|K_d\rangle$ is only one of the ground states of the system. It is shown that other ground states can be constructed by applying a few nonlocal $X$-type operators corresponding to noncontractible loops around the torus [9]. Therefore, it is concluded that the above degeneracy is robust against any local perturbation. Such a robustness is, in fact, a universal signature of all topological phases.

From a microscopic perspective, it has been shown that topological order in the $Z_d$ Kitaev state is related to a condensation of string nets with a long-range entanglement. It is especially simple to see string nets in Eq. (3). To this end, note that if we expand the operator of $\prod_v(1 + A_v + A_v^2 + \cdots + A_v^{d-1})$ where it is product on all vertices of the lattice, it will be a summation of all $X$-type stabilizers of the $Z_d$ Kitaev state which are constructed by products of $A_v$'s. On the other hand, each vertex operator of $A_v$ can be represented by a loop on the dual lattice, as shown in Fig. 2. However, since each power of $A_v$ in the form of $A_v^m$ for $m = \{1, 2, \ldots, d-1\}$ is also a stabilizer, we should represent such stabilizers by weighted loops, where the weight of each loop will correspond to a power of $m$. Furthermore, a product of weighted vertex operators for different vertices will also be a stabilizer. Such stabilizers are also represented by the product of weighted loops. In particular, if two weighted loops have a common edge, their product should be represented by a different structure in the sense that the weight of the common edge will be equal to the summation of weights of two initial loops (see Fig. 2). Generally, such structures are called string networks, and each stabilizer of the $Z_d$ Kitaev state corresponds to one such string network. In this way, it is simple to see that the $Z_d$ Kitaev state is a superposition of string networks, and therefore, it is called a string-net condensed state.

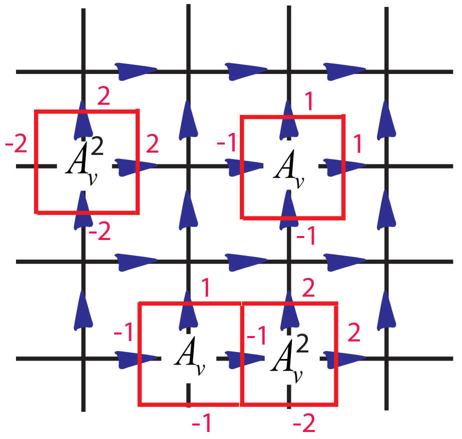

FIG. 2. A vertex operator is represented by a loop on the dual lattice where there is also a weight for each edge of the loop corresponding to the power of the $X$ operator in $A_v$. $A_v^2$ is also represented by a loop with different weights corresponding to the power of $X$ in $A_v^2$. A product of $A_v$ and $A_v^2$ corresponding to two neighboring vertices is represented by a string net, where the weight of the common edge of two initial loops is derived by a summation of the initial weights.

## III. LOCAL INVERTIBLE TRANSFORMATION ON THE $Z_d$ KITAEV STATE: A DEFORMED KITAEV STATE

In this section, we introduce a deformation of the $Z_d$ Kitaev state by applying an LI transformation; see also [15,16] for similar deformations. To this end, consider an invertible transformation on a single qudit in the form of $\exp\{\frac{\beta}{2}(Z + Z^{-1})\}$, where $\beta$ is a positive real number. Then we consider an LI transformation as the product of $\exp\{\frac{\beta}{2}(Z + Z^{-1})\}$ on all qudits of the $Z_d$ Kitaev state in the following form:

$$|K_d\rangle \to \exp\left\{\frac{\beta}{2}\sum_i \left(Z_i + Z_i^{-1}\right)\right\}|K_d\rangle. \quad (6)$$

Since the invertible transformation is not unitary, it does not preserve the norm of the quantum state. However, we can add a normalization factor to the final state to have a normalized state denoted by $|K_d(\beta)\rangle$. Note that for an arbitrary value of $\beta$ the above transformation can be considered a sequence of small transformations which gradually map the initial state to the final state. In particular, it is interesting to consider the final state when $\beta \to \infty$. To this end, note that since $Z = \sum_{m=0}^{d-1} \omega^m |m\rangle\langle m|$, one can show that $\exp\{\frac{\beta}{2}(Z + Z^{-1})\} = \sum_{m=0}^{d-1} \exp\{\beta \cos\frac{2\pi m}{d}\}|m\rangle\langle m|$, which can be written in the form $\exp\{\beta\}(|0\rangle\langle 0| + \exp\{\beta(\cos\frac{2\pi}{d} - 1)\}|1\rangle\langle 1| + \cdots + \exp\{\beta(\cos\frac{2\pi(d-1)}{d} - 1)\}|d-1\rangle\langle d-1|)$. Then, since $\cos x \leqslant 1$, for $\beta \to \infty$ the above operator converts to a projective operator of $|0\rangle\langle 0|$. Using this fact, we conclude that the invertible transformation on the $Z_d$ Kitaev state converts it to a trivial state of $|000\cdots 0\rangle$ for the limit of $\beta \to \infty$.

In this way, it seems that, by an infinite sequence of small invertible transformations, we will be able to convert a topological state to a trivial state. This means that in the space of quantum states on $N$ qudits, we have a transition between two different quantum phases. On the other hand, it is clear that $|K(\beta)\rangle$ can be considered the ground state of a Hamiltonian which is constructed by applying the same LI transformation to $H_0$ in the following form:

$$H_\beta = \exp\left\{\frac{\beta}{2}\sum_i \left(Z_i + Z_i^{-1}\right)\right\} H_0 \exp\left\{-\frac{\beta}{2}\sum_i \left(Z_i + Z_i^{-1}\right)\right\}. \tag{7}$$

Since the invertible transformation is a similarity transformation, it preserves the real energy spectrum of the initial $Z_d$ Kitaev model. Therefore, if we denote eigenstates of $H_0$ by $|K_i\rangle$, eigenstates of $H_\beta$ will be in the form of $\exp\{\frac{\beta}{2}\sum_i(Z_i + Z_i^{-1})\}|K_i\rangle$ with the same eigenvalues. In particular, $|K_d(\beta)\rangle$ is the ground state of the above Hamiltonian.

We should also emphasize that $H_\beta$ is, in fact, a non-Hermitian Hamiltonian with real eigenvalues and with eigenstates which are not orthogonal. However, it is not an important problem because it has been shown that one can define a different metric for the definition of the inner product in the sense that the above eigenstates are orthogonal [46]. Such non-Hermitian Hamiltonians are physically meaningful and are called quasi-Hermitian Hamiltonians [43]. Regardless of different physical motivations behind quasi-Hermitian Hamiltonians, using the above quasi-Hermitian Hamiltonian is important for us because we will be able to consider the phase structure of $|K_d(\beta)\rangle$ as a quantum phase transition in a model Hamiltonian. In particular, since $|K_d(\beta)\rangle$ is the ground state of the above quasi-Hermitian Hamiltonian, we can compute the ground-state fidelity as a measure for characterizing a quantum phase transition.

## IV. GROUND-STATE FIDELITY AND MAPPING TO THE $d$-STATE CLOCK MODEL

As we mentioned in the previous section, using a quantum Hamiltonian with an exact ground state a simple way to characterize the quantum phase transition is to compute the ground-state fidelity. The ground-state fidelity is defined in the form of the inner product of two consecutive ground states, $|K_d(\beta)\rangle$ and $|K_d(\beta+\delta\beta)\rangle$, in the form $F = \langle K_d(\beta)|K_d(\beta+\delta\beta)\rangle$. In order to compute this quantity, note that the normalization factor in $|K_d(\beta)\rangle$ should be in the form $1/[\langle K_d|\exp\{\beta\sum_i(Z_i + Z_i^{-1})\}|K_d\rangle]^{\frac{1}{2}}$. If we denote $\langle K_d|\exp\{\beta\sum_i(Z_i + Z_i^{-1})\}|K_d\rangle$, which is a function of $\beta$, by $\mathcal{Z}(\beta)$, the ground-state fidelity will take the following form:

$$F = \frac{\langle K_d|\exp\left\{(\beta+\delta\beta/2)\sum_i\left(Z_i + Z_i^{-1}\right)\right\}|K_d\rangle}{\sqrt{\mathcal{Z}(\beta)}\sqrt{\mathcal{Z}(\beta+\delta\beta)}}. \tag{8}$$

Now, note that the numerator in the above equation is the same as $\mathcal{Z}(\beta+\delta\beta/2)$. Therefore, the ground-state fidelity will have the following simple form in terms of $\mathcal{Z}$:

$$F = \frac{\mathcal{Z}(\beta+\delta\beta/2)}{\sqrt{\mathcal{Z}(\beta)}\sqrt{\mathcal{Z}(\beta+\delta\beta)}}. \tag{9}$$

Now, we use a statistical mechanical mapping [52] and prove that the function of $\mathcal{Z}$ is related to the partition function of a classical $d$-state clock model. To this end, consider a classical clock model on a square lattice where $d$-state variables of $\theta = 2\pi n/d$ , $n = 0, 1, \ldots, d-1$, live on vertices and the classical

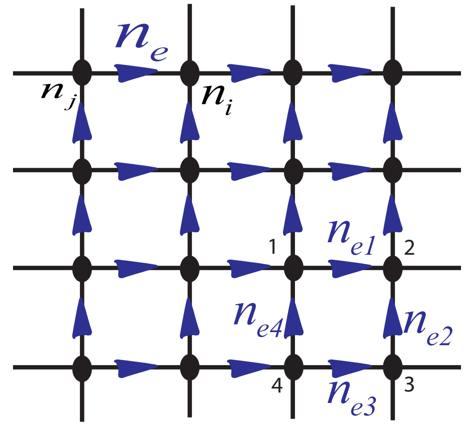


FIG. 3. The $d$-state clock model defined on a square lattice with $d$-state variables living on vertices. New edge variables of $n_e$ are defined that correspond to each edge of the lattice, in the form of $n_e = n_i - n_j$.

Hamiltonian is in the following form:

$$H_{cl} = -\sum_{\langle i,j\rangle}\cos(\theta_i - \theta_j), \tag{10}$$

where $\langle i, j\rangle$ refers to the interaction between the nearest neighbors. The partition function of such a system in terms of a finite temperature of $T$ has the following form:

$$\mathcal{Z}_{\text{clock}}(T) = \sum_{\{\theta_i\}}\exp\left\{\sum_{\langle i,j\rangle}\cos(\theta_i - \theta_j)/T\right\}, \tag{11}$$

where we set the Boltzmann constant $k_B$ equal to 1 and $\sum_{\{\theta_i\}}$ refers to the summation on all configurations of $d$-state variables. Then if we use the fact that $\cos(x) = \frac{\exp\{ix\}+\exp\{-ix\}}{2}$ and use the equation $\omega = \exp\{i2\pi/d\}$, we will have

$$\mathcal{Z}_{\text{clock}}(T) = \sum_{\{n_i\}}\exp\left\{\sum_{\langle i,j\rangle}\frac{\omega^{n_i-n_j} + \omega^{n_j-n_i}}{2T}\right\}, \tag{12}$$

where $n_i = 0, 1, \ldots, d-1$ refer to different values of the $d$-state variables of $\theta_i$. In the next step, we define new $d$-state variables corresponding to each edge of the square lattice in the form of $n_e = n_i - n_j$, which are called edge variables. We can replace these edge variables in the partition function relation. However, there is a point where new edge variables are not independent variables because of their definition in terms of $n_i$'s. For example, consider a square plaquette of the lattice as shown in Fig. 3, where there are four vertex variables, $n_1, n_2, n_3, n_4$. We consider a specific direction for each edge, as shown in Fig. 3, and accordingly, each edge variable is equal to a vertex variable living on the end point of the edge minus a vertex variable living on the first point of that edge, i.e., $n_{e1} = n_2 - n_1$, $n_{e2} = n_2 - n_3$, $n_{e3} = n_3 - n_4$, and $n_{e4} = n_1 - n_4$. By such a definition, it is concluded that there is a relation between edge variables corresponding to each plaquette in the form of $n_{e1} - n_{e2} - n_{e3} + n_{e4} = 0$. Using the same notation that we used for plaquette operators of the Kitaev state in Sec. II, the above constraint can be written in compact form as $\sum_{e\in\partial p}\sigma_e n_e = 0$. We apply these constraints using a set of $\delta$ functions in the partition function in the following form:

$$\mathcal{Z}_{\text{clock}}(T) = \sum_{\{n_e\}} \exp\left\{\sum_e \frac{\omega^{n_e}+\omega^{-n_e}}{2T}\right\}\prod_p \delta\left(\prod_{e\in\partial p}\omega^{\sigma_e n_e}, 1\right). \tag{13}$$

On the other hand, since $\omega = \exp\{i2\pi/d\}$, it is simple to check that $1+\omega+\omega^2+\cdots+\omega^{d-1}=0$. Therefore, it is concluded that the delta function $\delta(\omega^k, 1)$ can be written in the form of $\frac{1}{d}\sum_{m=0}^{d-1}\omega^{km}$. In this respect, we rewrite each $\delta$ function in Eq. (13) in the following form:

$$\delta\left(\prod_{e\in\partial p}\omega^{\sigma_e n_e}, 1\right) = \frac{1}{d}\sum_{m=0}^{d-1}\left(\prod_{e\in\partial p}\omega^{\sigma_e n_e}\right)^m. \tag{14}$$

Now we are ready to introduce a quantum formalism for the partition function of (13). To this end, note that $\omega^{n_e}$ is, in fact, an eigenvalue of the Pauli operator of $Z$ corresponding to the eigenstate of $|n_e\rangle$. Consequently, an arbitrary function in the form $\sum_{n_e} g(\omega^{n_e})$ can be rewritten in the form of $d\langle +|g(Z)|+\rangle$. Using the above equation, we are able to write the partition function in the following quantum language:

$$\mathcal{Z}_{\text{clock}}(T) = d^N\ {}^{N\otimes}\langle +|\exp\left\{\sum_e \frac{Z_e+Z_e^{-1}}{2T}\right\} \times \prod_p \frac{\sum_{m=0}^{d-1}\left(\prod_{e\in\partial p} Z_e^{\sigma_e}\right)^m}{d}|+\rangle^{\otimes N}. \tag{15}$$

Interestingly, the operator of $\prod_{e\in\partial p} Z^{\sigma_e}$ is the same as the plaquette operator of $B_p$ in the $Z_d$ Kitaev model. Therefore, it is concluded that the state of $\prod_p \frac{\sum_{m=0}^{d-1} B_p^m}{d}|+\rangle^{\otimes N}$ is the same as the $Z_d$ Kitaev state up to a normalization factor where it is stabilized by all $B_p$ and $A_v$ stabilizer operators. On the other hand, since $\prod_p \frac{\sum_{m=0}^{d-1} B_p^m}{\sqrt{d}}$ is a projective operator, it is concluded that it is equal to by replacing $(\prod_p \frac{\sum_{m=0}^{d-1} B_p^m}{\sqrt{d}})^2$ in Eq. (15), the partition function will be equal to an inner product in the following form:

$$\mathcal{Z}_{\text{clock}}(T) = d^{\frac{N}{2}}\langle K_d|\exp\left\{\sum_e \frac{Z_e+Z_e^{-1}}{2T}\right\}|K_d\rangle. \tag{16}$$

The above equation is exactly the same relation that we looked for. If we go back to Eqs. (8) and (9) for the ground-state fidelity, we can conclude that the normalization factor of $\mathcal{Z}(\beta)$ in the ground-state fidelity is the same as the partition function of the clock model in Eq. (16) where $\frac{1}{2T}$ in the classical model has been mapped to $\beta$ in the ground-state fidelity for the deformed Kitaev state. Furthermore, we should emphasize that the above mapping from the normalization factor of the deformed Kitaev state and partition function of the clock model is, in fact, a result of encoding Boltzmann weights of the clock model in amplitudes of the deformed Kitaev state. In other words, it is enough to write an expansion of the $Z_d$ Kitaev state in terms of string nets. Then if we apply the LI operator $\exp\{\frac{\beta}{2}\sum_i(Z_i+Z_i^{-1})\}$ to different terms of the above expansion, a square root of Boltzmann weights of the clock model appears.

Finally, we rewrite the ground-state fidelity in Eq. (9) in the following form:

$$F_{K-state}(\beta,\delta\beta) = \frac{\mathcal{Z}_{\text{clock}}(\beta+\delta\beta/2)}{\sqrt{\mathcal{Z}_{\text{clock}}(\beta)\mathcal{Z}_{\text{clock}}(\beta+\delta\beta)}}. \tag{17}$$

It is, in fact, a relation between the ground-state fidelity in the quantum model and the partition function of the clock model. On the other hand, since the parameter $\delta\beta$ in the above equation is very small, it is useful to perform a Taylor expansion for the state fidelity in terms of different powers of $\delta\beta$. By a simple calculation, we derive the following approximation for the ground-state fidelity:

$$F_{K-state}(\beta,\delta\beta) \simeq 1 - \frac{1}{8}\left(\frac{\partial^2 \ln(\mathcal{Z}_{\text{clock}})}{\partial\beta^2}\right)\delta\beta^2, \tag{18}$$

where we have ignored higher powers of $\delta\beta$. Furthermore, it is well known that the second derivation of the partition function is related to the heat capacity for any classical statistical model. Therefore, we find that the ground-state fidelity is related to the heat capacity of the clock model by the following relation:

$$F_{K-state}(\beta,\delta\beta) \simeq 1 - \frac{(C_v)_{\text{clock}}}{8\beta^2}\delta\beta^2. \tag{19}$$

We recall that we considered the ground-state fidelity to find a signature for the phase transition from the $Z_d$ Kitaev phase to the trivial phase. Now we have found that this quantity is related to the partition function and the heat capacity of a classical clock model. This means that if the partition function or the heat capacity of the classical model shows a singularity in the transition temperature $T_c$, the ground-state fidelity will also show a singularity in $\beta_c = \frac{1}{2T_c}$ which will be, indeed, a topological phase transition point in our quantum model.

## V. IDENTIFYING A KT-LIKE QUANTUM PHASE

In the previous section we established a mapping from the deformed Kitaev model to the clock model. In particular we found that the ground-state fidelity of the deformed Kitaev model is mapped to the partition function of the clock model (17). Then, by an expansion of the ground-state fidelity in terms of different powers of $\delta\beta$, we showed that the ground-state fidelity is related to the heat capacity of the clock model up to second order of $\delta\beta$ [Eq. (19)]. Using the mapping between the ground-state fidelity for the deformed Kitaev model and the heat capacity of the clock model, now we are able to consider the phase structure of our quantum system.

Fortunately, the clock model has been well studied in the literature in the sense that we have much information about its phase structure. Let us focus on the free energy and the heat capacity of this model, which are related to the partition function in the form of $A = -k_B T\ln(\mathcal{Z})$ and $C_v = k_B\beta^2\frac{\partial^2 ln\mathcal{Z}}{\partial\beta^2}$, respectively. It has been known that the clock model, for $d \geqslant 5$, shows two KT phase transitions [39–41] where a KT phase emerges between the ferromagnetic and paramagnetic phases (see Fig. 4). The indicator of the above KT phase transitions is the existence of an essential singularity in the free energy as a function of temperature at transition points denoted by

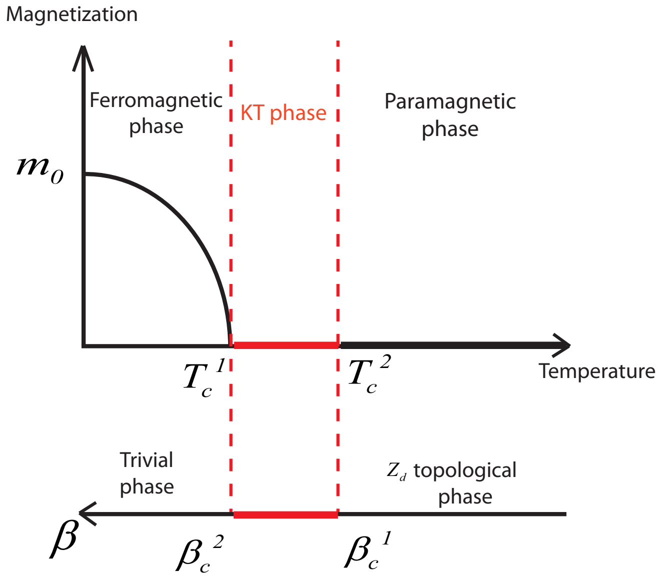


FIG. 4. A schematic of the phase diagram of the $d$-state clock model for $d \geqslant 5$ with two transition points, $T_c^1$ and $T_c^2$, where a KT phase emerges between the ferromagnetic and paramagnetic phases. Although the magnetization is zero in the KT phase, the correlation function $C(r)$ shows a power law behavior where it decays in the form of $r^{-\eta}$. Since $\beta$ in the deformed Kitaev model is equal to $\frac{1}{2T}$, the trivial phase and the $Z_d$ topological phase correspond to the ferromagnetic phase and the paramagnetic phase, respectively.

| | d-state clock model | | deformed Kitaev model | | reference |
|---|---|---|---|---|---|
| | $T_c^{(1)}$ | $T_c^{(2)}$ | $\beta_c^{(1)}$ | $\beta_c^{(2)}$ | |
| $d=2$ | 2.269 | 2.269 | 0.221 | 0.221 | [39] |
| $d=3$ | 1.492 | 1.492 | 0.336 | 0.336 | [39] |
| $d=4$ | 1.135 | 1.135 | 0.441 | 0.441 | [39] |
| $d=5$ | 0.906 | 0.952 | 0.525 | 0.552 | [40] |
| $d=6$ | 0.690 | 0.913 | 0.548 | 0.725 | [40] |
| $d=7$ | 0.531 | 0.907 | 0.551 | 0.942 | [40] |
| $d=8$ | 0.417 | 0.906 | 0.552 | 1.199 | [40] |
| $d\to\infty$ | 0 | 0.893 | 0.560 | $\infty$ | [50] |

FIG. 5. According to the denoted references, there are critical temperatures for the $d$-state clock model for $d \leqslant 8$ and $d \to \infty$. Note that temperatures are dimensionless in the form of $\frac{k_B T}{J}$ and we have set $J$ and $k_B$ equal to 1, where $J$ refers to the coupling constant of the clock model. The transition points of the quantum model are also determined by the equations $\beta_c^1 = \frac{1}{2T_c^2}$ and $\beta_c^2 = \frac{1}{2T_c^1}$ according to quantum-classical mapping.

$T_c^1$ and $T_c^2$ (see Fig. 5, where values of transition temperatures are given for a few values of $d$). The singular term of the free energy near transition points is in the form of $e^{-\frac{c}{t^{1/2}}}$, where $c$ is a positive constant and $t = \frac{T-T_c}{T_c}$ [53]. Such a singularity is called essential because all derivatives of finite order of the free energy with respect to $T$ are continuous and there is only a divergence in the derivative of infinite order. Furthermore, since the heat capacity is equal to the second derivation of the free energy, it is also concluded that there is an essential singularity in the heat capacity as a function of $T$ at the KT transition point, which is called a weak singularity [53–55].

Now, using the mapping between the ground-state fidelity and the partition function as well as the heat capacity, we conclude that there must be two singular points, $\beta_c^1 = \frac{1}{2T_c^2}$ and $\beta_c^2 = \frac{1}{2T_c^1}$, in the ground state of the deformed Kitaev model where the ground-state fidelity will show essential singularities as a function of $\beta$ (see Fig. 4). In this respect, we conclude that there are a trivial phase and a $Z_d$ topological phase in the quantum model corresponding to the ferromagnetic phase and the paramagnetic phase, respectively. However, for $d \geqslant 5$ there is not a simple phase transition from the $Z_d$ topological phase to the trivial phase, but they are separated from each other by an intermediate quantum phase.

Although singularities in the ground-state fidelity can reveal different natures of the intermediate quantum phase, we need to characterize this phase in terms of some observable. To this end, note that the trivial phase, the intermediate phase, and the $Z_d$ topological phase in the quantum model correspond to the ferromagnetic phase, the KT phase, and the paramagnetic phase in the clock model, respectively. On the other hand, it is known that three different phases in the $d$-state clock model can be characterized in terms of a correlation function in the form of $\langle \cos(\theta_k - \theta_l)\rangle$, where $\theta_k$ and $\theta_l$ are two arbitrary $d$-state variables. This quantity is a function of $r$, the distance between two of the above variables in the lattice, and we denote it by $C(r)$. In the ferromagnetic phase $C(r)$ shows a long-range order in which it goes to a nonzero value in the limit of $r \to \infty$, and it will be the same as the ferromagnetic order parameter. In the paramagnetic phase, the system does not have a long-range order, and specifically, $C(r)$ exponentially decays to zero. Interestingly, in the KT phase there is also no long-range order in the sense that the order parameter is equal to zero. However, the correlation function $C(r)$ decays to zero in an algebraical way in which it shows a power law behavior in the form of $r^{-\eta}$ [39]. In other words, the KT phase has a quasi-long-range order.

According to the above motivation, if we are able to find an observable in the quantum model corresponding to the correlation function in the clock model, such an observable can be considered a signature of the intermediate phase in the sense that it reveals different behaviors of the intermediate phase with trivial and $Z_d$ topological phases.

To this end, note that the correlation function $\langle \cos(\theta_k - \theta_l)\rangle = \langle \cos\frac{2\pi}{d}(n_k - n_l)\rangle$ is equal to the real part of $\exp\{i\frac{2\pi}{d}(n_k - n_l)\}$, which should be computed by the following relation:

$$\left\langle \exp\left\{ i\frac{2\pi}{d}(n_k - n_l)\right\}\right\rangle = \frac{\sum_{\{n_i\}} e^{i\frac{2\pi}{d}(n_k-n_l)} \exp\left\{\sum_{\langle i,j\rangle} \frac{\cos\frac{2\pi}{d}(n_i-n_j)}{T}\right\}}{\mathcal{Z}}. \quad (20)$$

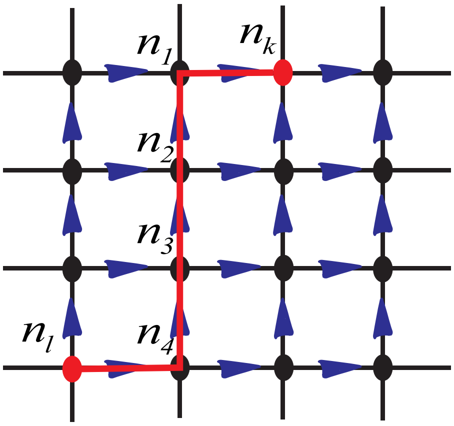


FIG. 6. A string $S$, denoted in red (light gray), connects two $d$-state variables of $n_k$ and $n_l$, where we consider a sequence of four $d$-state variables, $n_1$, $n_2$, $n_3$, and $n_4$, between them. By a mapping from classical vertex variables to quantum edge variables, the correlation function $\langle \exp\{i\frac{2\pi}{d}(n_k-n_l)\}\rangle$ is mapped to the expectation value of an operator of $\prod_{e\in S} Z_e$ corresponding to the string $S$.

Next, we rewrite the above equation in the quantum formalism by replacing vertex variables with edge variables. Everything is similar to the formalism that we used for the partition function in Sec. III. However, here we also have another term of $e^{i\frac{2\pi}{d}(n_k-n_l)}$ which should be rewritten in terms of edge variables. To this end, as shown in Fig. 6 as an example, we can consider a sequence, denoted by a string $S$, of $d$-state variables $n_1$ to $n_4$ between $n_k$ and $n_l$. Then we write $e^{i\frac{2\pi}{d}(n_k-n_l)}$ in the above equation in the following form:

$$e^{i\frac{2\pi}{d}(n_k-n_1)}e^{i\frac{2\pi}{d}(n_1-n_2)}e^{i\frac{2\pi}{d}(n_2-n_3)}e^{i\frac{2\pi}{d}(n_3-n_4)}e^{i\frac{2\pi}{d}(n_4-n_l)}. \quad (21)$$

In this way, after the change in variable, Eq. (20) is written in terms of edge variables $n_e$ in the following form:

$$\left\langle \exp\left\{i\frac{2\pi(n_k-n_l)}{d}\right\}\right\rangle = \frac{\sum_{\{n_e\}}\prod_{e\in S} e^{i\frac{2\pi n_e}{d}} e^{\sum_e \frac{\cos(\frac{2\pi n_e}{d})}{T}} \prod_p \delta\left(\sum_{e\in p}\sigma^e n_e, 0\right)}{\mathcal{Z}}. \quad (22)$$

Next, it is enough to replace variables of $\omega^{n_e}$ by Pauli operators of $Z$ using the same mechanism that we used to derive Eq. (16). However, there is only a difference where we will have an operator in the form of $\prod_{e\in S} Z_e$ corresponding to $\prod_{e\in S} e^{i(\frac{2\pi n_e}{d})}$ in the above equation. In this way, the correlation function will take on the following form in the quantum language:

$$\langle e^{i(\theta_k-\theta_l)}\rangle = \frac{\langle K_d|\left(\prod_{e\in S} Z_e\right)\exp\left\{\sum_e \frac{Z_e+Z_e^{-1}}{2T}\right\}|K_d\rangle}{\langle K_d|\exp\left\{\sum_e \frac{Z_e+Z_e^{-1}}{2T}\right\}|K_d\rangle}. \quad (23)$$

Interestingly, the above relation is, in fact, the same as the expectation value of the operator of $\prod_{e\in S} Z_e$, which we call a string operator, in the deformed Kitaev state, i.e., $\langle K_d(\beta)|\prod_{e\in S} Z_e|K_d(\beta)\rangle$. In this way, the expectation value of a string operator in the form of $\frac{\prod_{e\in S} Z_e+\prod_{e\in S} Z_e^{-1}}{2}$, which we call a string parameter, behaves similar to the correlation function in the clock model. This means that if $r$ is the distance between two end points of the string $S$, the string parameter shows different behaviors as a function of $r$ for three different phases of the quantum model. In particular, when $r\to\infty$, it goes to a nonzero value in the trivial phase, it exponentially decays to zero for the $Z_d$ topological phase, and finally, it shows a power law behavior in the intermediate phase.

On the other hand, note that it has been shown that two quantum states which can be transformed into each other by small local invertible transformations are in the same topological class [24]. Therefore, since the intermediate phase shows a singular transition to the trivial phase during the small local invertible transformation, it seems that the intermediate phase must be a topological (nontrivial) phase. However, note that the intermediate phase is also distinguished from the $Z_d$ topological phase by a power law behavior instead of an exponential one. Finally, since the intermediate phase corresponds to the KT phase in the clock model and there is also a power law behavior which is also observed in the correlation function in the KT phase, we call the intermediate phase a KT-like quantum phase.

We should emphasize that for $d\leqslant 4$, the clock model shows a simple ferromagnetic-paramagnetic phase transition point. Specifically, for $d=2$ and $d=4$ there is a single critical point in the Ising universality class, and for $d=3$ the clock model will be a three-state Potts model with a single critical point. Therefore, for the corresponding quantum model the KT-like quantum phase disappears, and we will have a simple topological phase transition from the $Z_d$ topological phase to the trivial phase.

Finally, we also have the phase transition points for the deformed Kitaev model according to well-known critical temperatures for the clock model. In Fig. 5, according to a few recent papers in the literature, we show transition temperatures of the clock model for $d\leqslant 8$ and the corresponding transition points for the deformed Kitaev model. Furthermore, the clock model will be the same as the $X$-$Y$ model in $d\to\infty$ with a well-known phase transition point [38,56]. In particular, in the $X$-$Y$ model there is no ferromagnetic phase at finite temperature. On the other hand, the ferromagnetic phase in the classical model corresponds to the trivial phase of the quantum model. This means that for the deformed Kitaev model in $d\to\infty$, there are only a $Z_d$ topological phase and a KT-like phase in which the trivial phase occurs at an infinite value of $\beta$.

It is also important to interpret the above transition points in the deformed Kitaev model as a measure of stability of topological phases. To this end, note that it is well known that topological phases are stable against small LI transformations [24]. Therefore, it is expected that the $Z_d$ topological phase shows stability against the LI transformation that we considered in the sense that it must remain in the topological phase for small values of $\beta$. Consequently, we expect that by increasing $\beta$ the above stability breaks and a phase transition occurs. Therefore, we can interpret $\beta_c^1$ as a measure of stability of the $Z_d$ topological phase. Furthermore, $\beta_c^2$ will also be interpreted as a measure of the stability of the KT-like quantum phase. By these interpretations and according to the table in Fig. 5, we conclude that the stability of both the $Z_d$ topological phase and the KT-like phase increases by increasing $d$. In particular, the KT-like phase shows an infinite stability for $d \to \infty$. This also means that the KT-like quantum phase is more stable than the $Z_d$ topological phase against the above LI transformation for large $d$.

## VI. DISCUSSION

In mapping between classical spin models and quantum entangled systems, an interesting problem is to find quantum phase transitions corresponding to different well-known classical phase transitions. In this paper, we found an extended topological phase transition corresponding to the Kosterlitz-Thouless phase transition in the classical $d$-state clock model. On the one hand, we mapped the ground-state fidelity in a deformed Kitaev model to the heat capacity of the clock model and showed that there are three different phases in the deformed Kitaev model where an intermediate phase emerged between the $Z_d$ topological phase and the trivial phase. On the other hand, we mapped the correlation function in the clock model to a string parameter in the quantum model to characterize the intermediate phase in terms of an observable. We believe that the intermediate phase that we called a KT-like quantum phase might have important properties which have not already been seen in other quantum systems. In particular, while we know that the $Z_d$ topological phase has long-range entanglement, it is important to consider entanglement in the KT-like quantum phase. Specifically, we note that since the KT-like phase corresponds to a classical phase with a quasi-long-range order, it might be a topological phase with quasi-long-range entanglement, a problem that should be considered in future works.

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