Classical criticality establishes quantum topological order

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We establish an important duality correspondence between topological order in quantum many-body systems and criticality in ferromagnetic classical spin systems. We show how such a correspondence leads to a classical and simple procedure for characterization of topological order in an important set of quantum entangled states, namely the Calderbank-Shor-Steane (CSS) states. To this end, we introduce a particular quantum Hamiltonian which allows us to consider the existence of a topological phase transition from quantum CSS states to a magnetized state. We study the ground state fidelity in order to find nonanalyticity in the wave function as a signature of a topological phase transition. Since hypergraphs can be used to map *any* arbitrary CSS state to a classical spin model, we show that fidelity of the quantum model defined on a hypergraph H is mapped to the heat capacity of the classical spin model defined on dual hypergraph $\tilde{H}$. Consequently, we show that a ferromagnetic-paramagnetic phase transition in a classical model is mapped to a topological phase transition in the corresponding quantum model. We also show that magnetization does not behave as a local order parameter at the transition point while the classical order parameter is mapped to a nonlocal measure on the quantum side, further indicating the nonlocal nature of the transition. Our procedure not only opens the door for identification of topological phases via the existence of a local and classical quantity, i.e., critical point, but also offers the potential to classify various topological phases through the concept of universality in phase transitions.

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I. INTRODUCTION

One of the most important problems in condensed matter physics is to characterize the different phases of matter which are related to various kinds of order present in physical systems. The quantum counterpart of this problem is in particular interesting and has attracted much attention over the past decades [1]. While most well-known orders are classified based on symmetry properties of the system, there are physical systems with a topological order [2–9] associated with topological properties of the system instead of symmetries. In particular, topological order has a nonlocal nature in a sense that there is no local order parameter to characterize a topological phase transition under a symmetry-breaking mechanism. Consequently, unlike symmetry-breaking phases, characterizing topological phases has remained a challenging problem to this date [10–13].

Furthermore, topological order, due to its nonlocal nature, has been also an important concept in quantum information theory. It is specifically important in quantum error correction as a way to overcome the decoherence problem in quantum computers. Using general quantum error-correcting codes, specifically Calderbank-Shor-Steane (CSS) codes [14–17], one usually needs an active protocol for error correction. However, due to the topological nature of certain CSS codes, they exhibit self-correction [18–21], thus protecting information in a natural way. Generally, topological nature of various CSS codes makes them an immediate candidate for fault-tolerant quantum computing [22–27]. Therefore, being able to ascertain whether an arbitrary CSS code is topological or not is of fundamental importance particularly in their feasibility as quantum memory. One standard method is to identify measures such as topological entanglement entropy [28] or gap stability [29]. While such measures can well capture certain physical aspects of topological order, they nonetheless have a nonlocal nature. Therefore, a practical procedure for characterizing topological order for an arbitrary CSS quantum state by such measures remains a challenging problem [11].

On the other hand, during the past decade certain interesting maps from quantum entangled states, specially topological ones, to partition functions of classical spin models have been introduced [30–32] which have led to new developments in quantum information theory as well as statistical mechanics [33–38]. In the light of such mappings, one may explore the consequences of the existence of a phase transition in the classical spin models for the corresponding quantum entangled states. In particular the nonanalytic behavior associated with classical phase transitions should have important ramifications on the quantum side [39,40].

In this paper, we propose and prove that the existence of topological order in an arbitrary CSS state is identified by a critical ferromagnetic phase transition in the classical (dual) spin model. We therefore propose a simple mathematical procedure: Criticality in the classical partition function establishes topological order in the corresponding CSS states. This is important since it provides a classical local measure for a quantum nonlocal phenomenon. In order to prove this, we propose a quantum Hamiltonian model, the

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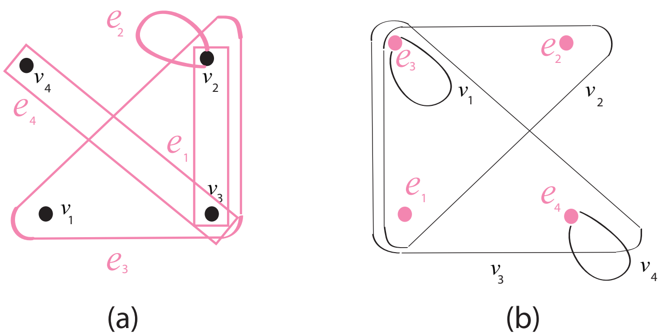

FIG. 1. (a) A simple hypergraph including four vertices denoted by blue (dark) circles and four hyperedges denoted by pink (light) curves. (b) Dual hypergraph of part (a).

CSS-magnetic (CSSM) model, which exhibits a transition as a function of control parameter (β), from the magnetized state to a CSS state. Due to the specific property of the ground state, we show that it cannot break any (spin) symmetries. We next show that the transition to the CSS state is indeed characterized by a singularity in the ground state fidelity and thus a quantum phase transition. We therefore conclude that the quantum phase transition must be of topological nature since it preserves the symmetry. As further evidence for the topological nature of the quantum phase transition, we also calculate the magnetization and show that it does not behave as a local (symmetry-breaking) order parameter, while the classical order parameter behaves as a nonlocal measure on the quantum side. In effect, we provide a duality between a (nonlocal) quantum topological phase transition and a (local) classical ferromagnetic phase transition.

II. CSS-MAGNETIC MODEL ON A HYPERGRAPH

A stabilizer state on N qubits is a positive eigenstate of N commutative operators belonging to the Pauli group. A CSS state is a particular set of stabilizer states which is stabilized by Z-type and X-type operators [16]. CSS states can be defined on hypergraphs [41–43]. A hypergraph $H = (V, E)$ is a set of vertices $V = \{v_1, v_2, ..., v_K\}$ and a set of hyperedges $E = \{e_1, e_2, ..., e_N\}$ where each hyperedge is equal to a subset of vertices, see Fig. 1(a). A set of hyperedges are called independent if no hyperedge is equal to collection of other hyperedges. For each hypergraph $H = (V, E)$, one can define a dual hypergraph $\tilde{H} = (\tilde{V}, \tilde{E})$ where $\tilde{V} = \{\tilde{v}_1, \tilde{v}_2, ..., \tilde{v}_N\}$ and $\tilde{E} = \{\tilde{e}_1, \tilde{e}_2, ..., \tilde{e}_K\}$ are vertices and hyperedges of $\tilde{H}$, respectively, and

$$\tilde{e}_i = \{\tilde{v}_m | v_i \in e_m \, in \, H\}, \tag{1}$$

where $v_i \in e_m \, in \, H$ refers to vertices belonging to the edge of e_m on H. Simply put, duality interchanges vertices and edges, see Fig. 1(b).

For a given hypergraph $H = (V, E)$, there is a CSS state in the following form:

$$|\text{CSS}_H\rangle = \frac{1}{2^{\frac{M}{2}}} \prod_{e \in E} (1 + A_e)|0\rangle^{\otimes N}, \tag{2}$$

where $|0\rangle$ is the positive eigenstate of the Pauli operator Z with N qubits living on vertices and $M \leqslant N$ is the number of independent hyperedges. A_e is an X-type stabilizer in the form of $\prod_{i \in e} X_i$ where $i \in e$ refers to all vertices belonging to e. Furthermore, there are $K = N - M$ number of Z-type stabilizers for the above state which commute with A_e [32]. A set of hyperedges corresponding to such Z-type stabilizers are defined as orthogonal hyperedges E^* where the CSS state of Eq. (2) can also be written in the following form:

$$|\text{CSS}_H\rangle = \frac{1}{2^{\frac{K}{2}}} \prod_{e^* \in E^*} (1 + B_{e^*})|+\rangle^{\otimes N}, \tag{3}$$

where $|+\rangle$ is the positive eigenstate of the Pauli operator X. e^* is a member of the set of orthogonal hyperedges E^* and B_{e*} is a Z-type stabilizer in the form of $\prod_{i \in e^*} Z_i$ where $i \in e^*$ refers to vertices belonging to e^*. Furthermore, it is simple to check that the above CSS state is also a nondegenerate ground state of a quantum CSS model with a Hamiltonian in the form of $h = -\sum_{e^* \in E^*} B_{e^*} - \sum_{e \in E} A_e$. We should note that while we can choose the X-type stabilizers as local operators, some Z-type stabilizers might be nonlocal. Therefore, the above Hamiltonian might be regarded as nonphysical. However, here we are only concerned with the existence of topological order in the ground state of such a Hamiltonian, namely the CSS state, and not whether such a Hamiltonian actually represents any physical system. In other words, corresponding to each CSS state we can construct a physical Hamiltonian by removing nonlocal operators from the Hamiltonian of h in a sense that there will be a degenerate ground space and the above CSS state will be only one of the ground states of the physical Hamiltonian. However, since topological order is a property of the wave function, we expect to see topological properties of the model in the above CSS state. Therefore, the existence of topological order in such a state means that its corresponding physical Hamiltonian will be a topological model with a topological degeneracy in the ground state.

We introduce an extended version of a CSS model, the CSS-magnetic (CSSM) model, corresponding to a given hypergraph $H = (V, E)$ in the following form:

$$\mathcal{H} = -\sum_{e^* \in E^*} B_{e^*} - \sum_{e \in E} A_e + \sum_{e \in E} U_e(\beta), \tag{4}$$

where $U_e(\beta) = \prod_{i \in e} \exp\{-\beta Z_i\}$ is a product operator corresponding to a hyperedge e with a tuning parameter β. If we expand this operator for a small value of β, it will correspond to a magnetic term in the first-order approximation. Additionally, the first two terms in Eq. (4) are stabilizers of a CSS state. Therefore, it is easy to see that the ground state of the above Hamiltonian goes through a transition from a CSS state ($\beta = 0$) to a magnetized state, $|0\rangle^{\otimes N}$, for $\beta \to \infty$. In this way, an important problem that needs be considered is the possibility of a quantum phase transition in this model. It is therefore important to identify the symmetry properties of the ground state of the CSSM model and ask if such a ground state exhibits any singular behavior as a function of β.

To find the ground state, we re-write the Hamiltonian in Eq. (4) in the following form:

$$\mathcal{H} = -\sum_{e* \in E*} B_{e*} + \sum_{e \in E} Q_e(\beta), \tag{5}$$

where $Q_e(\beta) = U_e - A_e$ which is a positive operator. One can check that $Q_e^2 = 2\cosh(\beta \sum_i Z_i)Q_e$ and since $\langle Q_e^2\rangle \geqslant 0$, one concludes that $\langle Q_e\rangle \geqslant 0$ for any arbitrary quantum state. This implies that the minimum eigenvalue of Q_e is zero.

On the other hand, since A_e and U_e commute with B_{e*}, it is clear that $[Q_e, B_{e^*}] = 0$, and therefore, an eigenstate of Q_e corresponding to an eigenvalue of zero will be the ground state of the CSSM. One can easily find the zero-eigenstate in the following form:

$$|G_H(\beta)\rangle = \frac{1}{\sqrt{\mathcal{Z}(\beta)}} \exp\{\frac{\beta}{2}\sum_i Z_i\}|\mathrm{CSS}_H\rangle, \tag{6}$$

where $\mathcal{Z}(\beta)$ is the normalization factor and $\sum_i$ in the exponential term refers to summation on all qubits. In order to show that the above state is in fact the ground state, first note that $U_e|G_H(\beta)\rangle = \frac{1}{\mathcal{Z}(\beta)}\exp\{-\frac{\beta}{2}\sum_{i\in e} Z_i\}\exp\{\frac{\beta}{2}\sum_{i\notin e} Z_i\}|\mathrm{CSS}_H\rangle$. Then, since $A_e \exp\{\frac{\beta}{2}\sum_{i\in e} Z_i\} = \exp\{-\frac{\beta}{2}\sum_{i\in e} Z_i\}A_e$, it follows that $A_e|G_H(\beta)\rangle = U_e|G_H(\beta)\rangle$ and therefore $Q_e|G_H(\beta)\rangle = 0$.

It remains to find the exact form of the normalization factor. We now show that $\mathcal{Z}(\beta)$ is the partition function of a classical ferromagnetic spin model defined on dual hypergraph $\tilde{H}$ with the following classical Hamiltonian:

$$\mathcal{H}_{cl} = -J\sum_{\tilde{e}\in\tilde{E}}\prod_{i\in\tilde{e}} s_i, \tag{7}$$

where J is the ferromagnetic coupling constant and $\prod_{i\in\tilde{e}} s_i$ refers to many-body interaction between binary spins belonging to a hyperedge $\tilde{e}$. Now, using Eq. (6), we have:

$$\mathcal{Z}(\beta) = \langle\mathrm{CSS}_H|\exp\{\beta\sum_i Z_i\}|\mathrm{CSS}_H\rangle. \tag{8}$$

On the other hand, according to Eq. (3), the CSS state can be written in terms of Z-type operators, and since B_{e*} operators commute with $\exp\{\beta\sum_i Z_i\}$ and all B_{e*}'s stabilize the CSS state, we will arrive at the following form for Eq. (8):

$$\mathcal{Z}(\beta) = 2^{\frac{K}{2}}\langle\alpha|\mathrm{CSS}_H\rangle, \tag{9}$$

where $|\alpha\rangle = \exp\{\beta\sum_i Z_i\}|+\rangle^{\otimes N} = 2^{-\frac{N}{2}}(e^{\beta}|0\rangle + e^{-\beta}|1\rangle)^{\otimes N}$. Indeed, such a product state of a CSS state defined on a hypergraph H is equal to the partition function of a classical ferromagnetic spin model defined on dual hypergraph $\tilde{H}$ up to a factor $2^{\frac{K}{2}-N}$, with β being related to the temperature in the classical model in the form of $T = \frac{J}{k_B\beta}$, as has been shown in Ref. [32]. In the following we set k_B and J to unity so that $T = \frac{1}{\beta}$.

Finally, we note that relation of the normalization factor in our model to the partition function of a classical model is in fact a result of specific direction of the quantum state. In other words, the unnormalized quantum state is in the form of $\exp\{\frac{\beta}{2}\sum_i Z_i\}|\mathrm{CSS}_H\rangle$. It is simple to check that if one expands the state $|\mathrm{CSS}_H\rangle$ in the computational basis of $|0\rangle$ and $|1\rangle$ and then applies the operator $\exp\{\frac{\beta}{2}\sum_i Z_i\}$, the above state will be a superposition of computational bases with weights which are related to the Boltzmann weights corresponding to different configurations of a classical spin model. It is exactly this reason that the normalization factor is mapped to the partition function as it is a summation of the Boltzmann weights.

Obtaining the exact form of the ground state, we are ready to consider the existence of a quantum phase transition in the CSSM. First, we consider the symmetries of the ground state in order to address the possibility of a symmetry breaking phase transition. We first consider the $\beta \to \infty$ limit where one can see that the ground state is a magnetized state, $|00\ldots0\rangle$. It is clear that, in this extreme, any Z-type operator is a symmetry operator of the ground state. However, when we decrease β to a finite value, only the Z-type operators which are stabilizers of the CSS state, i.e., B_{e^*}, remain as the symmetries of the ground state, independent of the *finite* value of β. Therefore, the only symmetry-breaking transition that might occur in the ground state must occur at $\beta \to \infty$, i.e., at zero temperature. All other finite β ground states of the CSSM model possess the symmetries of B_{e^*} operator. We conclude that any phase transition occurring at finite β cannot be a symmetry breaking transition and thus must be a topological phase transition. In fact, quantum topological phase transition is typically accompanied by long-range entanglement without any symmetry breaking property. The signature of such transitions are encoded in the ground state of the system and are usually studied using tools of quantum information theory such as measures of entanglement as well as fidelity [44–51]. Ground state fidelity as a function of β can encode such a transition and thus show a singular behavior. We next calculate such a quantity and show that is is equivalent to the heat capacity of the classical spin model which exhibits a singular behavior at the ferromagnetic phase transition at finite β, thus establishing the corresponding topological phase transition in the CSSM model.

III. GROUND STATE FIDELITY

The ground state fidelity will be a function of β and $\delta\beta$. However, since $\delta\beta$ is a very small quantity we can expand fidelity in terms of $\delta\beta$. We first consider the exact form of ground state fidelity as follows:

$$F = \frac{\langle\mathrm{CSS}_H|\exp\left\{\left(\beta+\frac{\delta\beta}{2}\right)\sum_i Z_i\right\}|\mathrm{CSS}_H\rangle}{\sqrt{\mathcal{Z}(\beta)\mathcal{Z}(\beta+\delta\beta)}}. \tag{10}$$

Since the inner product term in the above relation is again related to a partition function we will have:

$$F = \frac{\mathcal{Z}\left(\beta+\frac{\delta\beta}{2}\right)}{\sqrt{\mathcal{Z}(\beta)\mathcal{Z}(\beta+\delta\beta)}}. \tag{11}$$

After a Taylor expansion for the above equation, we find the following form up to the second order:

$$F(\beta,\delta\beta) \simeq 1 - \frac{1}{8}\left(\frac{\partial^2\ln(\mathcal{Z})}{\partial\beta^2}\right)\delta\beta^2, \tag{12}$$

where $\mathcal{Z}(\beta)$ is the partition function of a classical spin model on the hypergraph $\tilde{H}$. On the other hand, heat capacity of a classical spin model is given by $C_v = \frac{1}{T^2}\frac{\partial^2\ln(\mathcal{Z})}{\partial^2\beta}$ where β is the inverse temperature, $\beta = 1/T$. Therefore, we have shown that the ground state fidelity of the CSSM on a hypergraph H is related to the heat capacity of a classical spin model on dual hypergraph $\tilde{H}$:

$$F_H(\beta, \delta\beta) \simeq 1 - \frac{(C_v)_{\tilde{H}}}{8\beta^2}\delta\beta^2. \quad (13)$$

Now, it is known that if a classical spin model has a second-order phase transition at a critical temperature T_c, the heat capacity shows a singularity at T_c where it diverges according to $C_v \sim (T - T_c)^{-\alpha}$ where α is a critical exponent of the classical spin model [52]. We conclude that the ground state fidelity of a CSSM on hypergraph H shows a singularity at a critical value of β_c if its classical dual shows a phase transition at a finite temperature of $T_c = 1/\beta_c$. Therefore, a second-order phase transition in a ferromagnetic classical spin model establishes a quantum phase transition in the corresponding CSSM model, and since any finite β transition is a nonsymmetry breaking transition, it must therefore be a topological phase transition and thus indicating the existence of a topological phase in the CSS state. We should emphasize that such a procedure for identification of a topological phase transition works for any arbitrary CSS state. In the Appendix, we have provided two particular examples in order to clarify the usefulness of the procedure further.

IV. ORDER PARAMETER CONSIDERATIONS

Since CSS states with a well-defined thermodynamics limit are classified as gapped quantum liquids [13], and since such systems are known to have quantum phase transitions which are categorized as symmetry-breaking, first-order, trivial, or topological, we see that our method leads us to conclude that our quantum phase transition is indeed a topological one. However, we need to prove that our consideration of Z_2 symmetry is sufficient for the purpose of establishing a topological phase. To this end, we look for an indicator of topological phase transition in the CSSM in a form of an order parameter. First, let us consider the magnetization of the CSSM, defined as $m_H = \frac{\langle \sum_i Z_i \rangle}{N}$, which can be written as:

$$m_H = \frac{\langle \mathrm{CSS}_H | (\sum_i Z_i) \exp\{\beta \sum_i Z_i\} | \mathrm{CSS}_H \rangle}{N \mathcal{Z}(\beta)}. \quad (14)$$

We now replace the operator $\sum_i Z_i$ with a derivative of $\exp\{\beta \sum_i Z_i\}$. Then, the magnetization of the CSSM is related to the internal energy of a classical ferromagnetic spin model in the following form:

$$m_H = \frac{1}{N} \frac{1}{\mathcal{Z}(\beta)} \frac{\partial \mathcal{Z}}{\partial \beta} = -\frac{1}{N} \mathcal{E}_{\tilde{H}}, \quad (15)$$

where $\mathcal{E}_{\tilde{H}}$ refers to the internal energy of a classical ferromagnetic spin model defined on the hypergraph $\tilde{H}$. On the other hand, it is known that the internal energy of a ferromagnetic model does not behave like an order parameter as it displays a gradual (smooth) transition from a negative value to zero. Therefore, it is concluded that magnetization of the CSSM cannot be an order parameter, a fact that further implies the topological nature of the above transition.

On the other hand, one might expect to find an order parameter for the CSSM by considering the classical (local) order parameter, $\langle S_i \rangle$. In the mapping from classical partition function to CSS state, each multispin interaction of $\prod_{i \in e} S_i$ corresponding to the hyperedge e is mapped to a Pauli operator Z_e [32]. Therefore, if we perform an inverse mapping, each spin variable S_i will be equal to a product of Z_e operators. Therefore, the order parameter for the quantum phase transition in the CSSM model will be the expectation value of suitable products of Z_e operators. Note that while such an order parameter might be difficult to calculate, one can see that it must exist due to the nature of the inverse mapping. Now if such a quantum order parameter is local, it must possess Z_2 symmetry, which we have argued is impossible for our model. Therefore it must be nonlocal, thus proving that a nonlocal order parameter exists, which is sufficient to establish the existence of a topological phase.

V. DISCUSSIONS

The fact that quantum entangled states can be mapped to the partition function of classical spin models has many important consequences. One that has not attracted much attention in the literature is the consequence of singularities associated with criticality on the classical side. In this paper we have taken a step in this direction and have found that if the classical dual of a CSS state displays a critical point at a finite temperature T, the CSS state has topological order. We point out that evidence for such a correspondence already exists [32]. For example, the toric codes defined on arbitrary graphs are mapped to Ising models with ferromagnetic phase transition. GHZ states which do not have topological order are mapped to one-dimensional Ising models which do not show a phase transition. Another example is graph states without any topological order which are mapped to Ising models in the presence of magnetic field which do not show a phase transition. However, we have established that given an *arbitrary* CSS state, one can easily identify whether its classical counterpart has a ferromagnetic phase transition and thus conclude that it must have topological order. This procedure is simple and direct because the existence of a critical point is established either by simple observation or, in a more complicated case, by numerical simulations. Two concrete examples of such a procedure are discussed in the Appendix.

The problem of identifying topological phases is a challenging open problem. Our work offers some insights in this regard. For example, previously unknown topological states may be found by (inverse) mapping classically critical ferromagnetic models via hypergraphs. Also, our procedure can be used to check some controversial aspects of the topological nature of certain recently proposed CSS states such as X-cube model [53]. Furthermore, the fact that symmetry-breaking phase transition can be used to identify a symmetry preserving topological phase may have important consequences. As pointed out above, the potential for finding a nonlocal order parameter on the quantum side via mapping of a classical local order parameter offers an intriguing possibility. We also note that our CSSM model belongs to the well-known family of stochastic matrix form decomposition [30] where their classical-quantum correspondence has been studied for some two-dimensional models [36]. Our results may reveal other aspects of importance of such a family of states for studying topological properties of quantum states.

Another interesting possibility is the observation that fidelity was mapped to the heat capacity whose singularity is characterized by the exponent α. Standard statistical mechanics tells us that the divergence of heat capacity (and other nonanalytic behavior) at the critical point are *universal* in a sense that they depend only on the symmetries of the classical Hamiltonian and not on the details of interactions, etc. Thus, various different systems fall in the same universality class displaying the same exponents. It is well known that the scaling of quantum measures such as fidelity at the quantum phase transition point can be related to scaling of the correlation length and that the correlation length exponent can be used to define universality classes [54,55]. Therefore, the potential of applying the concept of universality, which is based on local symmetries of the classical model, to classify various topological phases offers an interesting prospect.

Finally, we note that the problem of topological phase transition and its relation to a wider class of quantum phase transitions has been studied by various authors before. Here, we have found a mechanism, by choosing a specific perturbation which preserves symmetry, to map a nonlocal quantum phase transition to a local classical phase transition and consequently use this mechanism as a diagnosis for the existence of topological phase in an important class of quantum states. Whether one can find similar mechanisms to embrace a more general class of quantum states poses an interesting possibility for future work.

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APPENDIX

In this Appendix, we consider two particular examples to show how a well-known classical ferromagnetic-paramagnetic phase transition can reveal a topological phase transition in a corresponding quantum CSSM model. In particular, we consider classical Ising models which have been well studied. In particular, since the one-dimensional classical Ising model does not show a phase transition at a finite temperature we expect that the corresponding quantum model does not show a topological phase transition. On the other hand, the two-dimensional Ising model has a finite temperature phase transition and there must be a topological phase transition in the corresponding quantum model.

Let us start with the one-dimensional Ising model where classical spins live in vertices of a one-dimensional periodic lattice and there is a two-body interaction between two neighboring spins. According to hypergraph duality explained in Ref. [32], this model is mapped to a GHZ state in the form of $\frac{1}{\sqrt{2}}(|++...+\rangle+|--...-\rangle)$ which is stabilized by X-type stabilizers in the form of X_iX_{i+1} and a nonlocal Z-type stabilizer in the form of $B=\prod_i Z_i$ where $\prod_i$ refers to all vertices of the lattice. Therefore, the corresponding CSSM model, according to the definition in Eq. (4), will be described by the following Hamiltonian:

$$H=-B-\sum_i X_iX_{i+1}+\sum_i e^{-\beta(Z_i+Z_{i+1})}. \tag{A1}$$

Note that it is similar to a quantum Ising model perturbed by a term in the form of $e^{-\beta(Z_i+Z_{i+1})}$. In particular, if one rewrites $e^{-\beta Z_i}$ in the form of $\cosh\beta-\sinh\beta Z_i$, the Hamiltonian will find the following form:

$$H=\cosh^2\beta-B-\sum_i X_iX_{i+1} \\ -(2\sinh\beta\,\cosh\beta)\sum_i Z_i+(\sinh^2\beta)\sum_i Z_iZ_{i+1}. \tag{A2}$$

It is clear that the above Hamiltonian is *not* an Ising model in a transverse field because of an additional term of $\sum_i Z_iZ_j$. In particular, it is not *a priori* clear if the above Hamiltonian can show a quantum phase transition at a finite value of β. However, our mapping can provide a solution to the above problem because the quantum Hamiltonian is mapped to a one-dimensional classical Ising model which has a phase transition at zero temperature. Therefore, according to our procedure, we conclude that the above quantum Hamiltonian has a phase transition point at infinite value of β.

On the other hand, our methodology can further explain the nature of this phase transition. According to our logic the exact ground state of the above quantum Hamiltonian will be in the following form:

$$|G(\beta)\rangle=e^{\frac{\beta}{2}\sum_i Z_i}(|+++...+\rangle+|---...-\rangle) \tag{A3}$$

up to a normalization factor. In particular, note that the above state is the ground state of our model even if we remove the nonlocal term B from the Hamiltonian. To see this better, consider a local quantum Hamiltonian without the nonlocal term. We will then have another (degenerate) ground state given by:

$$|G'(\beta)\rangle=e^{\frac{\beta}{2}\sum_i Z_i}(|+++...+\rangle-|---...-\rangle) \tag{A4}$$

up to a the normalization factor. On the other hand, using the local quantum Hamiltonian, we see that the ground state in the limit of $\beta\to\infty$ must be the state $|000...0\rangle$. However, one can check that the state (A3) goes to $|00...0\rangle$ while the state (A4) does not satisfy this condition. Accordingly, a sector of the ground subspace ($|G(\beta)\rangle$) is important for determining the phase transition point in a sense that it is the ground state for *all* values of β.

Now, our result shows that a symmetry breaking phase transition occurs at infinite value of β in a sense that the ground state is a nondegenerate magnetized state $|000...0\rangle$ at infinite value of β which has a Z_2 symmetry, and it breaks to a degenerate space for any other finite value of β. In other words, both states ($|G\rangle$ and $|G'\rangle$) are ground states of the local quantum Hamiltonian at any finite value of β and also any arbitrary superposition of them. Specifically, one can check that the following two quantum states are also ground states of the local quantum Hamiltonian, up to a normalization factor, for finite values of β:

$$|G_1(\beta)\rangle=e^{\frac{\beta}{2}\sum_i Z_i}|+++...+\rangle \\ |G_2(\beta)\rangle=e^{\frac{\beta}{2}\sum_i Z_i}|---...-\rangle. \tag{A5}$$

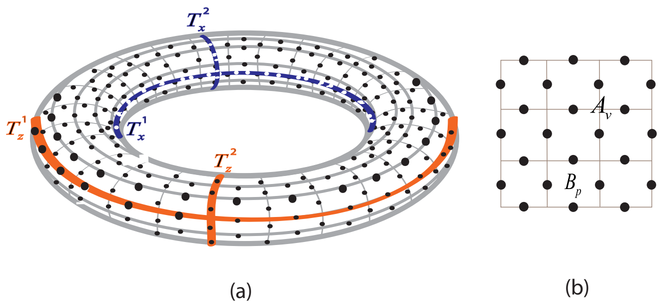

FIG. 2. (a) Toric code is defined on a torus where qubits live on edges of a square lattice. Corresponding to noncontractible loops around the torus, nonlocal X type and Z type operators are defined. (b) X-type and Z-type stabilizers are defined corresponding to vertices and plaquettes of the lattice, respectively.

It is clear that the above quantum states do not have a Z_2 symmetry, which means that in the thermodynamic limit the Z_2 symmetry of the system breaks as the system selects one of the above quantum states. Such a phase transition occurs as the system starts from the symmetric state of $|000...0\rangle$ and then the $\beta \to \infty$ is taken. We have therefore shown that in such quantum model there is no topological phase transition but a symmetry-breaking phase transition which occurs at infinite β. This means that the GHZ state does not have a topological order but it is a symmetry-breaking quantum phase.

As the second example, we consider the two-dimensional Ising model on a square lattice. As it has been shown in Ref. [32], such model should be mapped to a Toric code state on the same square lattice. Toric code model is defined on a torus, see Fig. 2, where X-type stabilizers of $A_v = \prod_{i\in v} X_i$ are defined corresponding to each vertex of the lattice and Z-type stabilizers of $B_p = \prod_{i\in\partial p} Z_i$ are defined corresponding to each plaquette of the lattice. The Hamiltonian describing toric code model is in the form of $H_1 = -\sum_p B_p - \sum_v A_v$ which has a fourfold degenerate ground state. One of the ground states is in the following form up to a normalization factor:

$$|\psi\rangle = \prod_v (1 + A_v)|00...0\rangle, \tag{A6}$$

where $|0\rangle$ in the positive eigenstate of the Pauli operator Z. Other degenerate ground states can also be constructed by two nonlocal X-type operators corresponding to two nontrivial loops around the torus, which we denote by T_x^1 and T_x^2 as shown in Fig. 2. Then, the four degenerate ground states will be in the following form:

$$|\psi_{ij}\rangle = \left(T_x^1\right)^i \left(T_x^2\right)^j |\psi\rangle, \tag{A7}$$

where $i, j = \{0, 1\}$. On the other hand, the above states can be distinguished from each other by two nonlocal Z-type operators T_z^1 and T_z^2 corresponding to nontrivial loops around the torus, see Fig. 2. Because of anticommutation relations between nonlocal X-type and Z-type operators, it is shown that the effect of T_z^1 and T_z^2 in the ground states of the toric code model will be in the following form:

$$T_z^1|\psi_{ij}\rangle = (-1)^j|\psi_{ij}\rangle \ , \ T_z^2|\psi_{ij}\rangle = (-1)^i|\psi_{ij}\rangle. \tag{A8}$$

Therefore, it means that $|\psi_{00}\rangle = |\psi\rangle$ is also stabilized by T_z^1 and T_z^2. Now, turning to the Hamiltonian of the Toric code model H_1, we add operators T_z^1 and T_z^2 to the H_1 to have a new Hamiltonian in the form of $H_2 = -\sum_p B_p - \sum_v A_v - T_z^1 - T_z^2$. It is clear that such a Hamiltonian has a unique ground state of $|\psi_{00}\rangle$ and there is no degeneracy for the above Hamiltonian.

Next, the quantum CSSM model, according to definition in Eq. (4), will be constructed by adding a term in the form of $\prod_{i\in\partial p} e^{-\beta Z_i}$ to the H_2. On the other hand, according to our procedure, the ground state fidelity of such a model will be mapped to heat capacity of a two-dimensional Ising model. Consequently, since the classical two-dimensional Ising model has a phase transition in a finite temperature, the above corresponding CSS model will show a quantum phase transition at a finite β. This is in agreement with the fact that the Toric code state has topological order. The quantum phase transition is not a symmetry breaking one but a topological phase transition.

Furthermore, this also shows why we use nonlocal Z-type stabilizers in H_2. To this end, note that although the Hamiltonian H_2 is nonlocal, it has a common ground state with the initial Hamiltonian H_1. On the other hand, since topological order is a property of the wave function, both models microscopically describe a topological order in their ground state wave function. Next, when we add the perturbation in the exponential form of $\sum_v e^{\beta\sum_{i\in v} Z_i}$ to H_1 and H_2, the state $|G(\beta)\rangle$ is still the ground state of both perturbed Hamiltonians in a sense that for $\beta = 0$, $|G(\beta)\rangle$ is equal to $|\psi_{00}\rangle$ and for $\beta \to \infty$ it is equal to magnetized state $|00...0\rangle$. In other words, even for the degenerate model of H_1, the quantum phase transition occurs in the sector of the Hilbert space which includes $|\psi_{00}\rangle$. It means that the above quantum phase transition is not related to the degeneracy of the ground state. It is for this reason that we consider nonlocal terms in the Hamiltonian in a sense that we are sure that the degeneracy does not play any role and there is no symmetry breaking phase transition in our CSSM models at a finite β.

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