Floquet-Drude conductivity

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A generalization of the Drude conductivity for systems which are exposed to periodic driving is presented. The probe bias is treated perturbatively by using the Kubo formula, whereas the external driving is included nonperturbatively using the Floquet theory. Using a different type of four-times Green functions disorder is approached diagrammatically, yielding a fully analytical expression for the Floquet-Drude conductivity. Furthermore, the Floquet Fermi "golden rule" is generalized to t-t' Floquet states, connecting the Floquet-Dyson series with scattering theory for Floquet states. It is shown that a low-energy approximation like the parabolic one fails significantly to give the correct conductivity in a system under driving.

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I. INTRODUCTION

Paul Drude published his theory of electric transport in metals as long ago as 1900 [1,2], which is today known as the Drude theory. To the present day several approaches have been developed to deepen the understanding of the microscopic mechanisms occurring in charge transport, including scattering theory using Fermi's "golden rule" [3,4] or quantum corrections to the Drude conductivity. The latter covers weak (anti)localization [5,6] in the form of geometry or spin dependent corrections [7–15]. In contrast to studies of static systems the development of lasers and masers generated a rising activity on explicitly time-dependent Hamiltonians, where the external field cannot be considered a small perturbation [16]. In the most recent decade, owing to the possibility of changing the topology of a system by means of external driving, the investigation of transport in driven systems increased [17–26]. This includes transport in driven systems [27,28], either with or without disorder [29–31], or the photovoltaic Hall effect [19]. Most works studying the renormalization of conductivity, due to an external driving, use a perturbative approach regarding the external driving [20,22]. We present a general formalism that allows the determination of the Drude conductivity in the presence of a nonperturbative external driving. We unify linear-response theory and Floquet formalism to account for the probe bias and an external driving, providing an alternative approach to the Keldysh formalism. Using a different type of four-times Green's-function formalism we derive both a Floquet-Dyson series and a generalized Floquet Fermi golden rule. To prove the consistency, both are shown to yield the same scattering time, a link that was missing so far. Even more important, the theory properly describes not only intra- but also inter-Floquet-replica scattering being completely neglected so far in literature. Finally, we present a closed analytical form for the Floquet-Drude conductivity and apply the developed theory to a parabolic approximation of the two-dimensional electron gas (2DEG) and the corresponding tight-binding model both with circularly polarized external driving. Regarding the 2DEG, the analysis shows that previous results have been overestimating the effect of the driving on the conductivity. The driven tight-binding model shows an entirely different driving dependency even in the low-energy limit. This observation is mainly caused by the different eigenstates rather than the similar spectra. In conclusion, low-energy models are likely to give an incorrect result for the conductivity in presence of an external driving.

II. KUBO FORMULA FOR PERIODICALLY DRIVEN SYSTEMS

Our first aim is to express the linear conductivity using Floquet states [16,32,33],

$$|\psi_\alpha(t)\rangle = \exp\left(-\frac{i}{\hbar}\varepsilon_\alpha t\right)|u_\alpha(t)\rangle, \tag{1}$$

where α is labeling a discrete set of quantum numbers and ε_α are the corresponding quasienergies [34,35]. The periodicity of the Floquet functions, which are eigenstates of the Floquet Hamiltonian

$$H_F(t) = H(t) - i\hbar\partial_t, \tag{2}$$

allow for the Fourier expansion $|u_\alpha(t)\rangle = \sum_n e^{-in\Omega t}|u_\alpha^n\rangle$ with Ω being the frequency of the external driving. The probe bias, with the corresponding vector potential $\mathcal{A}(\mathbf{q},\omega)$, is treated perturbatively in linear-response theory, i.e., using the Kubo formula in momentum space [36],

$$\begin{aligned}\langle J^a(\mathbf{q},\omega)\rangle = &\frac{i}{\hbar\mathfrak{V}}\sum_b\int_{-\infty}^{\infty}dt'\int_{-\infty}^{\infty}dt\{e^{i\omega t}\Theta(t-t')\\ &\times\langle[J^a(\mathbf{q},t),J^b(-\mathbf{q},t')]\rangle\mathcal{A}^b(\mathbf{q},t')\}\\ &-\frac{e^2n}{m}\mathcal{A}^a(\mathbf{q},\omega),\end{aligned} \tag{3}$$

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with $a, b \in \{x, y, z\}$, ω being the frequency of the response current, $\mathfrak{V}$ the volume of the system, $e < 0$ the electron charge, $\Theta(\cdot)$ the Heaviside function, m the effective electron mass, and n the electron density. $\langle\cdot\rangle$ denotes the statistical average with respect to the system's state which will in the presence of external driving not be in equilibrium. However, in what follows we shall assume the system to be in a stationary state so that occupation numbers of Floquet states are time independent [18,27,37–41]. Treating the external driving nonperturbatively, the current operators are expanded using Floquet states $|\psi_\alpha\rangle = c_\alpha^\dagger |0\rangle$,

$$J^{a,b}(\mathbf{q}, t) = \sum_{\alpha\beta} J^{a,b}_{\alpha\beta}(\mathbf{q}, t) c^\dagger_\alpha c_\beta, \tag{4}$$

with the fermionic annihilation (creation) operators $c_\alpha^{(\dagger)}$. The expansion coefficients are the matrix elements $J^{a,b}_{\alpha\beta}(\mathbf{q}, t) = \langle\psi_\alpha(t)|\, J^{a,b}(\mathbf{q})\, |\psi_\beta(t)\rangle$. The current expectation value is no longer a simple product of conductance and electric field $\mathbf{E} = -\partial_t \boldsymbol{\mathcal{A}}$ since it is convoluted over the bias frequency ω',

$$\langle J^a(\mathbf{q}, \omega)\rangle = \sum_b \int_{-\infty}^{\infty} d\omega'\, \bar{\sigma}^{ab}(\mathbf{q}, \omega, \omega') E^b(\mathbf{q}, \omega'). \tag{5}$$

Thus far, the conductivity tensor $\bar{\sigma}^{ab}$ depends on the bias frequency, the momentum $\mathbf{q}$, and on the resulting current frequency ω,

$$\begin{aligned} \bar{\sigma}^{ab}(\mathbf{q}, \omega, \omega') = &\frac{i}{\hbar\omega\mathfrak{V}} \sum_{\alpha\beta} \sum_{n_1..n_4=-\infty}^{\infty} (f_\alpha - f_\beta) \\ &\times \frac{\langle u_\alpha^{n_1}|\, j^a(\mathbf{q}) |u_\beta^{n_2}\rangle\langle u_\beta^{n_3}|\, j^b(-\mathbf{q}) |u_\alpha^{n_4}\rangle}{\omega + \frac{1}{\hbar}(\varepsilon_\alpha - \varepsilon_\beta) + (n_1 - n_2)\Omega + i0^+} \\ &\times \delta[\omega + (n_1 - n_2 + n_3 - n_4)\Omega - \omega'] \\ &+ i\frac{e^2 n}{m\omega}\delta_{ab}\delta(\omega - \omega'), \end{aligned} \tag{6}$$

with the single-particle current operator j^i and the distribution function $f_{\alpha,\beta} = \langle c^\dagger_{\alpha,\beta} c_{\alpha,\beta}\rangle$. In what follows we limit our calculations to a position independent driving field, i.e., $\mathbf{q} = \mathbf{0}$. Since we are ultimately interested in the DC conductivity, Eq. (5) simplifies to

$$\langle J^a(\mathbf{q}, \omega)\rangle = \sum_b \sigma^{ab}(\mathbf{q}, \omega) E^b(\mathbf{q}, \omega), \tag{7}$$

as shown in Appendix A. This allows us to express the real part of the longitudinal DC conductivity as

$$\operatorname*{Re}_{\omega\to 0}[\sigma^{xx}(0, \omega)] = \frac{\pi\hbar}{\mathfrak{V}}\left(\frac{e}{m}\right)^2 \int_{\lambda-\hbar\Omega/2}^{\lambda+\hbar\Omega/2} d\varepsilon \left(-\frac{\partial f}{\partial\varepsilon}\right)\sigma(\varepsilon) \tag{8}$$

along with the definition

$$\begin{aligned} \sigma(\varepsilon) = &\sum_{nn'=-\infty}^{\infty} \sum_{\alpha\beta} \langle u_\alpha^n|\, p^x\, |u_\beta^n\rangle\langle u_\beta^{n'}| p^x |u_\alpha^{n'}\rangle \\ &\times \delta(\varepsilon - \varepsilon_\alpha)\delta(\varepsilon - \varepsilon_\beta), \end{aligned} \tag{9}$$

and the abbreviation $\mathrm{Re}_{\omega\to 0}[\cdot]$ for $\lim_{\omega\to 0}\mathrm{Re}[\cdot]$. A similar result has already been derived in Ref. [18] using the Keldysh framework. In derivating Eq. (8) one requires the difference of two quasienergies to always be smaller than the photon energy of the external driving. Thus far, we have not used the convenient choice of the quasienergy [42] to be in $[-\hbar\Omega/2, \hbar\Omega/2)$, however we must choose a suitable, possibly momentum dependent function λ such that

$$\forall_\alpha : \lambda - \frac{\hbar\Omega}{2} \leqslant \varepsilon_\alpha < \lambda + \frac{\hbar\Omega}{2}. \tag{10}$$

III. FOUR-TIMES FLOQUET GREEN'S FUNCTION AND CONDUCTIVITY

In this section we set up a formalism using four-times Green's functions to express the result of the foregoing section for the conductivity in terms of Green's functions. These are the building blocks for the Floquet-Dyson equation. First, we define a t-t' state for the ℓth Floquet zone [16,43–45]:

$$\left|\psi_\alpha^\ell(t, t')\right\rangle = e^{-(i/\hbar)(\varepsilon_\alpha + \ell\hbar\Omega)t}\, |u_\alpha(t')\rangle\, e^{i\ell\Omega t'}, \tag{11}$$

recovering for $t = t'$ the Floquet state solution $|\psi_\alpha(t)\rangle$ of the time-dependent Schrödinger equation. For details of the t-t' formalism the authors refer to Appendix B and Refs. [16,46,47]. From the states given in Eq. (11) a bare four-times Green function is constructed:

$$\begin{aligned} \mathcal{G}_0^{r,a}(t_1, t_2, t_1', t_2') = &\mp i\Theta[\pm(t_1 - t_2)] \\ &\times \frac{1}{T}\sum_{\ell=-\infty}^{\infty}\sum_\alpha \left|\psi_\alpha^\ell(t_1, t_1')\right\rangle\left\langle\psi_\alpha^\ell(t_2, t_2')\right| \end{aligned} \tag{12}$$

with the period $T = 2\pi/\Omega$ and periodicity

$$\begin{aligned} \mathcal{G}_0^{r,a}(t_1, t_2, t_1', t_2') &= \mathcal{G}_0^{r,a}(t_1 + T, t_2 + T, t_1', t_2') \\ &= \mathcal{G}_0^{r,a}(t_1, t_2, t_1' + T, t_2' + T). \end{aligned} \tag{13}$$

This propagator fulfills

$$\begin{aligned} &\left[i\partial_{t_1} - \tfrac{1}{\hbar}H_F(t_1')\right]\mathcal{G}_0^{r,a}(t_1, t_2, t_1', t_2') \\ &\quad = \delta(t_1 - t_2)\sum_{s=-\infty}^{\infty}\delta(t_1' - t_2' + sT)\mathbb{1}. \end{aligned} \tag{14}$$

Fourier transforming the Green function and expanding the Floquet function into a Fourier series yields

$$\begin{aligned} \mathcal{G}_0^{r,a}(\varepsilon, t_1', t_2') = &\frac{1}{T}\sum_{n,n'=-\infty}^{\infty}\mathcal{G}_0^{r,a}(\varepsilon, n, n') \\ &\times e^{-in\Omega t_1'} e^{in'\Omega t_2'}. \end{aligned} \tag{15}$$

Generalizing the completeness relation of the Floquet functions to different times,

$$\begin{aligned} &\sum_\alpha |u_\alpha(t_1')\rangle\langle u_\alpha(t_2')| \sum_{\ell=-\infty}^{\infty}\delta(t_1' - t_2' + \ell T) \\ &\quad = \mathbb{1}\sum_{\ell=-\infty}^{\infty}\delta(t_1' - t_2' + \ell T), \end{aligned} \tag{16}$$

gives particular insight to the Lehmann representation of the four-times Floquet Green function:

$$\mathcal{G}_0^{r,a}(\varepsilon, t_1', t_2') = \frac{\mathbb{1}\sum_{\ell=-\infty}^{\infty}\delta(t_1' - t_2' + \ell T)}{\frac{1}{\hbar}\varepsilon - \frac{1}{\hbar}H_F(t_1')}. \tag{17}$$

One can show that the Fourier coefficients [46,47],

$$\mathcal{G}_0^{r,a}(\varepsilon,n,n') = \sum_{\ell=-\infty}^{\infty}\sum_{\alpha}\frac{|u_\alpha^{n+\ell}\rangle\langle u_\alpha^{n'+\ell}|}{\frac{1}{\hbar}\varepsilon-\frac{1}{\hbar}\varepsilon_\alpha-\ell\Omega\pm i0^+}, \tag{18}$$

are equal to the inverse of the Floquet matrix [27,29,30,48–53]. The periodicity of the Floquet eigenstates [54,55] suggests defining the unitary transformation $\boldsymbol{T}$ as in Sec. V of Ref. [56] which diagonalizes the Green function,

$$[\boldsymbol{D}(\varepsilon)]_{\alpha\beta}^{nn'} \equiv \left[\boldsymbol{T}^\dagger\boldsymbol{G}_0^{r,a}(\varepsilon)\boldsymbol{T}\right]_{\alpha\beta}^{nn'} \tag{19}$$

$$= \frac{\delta_{\alpha\beta}\delta_{nn'}}{\frac{1}{\hbar}\varepsilon-\frac{1}{\hbar}\varepsilon_\alpha-n\Omega\pm i0^+}, \tag{20}$$

where we denoted the matrix spanned by the Fourier components of the Green function as $[\boldsymbol{G}_0^{r,a}(\varepsilon)]_{nn'} = \mathcal{G}_0^{r,a}(\varepsilon,n,n')$. The Green function defined in Eq. (15) is used to express the conductivity from Eq. (8) as

$$\begin{aligned}&\operatorname*{Re}_{\omega\to0}[\sigma^{xx}(0,\omega)]\\&= \frac{-1}{4\pi\hbar\mathfrak{V}}\left(\frac{e}{m}\right)^2\int_{\lambda-\frac{\hbar\Omega}{2}}^{\lambda+\frac{\hbar\Omega}{2}}d\varepsilon\left(-\frac{\partial f}{\partial\varepsilon}\right)\int_0^T dt_1'\int_0^T dt_2'\\&\times\operatorname{tr}\left\{p^x\left[\mathcal{G}_0^r(\varepsilon,t_1',t_2')-\mathcal{G}_0^a(\varepsilon,t_1',t_2')\right]p^x(t_1'\leftrightarrow t_2')\right\}.\end{aligned} \tag{21}$$

A justification for the use of the t-t' formalism is given in Appendix B 3.

IV. FLOQUET-DYSON EQUATION

The focus of this section is to formulate a perturbative approach to include disorder in the expression of the conductivity described by bare propagators, i.e., Eq. (21). In the following, we will use the notation

$$\langle\mathbf{x}|\,\mathcal{G}^{r,a}(t_1,t_2,t_1',t_2')\,|\mathbf{x}'\rangle \equiv \mathcal{G}^{r,a}(t_1,t_2,\mathbf{x},\mathbf{x}',t_1',t_2'), \tag{22}$$

$$\langle\mathbf{x}|\,\mathcal{G}^{r,a}(\varepsilon,t_1',t_2')\,|\mathbf{x}'\rangle \equiv \mathcal{G}^{r,a}(\varepsilon,\mathbf{x},\mathbf{x}',t_1',t_2'), \tag{23}$$

for the matrix elements of the Green function in real space. The Green function for the system with an impurity potential $V(\mathbf{x}_1,t_1,t_1')$ at site $\mathbf{x}_1$ is supposed to fulfill

$$\begin{aligned}&\left[i\partial_{t_1}-\tfrac{1}{\hbar}\mathcal{H}_F(\mathbf{x}_1,t_1,t_1')\right]\mathcal{G}_{\mathrm{p}}^{r,a}(t_1,t_2,\mathbf{x}_1,\mathbf{x}_2,t_1',t_2')\\&= \delta(t_1-t_2)\delta(\mathbf{x}_1-\mathbf{x}_2)\sum_{s=-\infty}^{\infty}\delta(t_1'-t_2'+sT)\mathbb{1}\end{aligned} \tag{24}$$

with $\mathcal{H}_F(\mathbf{x}_1,t_1,t_1') = H_F(\mathbf{x}_1,t_1') + V(\mathbf{x}_1,t_1,t_1')$. We limit the calculation presented here to the time-independent potential $V(\mathbf{x})$, as including an explicit time dependence is straightforward; see Appendix C. Following the standard steps [36,57–59], we can derive a recursive integral expansion, i.e., a Dyson series, for the Green function, in the case of a system perturbed by impurities:

$$\begin{aligned}\mathcal{G}_{\mathrm{p}}^{r,a}(\varepsilon,\mathbf{x}_1,\mathbf{x}_2,n,n') &= \mathcal{G}_0^{r,a}(\varepsilon,\mathbf{x}_1,\mathbf{x}_2,n,n')\\&+\frac{1}{\hbar}\int_{V_x}d\mathbf{x}\sum_{n_1=-\infty}^{\infty}\mathcal{G}_{\mathrm{p}}^{r,a}(\varepsilon,\mathbf{x}_1,\mathbf{x},n,n_1)V(\mathbf{x})\\&\times\mathcal{G}_0^{r,a}(\varepsilon,\mathbf{x},\mathbf{x}_2,n_1,n').\end{aligned} \tag{25}$$

The impurity potential $V(\mathbf{x}) = \sum_i^{N_{\mathrm{imp}}} v(\mathbf{x}-\mathbf{r}_i)$ is assumed to be a Gaussian random potential, which is uncorrelated such that the impurity average yields $\langle v(\mathbf{x})v(\mathbf{x}')\rangle_{\mathrm{imp}} = \mathcal{V}_{\mathrm{imp}}\delta(\mathbf{x}-\mathbf{x}') \Leftrightarrow v(\mathbf{k}) = \mathcal{V}_{\mathrm{imp}}$ and $\langle v(\mathbf{x})\rangle_{\mathrm{imp}} = 0$ (white noise) [58,59]. Fourier transforming Eq. (25) into momentum space and performing a disorder average one arrives at the expression for the disorder averaged Green function:

$$\begin{aligned}\boldsymbol{G}^{r,a}(\varepsilon,\mathbf{k}) &= \boldsymbol{G}_0^{r,a}(\varepsilon,\mathbf{k}) + \boldsymbol{G}_0^{r,a}(\varepsilon,\mathbf{k})\\&\times\sum_{n=1}^{\infty}[\boldsymbol{\Sigma}^{r,a}(\varepsilon,\mathbf{k})\boldsymbol{G}_0^{r,a}(\varepsilon,\mathbf{k})]^n,\end{aligned} \tag{26}$$

where the self-energy $\boldsymbol{\Sigma}^{r,a}(\varepsilon,\mathbf{k})$ is the sum over all irreducible diagrams. Applying the transformation $\boldsymbol{T}$, the solution of the recursive Eq. (26) in the eigenbasis is governed by

$$\begin{aligned}&\boldsymbol{T}^\dagger(\mathbf{k})\boldsymbol{G}^{r,a}(\varepsilon,\mathbf{k})\boldsymbol{T}(\mathbf{k})\\&= [\boldsymbol{D}(\varepsilon,\mathbf{k})+\boldsymbol{T}^\dagger(\mathbf{k})\boldsymbol{\Sigma}^{r,a}(\varepsilon,\mathbf{k})\boldsymbol{T}(\mathbf{k})]^{-1},\end{aligned} \tag{27}$$

with the diagonal matrix $\boldsymbol{D}(\varepsilon,\mathbf{k})$ given in Eq. (19). The difference of the retarded and advanced self-energy in the first-order Born approximation (1BA) can be related to a scattering time derived within the framework of the Floquet Fermi golden rule for t-t' states (11) as shown in the next section.

V. GENERALIZED FLOQUET FERMI GOLDEN RULE

In the following the steps of the derivation of the Fermi golden rule for t-t' Floquet states are similar to the one applied in Refs. [25,36]. The difference lies in the use of the t-t' Floquet states, see Eq. (11), instead of the Floquet states. A t-t' state fulfills

$$i\hbar\frac{\partial}{\partial t}\left|\psi_\alpha^\ell(t,t')\right\rangle = H_F(t')\left|\psi_\alpha^\ell(t,t')\right\rangle. \tag{28}$$

The corresponding time-evolution operator fulfilling this Schrödinger equation is given by

$$U_0(t,t_0,t') = e^{-(i/\hbar)H_F(t')(t-t_0)}. \tag{29}$$

If a perturbation is switched on at time t_0 the Schrödinger equation becomes

$$i\hbar\frac{\partial}{\partial t}\left|\Psi_\alpha^\ell(t,t')\right\rangle = [H_F(t')+V(t,t')]\left|\Psi_\alpha^\ell(t,t')\right\rangle, \tag{30}$$

with the boundary condition $|\psi_\alpha^\ell(t,t')\rangle = |\Psi_\alpha^\ell(t,t')\rangle$ for $t\leqslant t_0$. Changing into the interaction picture with

$$\left|\Psi_\alpha^\ell(t,t')\right\rangle_I = U_0^\dagger(t,t_0,t')\left|\Psi_\alpha^\ell(t,t')\right\rangle, \tag{31}$$

$$V_I(t,t') = U_0^\dagger(t,t_0,t')V(t,t')U_0(t,t_0,t'), \tag{32}$$

one finds up to first order in the potential V

$$\left|\Psi_\alpha^\ell(t,t')\right\rangle_I \approx \left|\psi_\alpha^\ell(t_0,t')\right\rangle + \frac{1}{i\hbar}\int_{t_0}^t dt_1\, V_I(t_1,t')\left|\psi_\alpha^\ell(t_0,t')\right\rangle \tag{33}$$

and for the overlap

$$\begin{aligned}\left\langle\psi_\beta^{\ell'}(t,t'')\middle|\Psi_\alpha^\ell(t,t')\right\rangle &= \left\langle\psi_\beta^{\ell'}(t,t'')\middle|\psi_\alpha^\ell(t,t')\right\rangle \\ &+ \frac{1}{i\hbar}\int_{t_0}^t dt_1 \left\langle\psi_\beta^{\ell'}(t_1,t'')\middle|V(t_1,t')\middle|\psi_\alpha^\ell(t_1,t')\right\rangle.\end{aligned} \tag{34}$$

In the next step let us consider the matrix element where the t-t' Floquet states have the same time dependence but different Floquet indices,

$$a_{\alpha\beta}^{\ell\ell'}(t,t') = \sum_{n=-\infty}^{\infty} a_{\alpha\beta}^{\ell\ell'}(t,n)\, e^{in\Omega t'} \tag{35}$$

$$\begin{aligned}&= \left\langle\psi_\beta^\ell(t,t')\middle|\Psi_\alpha^{\ell'}(t,t')\right\rangle \\ &\approx \delta_{\alpha\beta}\, e^{i\Omega(\ell-\ell')(t'-t)} \\ &\quad + \frac{1}{i\hbar}\int_{t_0}^t dt_1 \left\langle\psi_\beta^\ell(t_1,t')|V(t_1,t')\middle|\psi_\alpha^{\ell'}(t_1,t')\right\rangle.\end{aligned} \tag{36}$$

The Fourier coefficients for a perturbation, which is time-independent in the second time argument, are governed by

$$a_{\alpha\beta}^{\ell\ell'}(t,n) = \frac{1}{T}\int_0^T dt'\, a_{\alpha\beta}^{\ell\ell'}(t,t')\, e^{in\Omega t'} \tag{37}$$

$$\begin{aligned}&= \delta_{\alpha\beta}\delta_{n,\ell-\ell'}\, e^{-in\Omega t} \\ &\quad + \frac{1}{i\hbar}\int_{t_0}^t dt_1\, e^{(i/\hbar)[\varepsilon_\alpha-\varepsilon_\beta+(\ell-\ell')\hbar\Omega]t_1} \\ &\quad \times \sum_{m=-\infty}^{\infty} \left\langle u_\alpha^{m+\ell+n}\middle|V(t_1)\middle|u_\beta^{m+\ell'}\right\rangle.\end{aligned} \tag{38}$$

We see that the transition amplitude is only a function of the difference of the Floquet indices, $a_{\alpha\beta}^{\ell\ell'}(t,t') = a_{\alpha\beta}^{(\ell-\ell')}(t,t')$. Analog to the last section, t_0 can be set to zero and for $\alpha \neq \beta$ Eq. (36) simplifies to

$$a_{\alpha\beta}^{\ell\ell'}(t,t') = -\frac{i}{\hbar}\int_0^t dt_1 \left\langle\psi_\beta^\ell(t_1,t')\middle|V(t_1,t')\middle|\psi_\alpha^{\ell'}(t_1,t')\right\rangle. \tag{39}$$

Now, let us assume a scattering event from a t-t' Floquet state into another t-t' Floquet state with constant quasienergy, given by

$$\left|\psi_\alpha^\ell(\varepsilon,t,t')\right\rangle \equiv e^{-(i/\hbar)(\varepsilon+\ell\hbar\Omega)t}\left|u_\alpha(t')\right\rangle e^{i\ell\Omega t'}. \tag{40}$$

The quasienergy is independent of the quantum number. This state is not an eigenstate of the Hamiltonian, nevertheless it fulfills

$$\left\langle\psi_\alpha^\ell(t,t')\middle|\psi_\beta^{\ell'}(\varepsilon,t,t')\right\rangle = \delta_{\alpha\beta} e^{-(i/\hbar)[\varepsilon-\varepsilon_\alpha+(\ell'-\ell)\hbar\Omega]t}\, e^{i\Omega(\ell'-\ell)t'}. \tag{41}$$

Hence, Eq. (39) remains valid if the final state is of the same form as in Eq. (40). Consider now a scattering event from a t-t' Floquet state $\psi_\alpha^\ell(\mathbf{k}',t,t')$ into a state with constant energy ε,

$$\begin{aligned}\psi_\alpha^\ell(\mathbf{k}',t,t') &= e^{-(i/\hbar)[\varepsilon_\alpha(\mathbf{k}')+\ell\hbar\Omega]t} u_\alpha(\mathbf{k}',t')^{\,i\ell\Omega t'} \\ &\rightsquigarrow e^{-(i/\hbar)[\varepsilon+\ell'\hbar\Omega]t} u_\beta(\mathbf{k},t')^{\,i\ell'\Omega t'}.\end{aligned} \tag{42}$$

The Fourier coefficient of the matrix element in the case of a scattering as in Eq. (42) for a time-independent perturbation is given by

$$\begin{aligned}&a_{\alpha\beta}^{\ell\ell'}(\mathbf{k},\mathbf{k}',t,n) \\ &= -i\frac{V_{\mathbf{kk}'}}{\hbar}\int_0^t dt'\, e^{(i/\hbar)[\varepsilon-\varepsilon_\alpha(\mathbf{k}')-(\ell-\ell')\hbar\Omega]t'} \\ &\quad \times \sum_{m=-\infty}^{\infty}\left[u_\beta^{m+\ell+n}(\mathbf{k})\right]^* u_\alpha^{m+\ell'}(\mathbf{k}') \\ &= -i\frac{V_{\mathbf{kk}'}}{\hbar}\int_0^t dt'\, e^{(i/\hbar)[\varepsilon-\varepsilon_\alpha(\mathbf{k}')-(\ell-\ell')\hbar\Omega]t'} c_{\beta\alpha}^{\ell-\ell'+n}(\mathbf{k},\mathbf{k}'),\end{aligned} \tag{43}$$

$$\text{with } c_{\alpha\beta}^n(\mathbf{k},\mathbf{k}') \equiv \sum_{m=-\infty}^{\infty}\left[u_\alpha^{m+n}(\mathbf{k})\right]^* u_\beta^m(\mathbf{k}'). \tag{44}$$

This allows for a definition of the transition probability matrix

$$\left[\mathbf{A}_{\alpha\beta}^{\ell\ell' jj'}(\mathbf{k},\mathbf{k}',t)\right]_{n,n'} := \sum_\gamma a_{\gamma\alpha}^{\ell\ell'}(\mathbf{k},\mathbf{k}',t,n)\left[a_{\gamma\beta}^{jj'}(\mathbf{k},\mathbf{k}',t,n')\right]^*. \tag{45}$$

Equivalently to the derivation of the Floquet Fermi golden rule presented in Ref. [42] [see Eq. (D10) in Appendix D], in the limit $t \to \infty$ the transition probability matrix becomes

$$\begin{aligned}&\left[\mathbf{A}_{\alpha\beta}^{\ell\ell' jj'}(\mathbf{k},\mathbf{k}',t)\right]_{n,n'} \\ &= 4\pi^2 V_{\mathbf{kk}'}^2 \sum_\gamma c_{\alpha\gamma}^{\ell-\ell'+n}(\mathbf{k},\mathbf{k}') \\ &\quad \times \delta[\varepsilon-\varepsilon_\gamma(\mathbf{k}')-(\ell-\ell')\hbar\Omega] \\ &\quad \times \left[c_{\beta\gamma}^{j-j'+n'}(\mathbf{k},\mathbf{k}')\right]^* \delta[\varepsilon-\varepsilon_\gamma(\mathbf{k}')-(j-j')\hbar\Omega].\end{aligned} \tag{46}$$

Since the quasienergies are always defined to be in the central Floquet zone, cf. Eq. (D13), the probability matrix simplifies to

$$\begin{aligned}&\left[\mathbf{A}_{\alpha\beta}^{\ell\ell' jj'}(\mathbf{k},\mathbf{k}',t)\right]_{n,n'} \\ &= 4\pi^2 V_{\mathbf{kk}'}^2 \sum_\gamma c_{\alpha\gamma}^{n}(\mathbf{k},\mathbf{k}') \\ &\quad \times \left[c_{\beta\gamma}^{n'}(\mathbf{k},\mathbf{k}')\right]^* \delta^2[\varepsilon-\varepsilon_\gamma(\mathbf{k}')].\end{aligned} \tag{47}$$

The square of the delta distribution can be rewritten as [25]

$$\delta^2(\varepsilon) = \delta(\varepsilon)\delta(0) = \frac{\delta(\varepsilon)}{2\pi\hbar}\lim_{t\to\infty}\int_{-t/2}^{t/2} dt' e^{(i/\hbar)0t'} \tag{48}$$

$$= \frac{\delta(\varepsilon)t}{2\pi\hbar}. \tag{49}$$

Using this relation and performing the time derivative of each matrix element yields

$$\Gamma^{nn'}_{\alpha\beta}(\mathbf{k},\mathbf{k}') \equiv \frac{d\big[\mathbf{A}^{\ell\ell' jj'}_{\alpha\beta}(\mathbf{k},\mathbf{k}',t)\big]_{n,n'}}{dt} \tag{50}$$

$$= \frac{2\pi}{\hbar} V^2_{\mathbf{k}\mathbf{k}'} \sum_\gamma c^n_{\alpha\gamma}(\mathbf{k},\mathbf{k}') \times \big[c^{n'}_{\beta\gamma}(\mathbf{k},\mathbf{k}')\big]^* \delta[\varepsilon - \varepsilon_\gamma(\mathbf{k}')]. \tag{51}$$

Finally, one can perform an impurity average and identify $\langle V^2_{\mathbf{k}\mathbf{k}'}\rangle_{\text{imp}} = \mathcal{V}_{\text{imp}}$. Summing the rate over all momenta one gets the inverse scatting time matrix,

$$\left(\frac{1}{\boldsymbol{\tau}(\varepsilon,\mathbf{k})}\right)^{nn'}_{\alpha\beta} \equiv \frac{1}{V_{\mathbf{k}'}} \sum_{\mathbf{k}'} \big\langle \Gamma^{nn'}_{\alpha\beta}(\mathbf{k},\mathbf{k}')\big\rangle_{\text{imp}} \tag{52}$$

$$= \frac{2\pi}{\hbar}\mathcal{V}_{\text{imp}} \frac{1}{V_{\mathbf{k}'}} \sum_{\mathbf{k}'}\sum_\gamma c^n_{\alpha\gamma}(\mathbf{k},\mathbf{k}')\big[c^{n'}_{\beta\gamma}(\mathbf{k},\mathbf{k}')\big]^* \times \delta[\varepsilon - \varepsilon_\gamma(\mathbf{k}')] \tag{53}$$

$$= i\big\{\boldsymbol{T}^\dagger(\mathbf{k})\big[\boldsymbol{\Sigma}^r_{1\text{BA}}(\varepsilon,\mathbf{k}) - \boldsymbol{\Sigma}^a_{1\text{BA}}(\varepsilon,\mathbf{k})\big]\boldsymbol{T}(\mathbf{k})\big\}^{nn'}_{\alpha\beta}. \tag{54}$$

This expression is equal to the result derived from the Dyson series for the Floquet Green function. To our knowledge, this remarkable connection has not been established before. It comprises both inter- as well as intra-Floquet band scattering [60,61]. One can provide a connection to previous studies by setting $n = n' = 0$ in Eq. (50), which results in the scattering rates given Refs. [17,42] [see Eq. (D20) in Appendix D]. For reasons of comparability, the Fermi golden rule for Floquet states is derived in detail in Appendix D. However, in general the relaxation rate matrix is not diagonal in the Floquet space nor can it be reduced to the $(n, n') = (0, 0)$ element only, as will be discussed later on. Also, one should notice that only on the diagonal is the difference of the retarded and advanced self-energy equal to the imaginary part of the retarded self-energy.

VI. FLOQUET-DRUDE CONDUCTIVITY

Now, let us consider only the self-energy corrections during the disorder average and perform the time integration in Eq. (21). The disorder is not supposed to change the eigenenergies of the bare system, hence we drop all off-diagonal elements of the self-energy. This allows us to proceed analytically and ultimately express the conductivity in a compact way,

$$\underset{\omega\to 0}{\text{Re}}\,[\sigma^{xx}(0,\omega)] = \frac{\hbar}{4\pi\mathfrak{V}}\left(\frac{e}{m}\right)^2 \int_{-\hbar\Omega/2}^{\hbar\Omega/2} d\varepsilon \left(-\frac{\partial f}{\partial\varepsilon}\right) \times \frac{1}{\mathfrak{V}_{\mathbf{k}}}\sum_{\mathbf{k}} k_x^2 \sum_\alpha \sum_{n=-\infty}^{\infty} \left[\left(\frac{1}{2\boldsymbol{\tau}(\varepsilon,\mathbf{k})}\right)^{nn}_{\alpha\alpha}\right]^2 \times \left\{\left(\tfrac{1}{\hbar}\varepsilon - \tfrac{1}{\hbar}\varepsilon_\alpha - n\Omega\right)^2 + \left[\left(\frac{1}{2\boldsymbol{\tau}(\varepsilon,\mathbf{k})}\right)^{nn}_{\alpha\alpha}\right]^2\right\}^{-2}. \tag{55}$$

VII. APPLICATION TO A 2DEG

For a simple nontrivial application of the method described above we chose a 2DEG from a direct semiconductor close to the Γ point under illumination with circularly polarized light. To allow for a direct comparison with existing theories on the topic of conductivity in driven systems we first apply our theory to the most simple model: The lowest s-type conduction band is approximated by a parabolic Hamiltonian $H = \mathbf{p}^2/2m$. The vector potential of the radiation,

$$\mathbf{A}(t) = A_x\cos(\Omega t)\hat{\mathbf{x}} + A_y\sin(\Omega t)\hat{\mathbf{y}}, \tag{56}$$

is coupled to the momentum via minimal coupling leading to the time-dependent Hamiltonian

$$H(t) = \frac{\hbar^2}{2m}\{\mathbf{k}^2 + \gamma^2 + 2\gamma[k_x\cos(\Omega t) + k_y\sin(\Omega t)]\}, \tag{57}$$

with $A_x = A_y =: A$ and the light parameter $\gamma := eA/\hbar$. The solution of the time-dependent Schrödinger equation was already given by Kibis [25]. To evaluate the expression for the conductivity, one has to specify the distribution function further. In the off-resonant regime, absorption of photons is suppressed, hence, a Fermi distribution can be assumed. However, since the parabolic spectrum is unbounded, it is not obvious how to set the Fermi energy ε_F for the driven parabolic spectrum [42]. Here, we truncate the momentum range to set the Fermi energy; see Appendix E. Evaluating Eq. (52) yields for the scattering time on the diagonal

$$\left(\frac{1}{\boldsymbol{\tau}(\varepsilon_F,\mathbf{k})}\right)^{nn} = \frac{\mathcal{V}_{\text{imp}}m}{\hbar^3}\sum_{m=-\infty}^{\infty} J^2_{m+n}(z_k)J^2_m(z_\varepsilon), \tag{58}$$

together with

$$z_k = \frac{\hbar^2\gamma k}{m}\Big/\hbar\Omega, \quad z_\varepsilon = \frac{\hbar^2\gamma\sqrt{2m\varepsilon_F/\hbar^2}}{m}\Big/\hbar\Omega. \tag{59}$$

We can further disregard all pairings of only retarded or only advanced Green's functions in Eq. (55), since they give a contribution of the order of $1/(\varepsilon_F\tau_0)$ with τ_0 being the scattering time of the undriven system [59]. Aside from the relation between the Fermi energy and the relaxation rate, in the driven case one finds an important relation between the relaxation rate and the driving frequency. This ratio controls whether or not only the $n = 0$ element in Eq. (58) is significant: If $\Omega\tau_0 \gg 1$, the broadening of the nonzero Floquet modes is small enough such that the leaking into the central Floquet zone is negligibly small, as depicted in Fig. 1. The theory presented here also makes the regime accessible where $\Omega\tau_0 \simeq 1$. In that case, the nonzero Floquet modes contribute significantly; see Fig. 2.

But even in the off resonant regime, $\Omega\tau_0 \gg 1$, previous studies [20] overestimate the effect of circular driving as will be shown in the following. The central entry of the product of the retarded Green function with an advanced one is

$$[\boldsymbol{G}^r(\varepsilon_F,\mathbf{k})\boldsymbol{G}^a(\varepsilon_F,\mathbf{k})]^{00} \approx 2\pi\hbar(\boldsymbol{\tau}(\varepsilon_F,\mathbf{k}))^{00}\delta(\varepsilon_F - \epsilon_{\mathbf{k}}). \tag{60}$$

Applying these simplifications to Eq. (55), one arrives at the conductivity

$$\underset{\omega\to 0}{\text{Re}}\,[\sigma^{xx}(0,\omega)] = \frac{1}{\mathfrak{V}4\pi}\left(\frac{e^2}{m}\right)k_F^2[\boldsymbol{\tau}(\varepsilon_F,k_F)]^{00}, \tag{61}$$

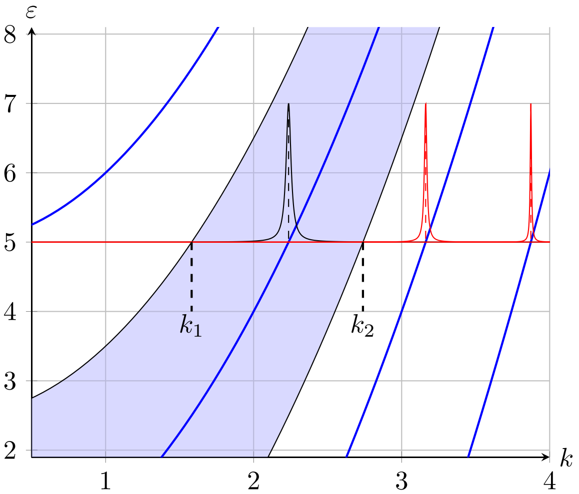

FIG. 1. The peaks show the broadening of the Floquet bands caused by the scattering time. The blue shaded area is the central Floquet zone. If $\Omega\tau_0 \gg 1$, the leaking of the nonzero Floquet modes (red curves) into the central Floquet zone is negligibly small.

where the scattering time is evaluated at the Fermi energy and Fermi wave vector $k_F = \sqrt{2m\varepsilon_F}/\hbar$. Hence, the ratio between conductivity without driving and dressed conductivity is given by

$$\frac{\underset{\omega\to 0}{\mathrm{Re}}\,[\sigma^{xx}(0,\omega)]}{\underset{\omega\to 0}{\mathrm{Re}}\,[\sigma^{xx}(0,\omega)]\big|_{\gamma=0}} = \frac{1}{\sum_{l=-\infty}^{\infty} J_l^4(z_\varepsilon)}. \tag{62}$$

In Fig. 3 we present the conductivity of a 2DEG irradiated by a circularly (σ_c) polarized light of intensity I. For comparison also the results in the case of linearly polarized light are shown. We conclude that for a parabolic spectrum approximation the central entry of the scattering time used in Eq. (61) is equal to the result which one yields from the Floquet Fermi

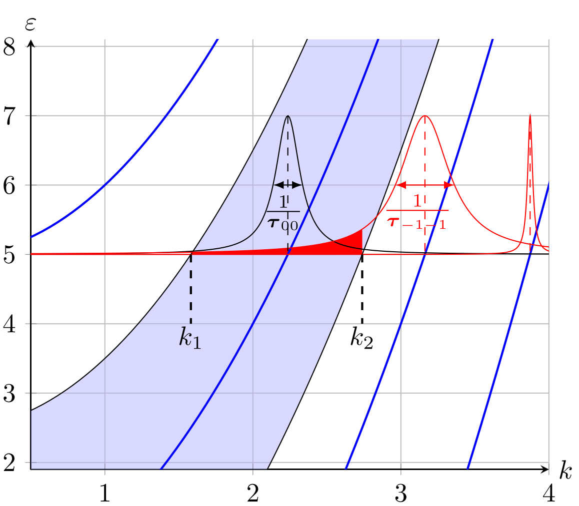

FIG. 2. The peaks show the broadening of the Floquet bands caused by the scattering time. The blue shaded area is the central Floquet zone. If $\Omega\tau_0 \simeq 1$, the nonzero Floquet modes are leaking into the central Floquet zone. The red shaded area marks the contribution of the -1 Floquet band to the conductivity.

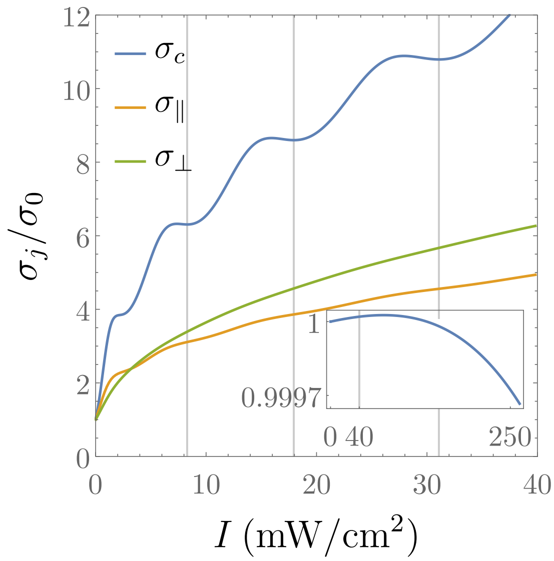

FIG. 3. The Drude conductivity of a 2DEG irradiated by a circularly (σ_c) and linearly polarized light of intensity I is plotted in comparison to the undriven case (σ_0). For linearly polarized light, $\sigma_\perp$ and $\sigma_\parallel$ are shown. The ratio $\sqrt{E_F/\mathrm{meV}}/[\Omega/(\mathrm{rad/s})]^2 = \sqrt{10}/10^{24}$ is chosen. The vertical lines indicate the minima of σ_c. The inset shows the result for σ_c for a square lattice with lattice constant $a = 5.6$ Å, $\hbar\Omega = 0.3$ THz, $E_F = 10$ meV, and hopping energy $g = 1.83$ eV. For the same parameters the parabolic result, Eq. (62), is $\sigma_c/\sigma_0 = 1 + 0.1I/(\mathrm{mW/cm^2}) + \mathcal{O}[\mathrm{I/(mW/cm^2)}]^2$.

golden rule [17,25,42]. It is used, e.g., by the authors of Ref. [20] to calculate the conductivity. However, the equation *ibid* overestimates the effect of the driving for the latter.

Our method allows us to go beyond the parabolic approximation. Let us examine a square lattice tight-binding model since here the spectrum is bounded. We define the lattice vectors as $\mathbf{a}_1 = a(1,0)^T$ and $\mathbf{a}_2 = a(0,1)^T$ and restrict the model to only nearest-neighbor hopping. Applying minimal coupling the corresponding Hamiltonian is given by

$$H(t) = -g\left[\sum_n e^{i\mathbf{k}\cdot\mathbf{a}_n} \cdot e^{i(e/\hbar)\mathbf{A}(t)\cdot\mathbf{a}_n} + \mathrm{H.c.}\right], \tag{63}$$

where g is the hopping parameter. The quasienergy ϵ can be obtained in an analytical form, see Appendix F,

$$\epsilon = -2g\left[J_0\left(a\frac{eA_x}{\hbar}\right)\cos(k_x a) + J_0\left(a\frac{eA_y}{\hbar}\right)\cos(k_y a)\right]. \tag{64}$$

In this case it can be explicitly shown that the current, averaged over one period T, is given by $\bar{\mathbf{J}}(\mathbf{k}) = -e\nabla_{\mathbf{k}}\epsilon(\mathbf{k})/\hbar$; see Appendix G. With this at hand and Eq. (55) the Drude conductivity can be readily calculated applying the triangle integration method [62–64]. The result for circularly polarized light is plotted in the inset of Fig. 3. It shows that in the case of a bounded spectrum the features appear at different driving intensities and can even disappear completely for reasonable intensities and driving frequencies in the THz regime. A more detailed comparison between both models is given in Appendix F 2. The inverse scattering times for exemplary parameters for both models of the 2DEG are depicted in

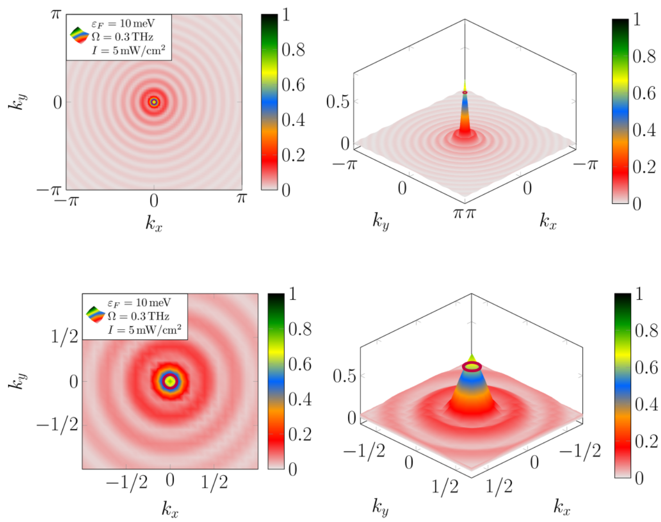

FIG. 4. Ratio between the central entry of the dressed inverse scattering time with the bare scattering time $[1/\boldsymbol{\tau}_{00}(\mathbf{k})]/[1/\boldsymbol{\tau}_{00}(\mathbf{k})|_{\gamma=0}]$ for the parabolic dispersion. The Fermi energy was set to 10 meV, the driving frequency to 0.3 THz, and the intensity to 5 mW/cm^2. The Fermi energy is shown as purple contour.

Figs. 4 and 5. Figure 4 shows the normalized inverse scattering time for the parabolic dispersion for some representative parameters. In the presence of the driving the inverse scattering time is significantly changed compared to the static case complying with the change of the conductivity depicted in Fig. 3. To support the comparability of the effective model and the tight-binding description the parameters are chosen to be equal for Fig. 5 and Fig. 4. In contrast to the parabolic dispersion the scattering time for the square lattice is barely

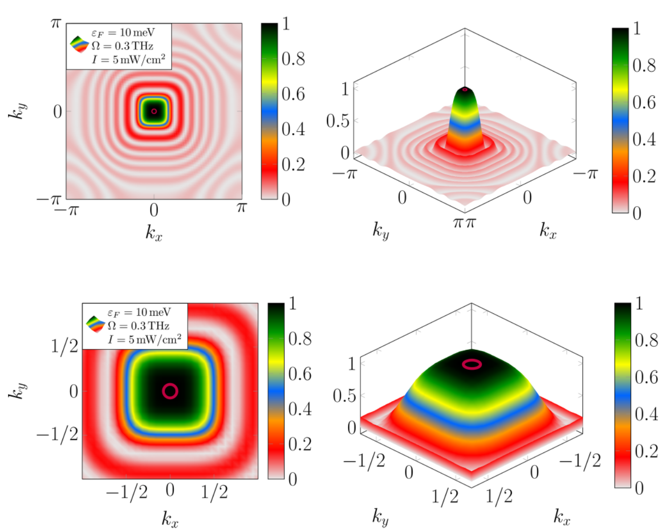

FIG. 5. Ratio between the central entry of the dressed inverse scattering time with the bare scattering time $[1/\boldsymbol{\tau}_{00}(\mathbf{k})]/[1/\boldsymbol{\tau}_{00}(\mathbf{k})|_{\gamma=0}]$ for the tight-binding model. The Fermi energy was set to 10 meV, the driving frequency to 0.3 THz, and the intensity to 5 mW/cm^2. The Fermi energy is shown as purple contour. It can be seen that the scattering time compared on the Fermi contour is barely changed by the driving.

changed due to the driving at the Fermi energy (see Fermi contour).

VIII. CONCLUSION

In this paper the Drude conductivity under an external driving has been investigated, including a formalism which allows for a rigorous derivation of the conductivity from an appropriate Dyson series by means of the Floquet formalism. The considerations are based on linear-response theory regarding the probe bias, whereas the external driving is incorporated nonperturbatively. Providing a new type of four-time Floquet Green functions, we were able to express the conductivity in terms of the aforementioned Green functions. This representation was so far missing. This type of Green functions allows for the formulation of a Dyson series, leading under the assumption of Gaussian white noise to the Floquet self-energy in Born approximation. To prove consistency of the presented considerations a generalized Floquet Fermi golden rule was derived yielding the same result for the scattering time as the Dyson series, a connetion that was missing so far. Neglecting coherent scattering processes leads to a compact expression for the Floquet-Drude conductivity. Due to its generality, the presented formalism can be applied to a broad class of materials. As a first application we investigated the conductivity for two basic models for a 2DEG. We have shown that the effective continuum model yields a strikingly different prediction for the conductivity compared to the tight-binding approach. The deviations of the predictions for the conductivity are caused by the different eigenstates of the models rather than the similar spectra.

ACKNOWLEDGMENT

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APPENDIX A: FLOQUET KUBO FORMULA FOR THE LINEAR CONDUCTIVITY

In a driven system the current is not simply a product of resistance and electric field; see Eq. (5) together with Eq. (6). Rather, the conductivity depends on both the frequency spectrum of the bias ω' as well as a response frequency ω. Furthermore, we introduce

$$\omega \equiv \tilde{\omega} + p\Omega \quad \Leftrightarrow \quad |\tilde{\omega}| \leqslant \Omega/2 \text{ with } p \in \mathbb{Z}, \tag{A1}$$

and require that the electric field E^b depends only on frequencies $|\tilde{\omega}'| \leqslant \Omega/2$. With this, Eq. (5) becomes

$$\langle J^a(\mathbf{q}, \tilde{\omega} + p\Omega)\rangle = \sum_b \int_{-\Omega/2}^{\Omega/2} d\tilde{\omega}'\, \bar{\sigma}^{ab} \times (\mathbf{q}, \tilde{\omega} + p\Omega, \tilde{\omega}')E^b(\mathbf{q}, \tilde{\omega}'), \tag{A2}$$

where the conductivity tensor is given by

$$\bar{\sigma}^{ab}(\mathbf{q}, \tilde{\omega} + p\Omega, \tilde{\omega}') = \frac{i}{\hbar(\tilde{\omega} + p\Omega)\mathfrak{V}} \sum_{\alpha\beta} \sum_{n_1..n_4=-\infty}^{\infty}$$

$$\times \frac{f_\alpha - f_\beta}{\tilde{\omega} + p\Omega + \frac{1}{\hbar}(\varepsilon_\alpha - \varepsilon_\beta) + (n_1 - n_2)\Omega + i0^+}$$
$$\times \langle u_\alpha^{n_1} | \mathbf{j}^\ell(\mathbf{q}) | u_\beta^{n_2} \rangle \langle u_\beta^{n_3} | \mathbf{j}^j(-\mathbf{q}) | u_\alpha^{n_4} \rangle$$
$$\times \delta[\tilde{\omega} + p\Omega + (n_1 - n_2 + n_3 - n_4)\Omega - \tilde{\omega}']$$
$$+ i \frac{e^2 n}{m(\tilde{\omega} + p\Omega)} \delta_{\ell j} \delta(\tilde{\omega} + p\Omega - \tilde{\omega}'). \tag{A3}$$

The argument of the δ distribution of the first term can become zero if and only if

$$n_1 - n_2 + n_3 - n_4 = -p. \tag{A4}$$

Since we are ultimately interested in the DC limit, we consider only the case where $p = 0$. Taking only the real part of the longitudinal conductivity one yields

$$\mathrm{Re}\,[\sigma^{xx}(0, \tilde{\omega})] = \frac{\pi}{\mathfrak{V}} \left(\frac{e}{m}\right)^2 \sum_{n,n'=-\infty}^{\infty} \sum_{\alpha\beta} \Bigg[\frac{f_\alpha - f_\beta}{\hbar\omega} \times \langle u_\alpha^n | p^x | u_\beta^n \rangle \langle u_\beta^{n'} | p^x | u_\alpha^{n'} \rangle \, \delta\big[\omega + \tfrac{1}{\hbar}(\varepsilon_\alpha - \varepsilon_\beta)\big] \Bigg]. \tag{A5}$$

In the limit of $\tilde{\omega} \to 0$ one ends up with Eq. (8).

APPENDIX B: t-t' FORMALISM

1. Separating the periodic from the aperiodic time dependence

In the t-t' formalism one starts from the Floquet states $|\psi_\alpha(t)\rangle = \exp(-\frac{i}{\hbar}\varepsilon_\alpha t)\,|u_\alpha(t)\rangle$ but formally discriminates the time dependence of the exponential from periodic time dependence as

$$|\psi_\alpha(t', t)\rangle = e^{-(i/\hbar)\varepsilon_\alpha t} |u_\alpha(t')\rangle, \tag{B1}$$

where obviously

$$|\psi_\alpha(t, t)\rangle = |\psi_\alpha(t)\rangle. \tag{B2}$$

The advantage of this artifice lies in the fact that the evolution of the states as a function of t is governed by the operator

$$U_F(t', t) = e^{-(i/\hbar)H_F(t')t}, \tag{B3}$$

i.e.,

$$|\psi_\alpha(t', t)\rangle = U_F(t', t - t_0)|\psi_\alpha(t', t_0)\rangle \tag{B4}$$
$$= e^{-(i/\hbar)\varepsilon_\alpha t_0} e^{-(i/\hbar)H_F(t')(t-t_0)} |u_\alpha(t')\rangle \tag{B5}$$
$$= e^{-(i/\hbar)\varepsilon_\alpha t} |u_\alpha(t')\rangle, \tag{B6}$$

which avoids any time ordering.

On the space of all states depending periodically with period T on a parameter t' having dimension of time, we define the scalar product

$$(\varphi|\chi) = \frac{1}{T}\int_0^T dt' \,\langle \varphi(t')|\chi(t')\rangle \tag{B7}$$
$$= \frac{1}{T}\int_0^T dt' \,\langle \varphi|t'\rangle\langle t'|\chi\rangle, \tag{B8}$$

which is equal to the scalar product introduced by Sambe [45]. The notation

$$\langle t'|\psi\rangle := |\psi(t')\rangle \tag{B9}$$

suggests to consider t' as a coordinate rather than a time parameter. The corresponding operator $\hat{t}'$ can be defined to act multiplicatively on the above wave functions,

$$\hat{t}'|\psi(t')\rangle = \langle t'|\hat{t}'|\psi\rangle = t'|\psi(t')\rangle, \tag{B10}$$

and the canonically conjugate operator is

$$\hat{w} := -i\hbar\partial_{t'} = H_F(t') - H(t') \Rightarrow [\hat{w}, \hat{t}'] = -i\hbar \tag{B11}$$

with a complete system of orthonormalized periodic eigenfunctions

$$\langle t'|l\rangle = e^{-i\Omega l t'},\ \ \hat{w}|l\rangle = l\hbar\Omega|l\rangle,\ \ (k|l) = \delta_{kl} \tag{B12}$$

with $k, l \in \mathbb{Z}$,

$$\sum_{l=-\infty}^{\infty} \langle t_1'|l\rangle\langle l|t_2'\rangle = T \sum_{s=-\infty}^{\infty} \delta(t_1' - t_2' + sT) \tag{B13}$$
$$= \langle t_1'|t_2'\rangle. \tag{B14}$$

In obtaining the completeness relation we have taken into account the Fourier expansion of the Dirac comb,

$$\sum_{r=-\infty}^{\infty} e^{ir\Omega t} = T \sum_{s=-\infty}^{\infty} \delta(t + sT). \tag{B15}$$

Switching between the two pertaining representations amounts, up to signs and prefactors, in the usual Fourier expansion,

$$\langle l|\psi\rangle = \frac{1}{T}\int_0^T dt' \,\langle l|t'\rangle\langle t'|\psi\rangle \tag{B16}$$
$$= \frac{1}{T}\int_0^T dt' \, e^{i\Omega l t'}\langle t'|\psi\rangle \tag{B17}$$
$$\Leftrightarrow\ \langle t'|\psi\rangle = \sum_{l=-\infty}^{\infty} \langle t'|l\rangle\langle l|\psi\rangle \tag{B18}$$
$$= \sum_{l=-\infty}^{\infty} e^{-i\Omega l t'}\langle l|\psi\rangle. \tag{B19}$$

Finally, the analogs of the wave functions $\psi_\alpha(q, t) = \langle q|\psi_\alpha(t)\rangle$ read in the t-t' formalism

$$\psi_\alpha(q, t', t) = \langle q|\psi_\alpha(t', t)\rangle = \langle q, t'|\psi_\alpha(t)\rangle. \tag{B20}$$

2. Field operators and one-particle Green functions

Generalizing the states (B1) we define

$$\big|\phi_\alpha^r(t', t)\big\rangle = e^{ir\Omega(t'-t)}|\psi_\alpha(t', t)\rangle \tag{B21}$$
$$= e^{-(i/\hbar)(\varepsilon_\alpha + r\hbar\Omega)t} e^{ir\Omega t'}|u_\alpha(t')\rangle \tag{B22}$$

with

$$\phi_\alpha^r(q, t', t) = \big\langle q\big|\phi_\alpha^r(t', t)\big\rangle = \big\langle q, t'\big|\phi_\alpha^r(t)\big\rangle \tag{B23}$$

and the simple properties

$$\big|\phi_\alpha^r(t', t)\big\rangle = U_F(t', t - t_0)\big|\phi_\alpha^r(t', t_0)\big\rangle, \tag{B24}$$

$$\big(\phi_\alpha^r(t)\big|\phi_\beta^s(t)\big) = \delta_{\alpha\beta}\delta_{rs}, \tag{B25}$$

$$\sum_\alpha \sum_{r=-\infty}^{\infty} \big|\phi_\alpha^r(t_1',t)\big\rangle\big\langle\phi_\alpha^r(t_2',t)\big| = \mathbb{1}\, T \sum_{s=-\infty}^{\infty} \delta(t_1'-t_2'+sT). \tag{B26}$$

In second quantization, this allows us to define a system of creation and annihilation operators $b_{\alpha r}^\dagger(t)$, $b_{\alpha r}(t)$ with

$$\big|\phi_\alpha^r(t)\big\rangle = b_{\alpha r}^\dagger(t)|0\rangle, \quad b_{\alpha r}(t)|0\rangle = 0 \tag{B27}$$

and

$$[b_{\alpha r}(t), b_{\beta s}^\dagger(t)]_\pm = \delta_{\alpha\beta}\delta_{rs}, \tag{B28}$$

$$[b_{\alpha r}(t), b_{\beta s}(t)]_\pm = [b_{\alpha r}^\dagger(t), b_{\beta s}^\dagger(t)]_\pm = 0. \tag{B29}$$

Field operators can be constructed as

$$\Phi(q,t',t) = \sum_\alpha \sum_{r=-\infty}^{\infty} \phi_\alpha^r(q,t',t) b_{\alpha r}(t), \tag{B30}$$

fulfilling

$$[\Phi(q_1,t_1',t), \Phi^\dagger(q_1,t_2',t)]_\pm = \delta(q_1-q_2)\, T \sum_{s=-\infty}^{\infty} \delta(t_1'-t_2'+sT) \tag{B31}$$

with again all other (anti)commutators at equal times t being zero, and

$$\langle q_2,t_2'|\Phi^\dagger(q_1,t_1',t)|0\rangle = \delta(q_1-q_2)\, T \sum_{s=-\infty}^{\infty} \delta(t_1'-t_2'+sT). \tag{B32}$$

The Floquet Hamiltonian H_F can be formulated as

$$H_F(t) = \sum_\alpha \sum_{r=-\infty}^{\infty} (\varepsilon_\alpha + r\hbar\Omega) b_{\alpha r}^\dagger(t) b_{\alpha r}(t) \tag{B33}$$

and is neither bounded from below nor from above. Going over to the Heisenberg picture,

$$\Phi_H(q,t',t) = U_F^\dagger(t',t)\Phi(q,t',t)U_F(t',t) = \sum_\alpha \sum_{r=-\infty}^{\infty} \phi_\alpha^r(q,t',t) b_{\alpha r}(0), \tag{B34}$$

we yield the retarded/advanced one-particle Green function

$$\begin{aligned}\mathcal{G}^{r/a}&(q_1,t_1',t_1,q_2,t_2',t_2)\\ &= \mp i\Theta[\pm(t_1-t_2)]\frac{1}{T}\langle[\Phi_H(q_1,t_1',t_1), \Phi_H^\dagger(q_2,t_2',t_2)]_\epsilon\rangle\\ &= \mp i\Theta[\pm(t_1-t_2)]\frac{1}{T}\\ &\quad\times \sum_{r=-\infty}^{\infty}\sum_\alpha \phi_\alpha^r(q_1,t_1',t_1)\big[\phi_\alpha^r(q_2,t_2',t_2)\big]^*\\ &= \mp i\Theta[\pm(t_1-t_2)]\frac{1}{T}\sum_{r=-\infty}^{\infty}\sum_\alpha \big[e^{-(i/\hbar)(\varepsilon_\alpha+r\hbar\Omega)(t_1-t_2)}\\ &\quad\times \langle q_1|u_\alpha(t_1')\rangle\langle u_\alpha(t_2')|q_2\rangle e^{ir\Omega(t_1'-t_2')}\big],\end{aligned} \tag{B35}$$

or, formulated as a Green operator,

$$\begin{aligned}\hat{\mathcal{G}}^{r/a}&(t_1',t_1,t_2',t_2)\\ &= \mp i\Theta[\pm(t_1-t_2)]\frac{1}{T}\sum_{r=-\infty}^{\infty}\sum_\alpha \big|\phi_\alpha^r(t_1',t_1)\big\rangle\big\langle\phi_\alpha^r(t_2',t_2)\big|\\ &= \mp i\Theta[\pm(t_1-t_2)]\frac{1}{T}\sum_{r=-\infty}^{\infty}\sum_\alpha \big[e^{-(i/\hbar)(\varepsilon_\alpha+r\hbar\Omega)(t_1-t_2)}\\ &\quad\times |u_\alpha(t_1')\rangle\langle u_\alpha(t_2')| e^{ir\Omega(t_1'-t_2')}\big].\end{aligned} \tag{B36}$$

These quantities have the significant property

$$\begin{aligned}&\left(i\partial_{t_1} - \frac{1}{\hbar}H_F(t_1')\right)\hat{\mathcal{G}}^{r/a}(t_1,t_2,t_1',t_2')\\ &= \delta(t_1-t_2)\sum_\alpha |u_\alpha(t_1')\rangle\langle u_\alpha(t_2')|\frac{1}{T}\sum_{r=-\infty}^{\infty} e^{ir\Omega(t_1'-t_2')}\\ &= \delta(t_1-t_2)\mathbb{1}\sum_{s=-\infty}^{\infty}\delta(t_1'-t_2'+sT),\end{aligned} \tag{B37}$$

where we have used Eqs. (B15) and the completeness relation of the Floquet functions $|u_\alpha(t)\rangle$. As the expressions (B36) depend only on the difference $t_1 - t_2$ and are periodic in t_1', t_2' we can go over to Fourier components as

$$\begin{aligned}\hat{\mathcal{G}}^{r/a}&(\omega,n_1,n_2)\\ &= \int_{-\infty}^{\infty} dt\, e^{i\omega t}\sum_{n_1,n_2=-\infty}^{\infty} e^{-in_1\Omega t_1'+in_2\Omega t_2'}\hat{\mathcal{G}}^{r/a}(t_1',t,t_2',0)\\ &= \frac{1}{T}\sum_{r=-\infty}^{\infty}\sum_\alpha \frac{\big|u_\alpha^{n_1+r}\big\rangle\big\langle u_\alpha^{n_2+r}\big|}{\omega - \frac{1}{\hbar}(\varepsilon_\alpha + r\hbar\Omega) \pm i0^+},\end{aligned} \tag{B38}$$

where the last line follows from

$$\Theta(\pm t) = \frac{\pm i}{2\pi}\int_{-\infty}^{\infty} d\omega\, \frac{e^{-i\omega t}}{\omega \pm i0^+}. \tag{B39}$$

Moreover, with the spectral density

$$\begin{aligned}A&(q_1,t_1',t_1,q_2,t_2',t_2)\\ &= \frac{1}{2\pi T}\langle[\Phi_H(q_1,t_1',t_1), \Phi_H^\dagger(q_2,t_2',t_2)]_\epsilon\rangle\\ &= \frac{1}{2\pi T}\sum_{r=-\infty}^{\infty}\sum_\alpha \big[e^{-(i/\hbar)(\varepsilon_\alpha+r\hbar\Omega)(t_1-t_2)}\\ &\quad\times \langle q_1|u_\alpha(t_1')\rangle\langle u_\alpha(t_2')|q_2\rangle e^{ir\Omega(t_1'-t_2')}\big],\end{aligned} \tag{B40}$$

having Fourier components

$$\begin{aligned}A&(\omega,q_1,t_1',q_2,t_2')\\ &= \frac{1}{T}\sum_{r=-\infty}^{\infty}\sum_\alpha \{\delta[\omega - (\varepsilon_\alpha + r\hbar\Omega)]\\ &\quad\times \langle q_1|u_\alpha(t_1')\rangle\langle u_\alpha(t_2')|q_2\rangle e^{ir\Omega(t_1'-t_2')}\},\end{aligned} \tag{B41}$$

we obtain the familiar Lehmann representation of the Green function,

$$\mathcal{G}^{r/a}(\omega, q_1, t_1', q_2, t_2') = \int_{-\infty}^{\infty} d\omega' \frac{A(\omega, q_1, t_1', q_2, t_2')}{\omega - \omega' \pm i0^+}. \quad \text{(B42)}$$

In summary, when treating t' not as a time parameter but rather as a state coordinate, the remaining time evolution in t is governed by the Floquet Hamiltonian being independent of t. Thus, we are left with an effectively time-independent Hamiltonian, and many formal manipulations known for such a situation work just in the same way. Note, however, that (i) the physical case still requires $t = t'$, and (ii) the Floquet Hamiltonian (B33) fails to be bounded from below.

3. Justification of the t-t' formalism

The necessity to use the t-t' formalism is far from obvious, hence, we would like to clarify why it is useful. We start with the Green function of the time-dependent Schrödinger equation

$$G_0^{r,a}(t_1, t_2) = \mp i\Theta[\pm(t_1 - t_2)] \sum_\alpha |\psi_\alpha(t_1)\rangle\langle\psi_\alpha(t_2)|. \quad \text{(B43)}$$

It is desirable to draw on the result from the Kubo formula in the static case where the conductivity can be written as a trace over the difference of retarded and advanced Green's functions. The ansatz is therefore

$$\tilde{\sigma} \equiv \mathrm{tr}\big\{p^x\big[G_0^r(\cdot) - G_0^a(\cdot)\big]p^x\big[G_0^r(\cdot) - G_0^a(\cdot)\big]\big\}. \quad \text{(B44)}$$

The time dependencies of the Green functions $G^{r,a}(\cdot)$ are left blank intentionally, since the above expression should only show the desired structure. If $\tilde{\sigma}$ is assumed to have the above form, there are only a few possible time dependencies of the Green functions. In the following some attempts, using the Green function of the time-dependent Schrödinger equation, are presented that fail to reproduce the result obtained directly from Eq. (9). Using the two-time Green function together with the same time ordering as in Eq. (21) yields the expression

$$\begin{aligned}\sigma(t_1, t_2) = \mathrm{tr}\big\{&p^x\big[G_0^r(t_1, t_2) - G_0^a(t_1, t_2)\big] \\ &\times\ p^x\big[G_0^r(t_2, t_1) - G_0^a(t_2, t_1)\big]\big\}.\end{aligned} \quad \text{(B45)}$$

Introducing Wigner coordinates

$$t = t_1 - t_2, \quad T = \frac{t_1 + t_2}{2}, \quad \text{(B46)}$$

where the relative time is Fourier transformed into energy space and the mean time is averaged over one driving cycle leading to

$$\begin{aligned}\sigma(\varepsilon) = \sum_{\alpha\beta} \sum_{nn'=-\infty}^{\infty} &\langle u_\alpha^n|p^x|u_\beta^n\rangle\langle u_\beta^{n'}|p^x|u_\alpha^{n'}\rangle \\ &\times\ \delta(\varepsilon + \varepsilon_\alpha - \varepsilon_\beta).\end{aligned} \quad \text{(B47)}$$

While the ordering of the Floquet indices is correct, the above quantity is proportional to a single delta distribution and thus not equal to Eq. (9). Any different time ordering in Eq. (B45) leads to an incorrect ordering of the Floquet indices.

Next, the attempt is analyzed where the Green functions depend on different relative times and the common mean time is averaged,

$$\begin{aligned}\sigma(t, t') = \frac{1}{T}\int_0^T dT\ \mathrm{tr}\big\{&p^x\big[G_0^r(t, T) - G_0^a(t, T)\big] \\ &\times\ p^x\big[G_0^r(t', T) - G_0^a(t', T)\big]\big\}.\end{aligned} \quad \text{(B48)}$$

Now perform a Fourier transformation of both relative times onto the same energy,

$$\sigma(\varepsilon) = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt'\ e^{(i/\hbar)\varepsilon(t+t')}\sigma(t, t'), \quad \text{(B49)}$$

which leads to the correct result for the conductivity if and only if the distribution function fulfills

$$\frac{\partial f}{\partial\big[\varepsilon + (n + n')\frac{\Omega}{2}\big]} = \frac{\partial f}{\partial \varepsilon} \quad \forall\, n, n' \in \mathbb{Z}. \quad \text{(B50)}$$

The last equation requires a half integer periodicity of the derivative of the distribution function, which is likely not to be fulfilled by an actual distribution function. Rather curious, defining a function by neglecting the summation over the Floquet indices in Eq. (12), namely

$$\tilde{G}^{r,a}(\varepsilon, t_1', t_2') = \frac{1}{T}\sum_\alpha \frac{|u_\alpha(t_1')\rangle\ \langle u_\alpha(t_2')|}{\frac{1}{\hbar}\varepsilon - \frac{1}{\hbar}\varepsilon_\alpha \pm i0^+}, \quad \text{(B51)}$$

allows us to reproduce the result form the Kubo formula

$$\begin{aligned}\sigma(\varepsilon) = \int_0^T dt_1' \int_0^T dt_2'\ \mathrm{tr}\{&p^x[\tilde{G}^r(\varepsilon, t_1', t_2') - \tilde{G}^a(\varepsilon, t_1', t_2')] \\ &\times\ p^x[\tilde{G}^r(\varepsilon, t_2', t_1') - \tilde{G}^a(\varepsilon, t_2', t_1')]\}.\end{aligned} \quad \text{(B52)}$$

However, the function given in Eq. (B51) is not a Green function of the Schrödinger equation or of the t-t' Schödinger equation.

The results presented in this section are not intended to prove that it is not possible to express Eq. (9) in terms of the Green function of the time-dependent Schrödinger equation. Nevertheless, it demonstrates that a naive ansatz fails. If the structure given in Eq. (B44) is desired, one way to solve the problem is to use the four-time Green functions introduced in Sec. III.

APPENDIX C: DYSON SERIES FOR TIME-DEPENDENT PERTURBATIONS

Here, we use the same notation as in Eqs. (22). As already shown in the latter, the bare Green function $\mathcal{G}_0$ fulfills the equation

$$\begin{aligned}&\big[i\partial_{t_1} - \tfrac{1}{\hbar}H_F^0(\mathbf{x}_1, t_1')\big]\mathcal{G}_0^{r,a}(t_1, t_2, \mathbf{x}_1, \mathbf{x}_2, t_1', t_2') \\ &\quad = \delta(t_1 - t_2)\delta(\mathbf{x}_1 - \mathbf{x}_2)\sum_{s=-\infty}^{\infty}\delta(t_1' - t_2' + sT)\mathbb{1}\end{aligned} \quad \text{(C1)}$$

with the definition of the Floquet Hamiltonian for the unperturbed system

$$H_F^0(\mathbf{x}_1, t_1') = H(\mathbf{x}_1, t_1') - i\hbar\partial_{t_1'}. \quad \text{(C2)}$$

The Hamiltonian for the perturbed one has the form

$$\mathcal{H}_F(\mathbf{x}_1, t_1, t_1') = H_F^0(\mathbf{x}_1, t_1') + V(\mathbf{x}_1, t_1, t_1'), \quad \text{(C3)}$$

where we stress that the dependency on t_1 is fully kept by the potential. Obviously, the bare Green function $\mathcal{G}_0$ fulfills

$$\begin{aligned}&\big[i\partial_{t_1} - \tfrac{1}{\hbar}\mathcal{H}_F(\mathbf{x}_1, t_1, t_1')\\&\quad + \tfrac{1}{\hbar}V(\mathbf{x}_1, t_1, t_1')\big]\mathcal{G}_0^{r,a}(t_1, t_2, \mathbf{x}_1, \mathbf{x}_2, t_1', t_2')\\&= \delta(t_1 - t_2)\delta(\mathbf{x}_1 - \mathbf{x}_2)\sum_{s=-\infty}^{\infty}\delta(t_1' - t_2' + sT)\mathbb{1}.\end{aligned} \tag{C4}$$

We are interested in the Green function of the perturbed system $\mathcal{G}_{\rm p}$ being a solution of

$$\begin{aligned}&\big[i\partial_{t_1} - \tfrac{1}{\hbar}\mathcal{H}_F(\mathbf{x}_1, t_1, t_1')\big]\mathcal{G}_{\rm p}^{r,a}(t_1, t_2, \mathbf{x}_1, \mathbf{x}_2, t_1', t_2')\\&= \delta(t_1 - t_2)\delta(\mathbf{x}_1 - \mathbf{x}_2)\sum_{s=-\infty}^{\infty}\delta(t_1' - t_2' + sT)\mathbb{1}.\end{aligned} \tag{C5}$$

Equating Eqs. (C4) and (C5) we get

$$\begin{aligned}&\big[i\partial_{t_1} - \tfrac{1}{\hbar}\mathcal{H}_F(\mathbf{x}_1, t_1, t_1')\big]\mathcal{G}_{\rm p}^{r,a}(t_1, t_2, \mathbf{x}_1, \mathbf{x}_2, t_1', t_2')\\&= \big[i\partial_{t_1} - \tfrac{1}{\hbar}\mathcal{H}_F(\mathbf{x}_1, t_1, t_1')\big]\mathcal{G}_0^{r,a}(t_1, t_2, \mathbf{x}_1, \mathbf{x}_2, t_1', t_2')\\&\quad + \tfrac{1}{\hbar}V(\mathbf{x}_1, t_1, t_1')\,\mathcal{G}_0^{r,a}(t_1, t_2, \mathbf{x}_1, \mathbf{x}_2, t_1', t_2').\end{aligned} \tag{C6}$$

The bare Green function is periodic in both t_1', and t_2',

$$\mathcal{G}_0^{r,a}(t, t_2, \mathbf{x}, \mathbf{x}_2, t_1' + T, t_2' + T) = \mathcal{G}_0^{r,a}(t, t_2, \mathbf{x}, \mathbf{x}_2, t_1', t_2'). \tag{C7}$$

Therefore, without loss of generality one can use the restriction

$$t_1', t_2' \in \big[-\tfrac{T}{2}, \tfrac{T}{2}\big]. \tag{C8}$$

Making use of the periodicity of the Green function and requiring that the potential is as well periodic in the second time argument,

$$V(\mathbf{x}, t_1, t_1' + T) = V(\mathbf{x}, t_1, t_1'), \tag{C9}$$

one can show that

$$\begin{aligned}&\int_{V_{\rm x}} d\mathbf{x}\int_{-\infty}^{\infty}dt\int_{-T/2}^{T/2}dt'\,\delta(t_1 - t)\delta(\mathbf{x}_1 - \mathbf{x})\\&\quad\times\sum_{s=-\infty}^{\infty}\delta(t_1' - t' + sT)V(\mathbf{x}, t, t')\mathcal{G}_0^{r,a}(t, t_2, \mathbf{x}, \mathbf{x}_2, t', t_2')\\&= \int_{-T/2}^{T/2}dt'\sum_{s=-\infty}^{\infty}\delta(t_1' - t' + sT)V(\mathbf{x}_1, t_1, t')\\&\quad\times\mathcal{G}_0^{r,a}(t_1, t_2, \mathbf{x}_1, \mathbf{x}_2, t', t_2')\\&= V(\mathbf{x}_1, t_1, t_1')\mathcal{G}_0^{r,a}(t_1, t_2, \mathbf{x}_1, \mathbf{x}_2, t_1', t_2'),\end{aligned} \tag{C10}$$

where in the last step we have used that the argument of the delta distribution can only be zero if $s = 0$. Comparing this equation with Eq. (C6) one finds a Dyson expansion for the Green function of the perturbed system,

$$\begin{aligned}\mathcal{G}_{\rm p}^{r,a}(t_1, t_2, \mathbf{x}_1, \mathbf{x}_2, t_1', t_2') &= \mathcal{G}_0^{r,a}(t_1, t_2, \mathbf{x}_1, \mathbf{x}_2, t_1', t_2')\\&\quad + \frac{1}{\hbar}\int_{V_{\rm x}} d\mathbf{x}\int_{-\infty}^{\infty}dt\int_{-T/2}^{T/2}dt'\,\mathcal{G}_{\rm p}^{r,a}(t_1, t, \mathbf{x}_1, \mathbf{x}, t_1', t')\\&\quad\times V(\mathbf{x}, t, t')\mathcal{G}_0^{r,a}(t, t_2, \mathbf{x}, \mathbf{x}_2, t', t_2').\end{aligned} \tag{C11}$$

If one assumes that the potential depends only on the periodic time component

$$V(\mathbf{x}, t, t') = V(\mathbf{x}, t') \Leftrightarrow H_F(\mathbf{x}, t, t') = H_F(\mathbf{x}, t') \tag{C12}$$

the Green function depends only on the difference $t_1 - t_2$,

$$\mathcal{G}_{\rm p}^{r,a}(t_1, t_2, \mathbf{x}_1, \mathbf{x}_2, t_1', t_2') = \mathcal{G}_{\rm p}^{r,a}(t_1 - t_2, \mathbf{x}_1, \mathbf{x}_2, t_1', t_2'). \tag{C13}$$

Applying Fourier transform on Eq. (C11) with respect to $t_1 - t_2$, one yields in energy space

$$\begin{aligned}\mathcal{G}_{\rm p}^{r,a}(\varepsilon, \mathbf{x}_1, \mathbf{x}_2, t_1', t_2') &= \mathcal{G}_0^{r,a}(\varepsilon, \mathbf{x}_1, \mathbf{x}_2, t_1', t_2')\\&\quad + \frac{1}{\hbar}\int_{V_{\rm x}} d\mathbf{x}\int_{-T/2}^{T/2}dt'\,\mathcal{G}_{\rm p}^{r,a}(\varepsilon, \mathbf{x}_1, \mathbf{x}, t_1', t')\\&\quad\times V(\mathbf{x}, t')\mathcal{G}_0^{r,a}(\varepsilon, \mathbf{x}, \mathbf{x}_2, t', t_2'),\end{aligned} \tag{C14}$$

where the explicit form of the bare Green function is given by

$$\begin{aligned}&\mathcal{G}_0^{r,a}(\varepsilon, \mathbf{x}_1, \mathbf{x}_2, t_1', t_2')\\&= \frac{1}{T}\sum_{r=-\infty}^{\infty}\sum_{\alpha}\frac{u_\alpha(\mathbf{x}_1, t_1')[u_\alpha(\mathbf{x}_2, t_2')]^*}{\frac{1}{\hbar}\varepsilon - \frac{1}{\hbar}\varepsilon_\alpha - r\Omega \pm i0^+}\,e^{ir\Omega(t_1' - t_2')}\\&= \frac{1}{T}\sum_{nn'}\mathcal{G}_0^{r,a}(\varepsilon, \mathbf{x}_1, \mathbf{x}_2, n, n')e^{-in\Omega t_1'}e^{in'\Omega t_2'}.\end{aligned} \tag{C15}$$

The Fourier coefficients are given by

$$\mathcal{G}_0^{r,a}(\varepsilon, \mathbf{x}_1, \mathbf{x}_2, n, n') = \sum_{r=-\infty}^{\infty}\sum_{\alpha}\frac{u_\alpha^{n+r}(\mathbf{x}_1)\big[u_\alpha^{n'+r}(\mathbf{x}_2)\big]^*}{\frac{1}{\hbar}\varepsilon - \frac{1}{\hbar}\varepsilon_\alpha - r\Omega \pm i0^+}, \tag{C16}$$

where we used the shortened notation

$$u_\alpha(\mathbf{x}_1, t_1') \equiv \langle\mathbf{x}_1|u_\alpha(t_1')\rangle,\ \ u_\alpha^n(\mathbf{x}_1) \equiv \langle\mathbf{x}_1|u_\alpha^n\rangle. \tag{C17}$$

Since we required the potential to be periodic in the second time argument it can be expanded in a Fourier series,

$$V(\mathbf{x}, t') = \sum_{n=-\infty}^{\infty}V_n(\mathbf{x})e^{-in\Omega t'}. \tag{C18}$$

This allows us to rewrite Eq. (C14) and perform the remaining time integration,

$$\mathcal{G}_{\rm p}^{r,a}(\varepsilon, \mathbf{x}_1, \mathbf{x}_2, n, n') = \mathcal{G}_0^{r,a}(\varepsilon, \mathbf{x}_1, \mathbf{x}_2, n, n') \tag{C19}$$

$$\begin{aligned}&+ \frac{1}{\hbar}\int_{V_{\rm x}} d\mathbf{x}\sum_{n_1, n_2=-\infty}^{\infty}\mathcal{G}_{\rm p}^{r,a}(\varepsilon, \mathbf{x}_1, \mathbf{x}, n, n_1)\\&\times V_{n_1 - n_2}(\mathbf{x})\mathcal{G}_0^{r,a}(\varepsilon, \mathbf{x}, \mathbf{x}_2, n_2, n').\end{aligned} \tag{C20}$$

APPENDIX D: FLOQUET FERMI GOLDEN RULE

A generalization of Fermi's golden rule [3,4,36,58,65] to time periodic Hamiltonians, i.e., the Floquet Fermi golden rule, was already derived by Kitagawa *et al.* in Ref. [17]. However, a detailed derivation and discussion of the "scattering theory for Floquet-Bloch states" is given in Ref. [42]. The Floquet Fermi golden rule was used by Kibis in Ref. [25] in order to explain the suppression of backscattering of conduction electrons in the presence of a high-frequency electric field. In regard to Fermi's golden rule for the t-t' Floquet states, the derivation of the Floquet Fermi golden rule is presented here in detail. It is assumed that the solution of the time-dependent Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}|\psi_\alpha(t)\rangle = H(t)|\psi_\alpha(t)\rangle \tag{D1}$$

and the corresponding time evolution operator $U_0(t, t_0)$ are known. In the presence of a time-dependent perturbation $V(t)$ the Schrödinger equation becomes

$$i\hbar\frac{\partial}{\partial t}|\Psi_\alpha(t)\rangle = [H(t)+V(t)]|\Psi_\alpha(t)\rangle. \tag{D2}$$

The potential $V(t)$ is switched on at a reference time t_0 such that the solutions of the Schrödinger equation coincide for times $t \leqslant t_0$,

$$|\psi_\alpha(t)\rangle = |\Psi_\alpha(t)\rangle \quad \text{for} \quad t \leqslant t_0. \tag{D3}$$

At times $t \leqslant t_0$ the particle is assumed to be in an eigenstate of the unperturbed Hamiltonian. Standard perturbation theory leads to the transition amplitude

$$\langle\psi_\beta(t)|\Psi_\alpha(t)\rangle = \delta_{\alpha\beta} + \frac{1}{i\hbar}\int_{t_0}^{t} dt'\,\langle\psi_\beta(t')|V(t')|\psi_\alpha(t')\rangle \tag{D4}$$

up to first order in the potential. Without loss of generality t_0 can be set to zero and for $\alpha \neq \beta$ the first nontrivial order of Eq. (D4) is given by

$$a_{\alpha\beta}(t) = -\frac{i}{\hbar}\int_0^t dt'\,\langle\psi_\beta(t')|V(t')|\psi_\alpha(t')\rangle. \tag{D5}$$

This formula, the Floquet Fermi golden rule, is equal to Eq. (10) of Ref. [25]. To proceed further, scattering from a Floquet state into a state with constant quasienergy

$$|\psi_\alpha(\varepsilon, t)\rangle = e^{-(i/\hbar)\varepsilon t}|u_\alpha(t)\rangle \tag{D6}$$

is considered. The quasienergy ε is independent of the quantum number. Hence, this state is not an eigenstate of the Hamiltonian, nevertheless it fulfills

$$\langle\psi_\alpha(t)|\psi_\beta(\varepsilon, t)\rangle = \delta_{\alpha\beta}\,e^{(i/\hbar)(\varepsilon_\alpha-\varepsilon)t}. \tag{D7}$$

Consequently, Eq. (D5) remains valid if the final state is $|\psi_\alpha(\varepsilon, t)\rangle$. Now, consider a scattering event from a Floquet state $\psi_\alpha(\mathbf{k}', t)$ into a state with constant energy ε,

$$\psi_\alpha(\mathbf{k}', t) = e^{-(i/\hbar)\varepsilon_\alpha(\mathbf{k}')t}u_\alpha(\mathbf{k}', t) \rightsquigarrow e^{-(i/\hbar)\varepsilon t}u_\beta(\mathbf{k}, t). \tag{D8}$$

If the perturbation $V(t)$ is time independent Eq. (D5) becomes

$$a_{\alpha\beta}(\mathbf{k}, \mathbf{k}', t) = -i\frac{V_{\mathbf{kk}'}}{\hbar}\sum_{nn'=-\infty}^{\infty}\int_0^t dt'\,e^{(i/\hbar)[\varepsilon-\varepsilon_\alpha(\mathbf{k}')-(n-n')\hbar\Omega]t'} \times \left[u_\beta^{n'}(\mathbf{k})\right]^* u_\alpha^n(\mathbf{k}') \tag{D9}$$

with $V_{\mathbf{kk}'} = \langle\varphi_{\mathbf{k},\mathbf{r}}|V(\mathbf{r})|\varphi_{\mathbf{k}',\mathbf{r}}\rangle$ where $\varphi_{\mathbf{k},\mathbf{r}} = \exp(-i\mathbf{k}\cdot\mathbf{r})/\sqrt{\mathfrak{V}}$.

Shifting t' by $-t/2$ yields the probability density

$$\begin{aligned}|a_{\alpha\beta}(\mathbf{k}, \mathbf{k}', t)|^2 &= \frac{V_{\mathbf{kk}'}^2}{\hbar^2}\Bigg|\sum_{nn'=-\infty}^{\infty}e^{(i/2\hbar)[\varepsilon-\varepsilon_\alpha(\mathbf{k}')-(n-n')\hbar\Omega]t} \\ &\quad\times \left[u_\beta^{n'}(\mathbf{k})\right]^* u_\alpha^n(\mathbf{k}')\int_{-t/2}^{t/2}dt'\,e^{(i/\hbar)[\varepsilon-\varepsilon_\alpha(\mathbf{k}')-(n-n')\hbar\Omega]t'}\Bigg|^2.\end{aligned} \tag{D10}$$

In the long-time limit $t \to \infty$ this simplifies to

$$|a_{\alpha\beta}(\mathbf{k}, \mathbf{k}', t)|^2 = 4\pi^2 V_{\mathbf{kk}'}^2\Bigg|\sum_{nn'=-\infty}^{\infty}\left[u_\beta^{n'}(\mathbf{k})\right]^* u_\alpha^n(\mathbf{k}') \times \delta[\varepsilon-\varepsilon_\alpha(\mathbf{k}')-(n-n')\hbar\Omega]\Bigg|^2. \tag{D11}$$

The quasienergies ε and $\varepsilon_\alpha(\mathbf{k})$ are chosen to be in the central Floquet zone such that

$$\forall_{\mathbf{k}}:\ |\varepsilon-\varepsilon_\alpha(\mathbf{k})| \leqslant \hbar\Omega, \tag{D12}$$

$$\begin{aligned}&\delta(\varepsilon-\varepsilon_\alpha-n\hbar\Omega)\delta(\varepsilon-\varepsilon_\alpha-m\hbar\Omega) \\ &= \delta^2(\varepsilon-\varepsilon_\alpha-n\hbar\Omega)\delta_{nm}.\end{aligned} \tag{D13}$$

Hence, Eq. (D11) becomes

$$|a_{\alpha\beta}(\mathbf{k}, \mathbf{k}', t)|^2 = 4\pi^2 V_{\mathbf{kk}'}^2\sum_{n=-\infty}^{\infty}c_{\beta\alpha}^{-n}(\mathbf{k}, \mathbf{k}')\left[c_{\beta\alpha}^{-n}(\mathbf{k}, \mathbf{k}')\right]^* \times \delta^2[\varepsilon-\varepsilon_\alpha(\mathbf{k}')-n\hbar\Omega] \tag{D14}$$

with

$$c_{\alpha\beta}^n(\mathbf{k}, \mathbf{k}') := \sum_{m=-\infty}^{\infty}\left[u_\alpha^{m+n}(\mathbf{k})\right]^* u_\beta^m(\mathbf{k}'). \tag{D15}$$

The square of the delta distribution can be rewritten as shown in Eq. (48). The transition probability is then

$$\Gamma_{\alpha\beta}(\mathbf{k}, \mathbf{k}') := \frac{d|a_{\alpha\beta}(\mathbf{k}, \mathbf{k}', t)|^2}{dt} \tag{D16}$$

$$= \frac{2\pi}{\hbar}V_{\mathbf{kk}'}^2\sum_{n=-\infty}^{\infty}c_{\beta\alpha}^{-n}(\mathbf{k}, \mathbf{k}')[c_{\beta\alpha}^{-n}(\mathbf{k}, \mathbf{k}')]^* \times \delta[\varepsilon-\varepsilon_\alpha(\mathbf{k}')-n\hbar\Omega]. \tag{D17}$$

The delta distribution can only have support if $n = 0$. Performing an impurity average according to the main text leads to $\langle V_{\mathbf{kk}'}^2\rangle_{\text{imp}} = \mathcal{V}_{\text{imp}}$ such that

$$\langle\Gamma_{\alpha\beta}(\mathbf{k}, \mathbf{k}')\rangle_{\text{imp}} = \langle\Gamma_{\alpha\beta}(\mathbf{k}, \mathbf{k}')\rangle_{\text{imp}} \tag{D18}$$

$$= \frac{2\pi}{\hbar}\mathcal{V}_{\text{imp}}\left|c_{\beta\alpha}^0(\mathbf{k}, \mathbf{k}')\right|^2\delta[\varepsilon-\varepsilon_\alpha(\mathbf{k}')]. \tag{D19}$$

The scattering time is then governed by the sum over all initial states and the sum over all momenta,

$$\frac{1}{\tau_\beta(\varepsilon, \mathbf{k})} = \frac{1}{V_{\mathbf{k}'}}\sum_{\mathbf{k}'}\sum_\alpha\langle\Gamma_{\alpha\beta}(\mathbf{k}, \mathbf{k}')\rangle_{\text{imp}} \tag{D20}$$

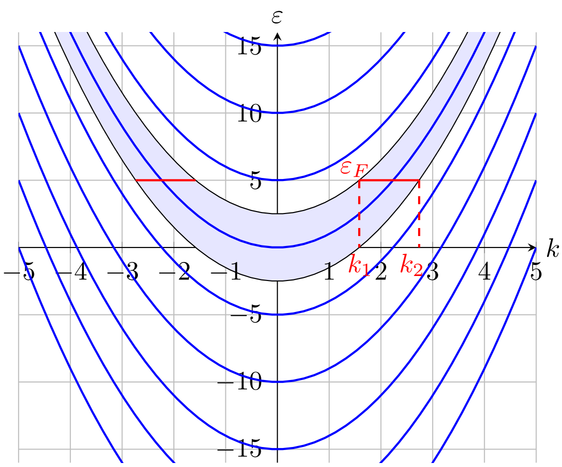

FIG. 6. The Floquet zones are chosen to wrap around the parabolas. The quasi-Fermi energy is only defined in a certain momentum range, i.e., $k \in [k_1, k_2)$, in the central Floquet zone.

$$= \frac{2\pi}{\hbar}\mathcal{V}_{\text{imp}}\frac{1}{V_{\mathbf{k}'}}\sum_{\mathbf{k}'}\sum_{\alpha}\left|c^0_{\beta\alpha}(\mathbf{k},\mathbf{k}')\right|^2\delta[\varepsilon-\varepsilon_\alpha(\mathbf{k}')]. \quad \text{(D21)}$$

The last equation is the Floquet Fermi golden rule.

APPENDIX E: DEFINITION OF THE FLOQUET ZONE

In the following we would like focus on the a parabolic spectrum and describe the appropriate choice of the function λ, which defines the boundary for the quasienergy ε_α,

$$\forall_\alpha : \lambda - \frac{\hbar\Omega}{2} \leqslant \varepsilon_\alpha < \lambda + \frac{\hbar\Omega}{2}. \quad \text{(E1)}$$

Since the spectrum is not bounded, one has to choose the Floquet zone as indicated in Fig. 6.

This limits the validity of the calculation to the $\Omega\tau_0 \gg 1$ regime as will be clear in the following. However, this limitation is only a peculiarity of the unbounded spectrum: In the derivation of the main text we defined the quasienergies to fulfill

$$\forall_{\mathbf{k},\mathbf{k}'} : |\varepsilon_\alpha(\mathbf{k}) - \varepsilon_\beta(\mathbf{k}')| < \hbar\Omega. \quad \text{(E2)}$$

As a consequence, in a system with a single band the condition forces the band width to be smaller than $\hbar\Omega$. Obviously this cannot be fulfilled by the parabolic spectrum. In the latter case, the momentum range where the quasi-Fermi energy is defined has to be truncated, as depicted with a red line in Fig. 6: k_1 and k_2 are functions of the driving frequency Ω. For decreasing Ω the momenta k_1 and k_2 move closer together. If the momentum range $k \in [k_1, k_2)$ is of the order of the broadening of the Green function, the truncation leads to an incorrect result for the conductivity.

APPENDIX F: EXAMPLE: DRIVEN SQUARE LATTICE

1. Closed expression for the Floquet functions

In this section we derive a fully analytic solution of the time-dependent Schrödinger equation for the quadratic lattice with time-periodic driving and compare it with the results for the parabolic dispersion. As in the main text, we define the two lattice vectors of the square lattice with a lattice constant a as

$$\mathbf{a}_1 = a\begin{pmatrix}1\\0\end{pmatrix}, \quad \mathbf{a}_2 = a\begin{pmatrix}0\\1\end{pmatrix} \quad \text{(F1)}$$

and choose the vector potential as

$$A = \begin{pmatrix}A_x\sin(\Omega t)\\A_y\cos(\Omega t)\end{pmatrix}, \quad \text{(F2)}$$

which allows us to tune the polarization between linear, elliptic, and circular for appropriate choices of amplitudes A_x, A_y. The time-dependent tight-binding Hamiltonian with hopping parameter g and a limitation to nearest-neighbor hopping is this given by

$$H(t) = -g[e^{i\mathbf{k}\cdot\mathbf{a}_1}\cdot e^{i(e/\hbar)A\cdot\mathbf{a}_1} + e^{i\mathbf{k}\cdot\mathbf{a}_2}\cdot e^{i(e/\hbar)A\cdot\mathbf{a}_2} + \text{H.c.}]. \quad \text{(F3)}$$

In the following we will make use of the identities

$$\int dt\, e^{i\gamma\sin(\Omega t)} = \sum_{n=-\infty}^{\infty} J_n(\gamma)\int dt\, e^{in\Omega t} \quad \text{(F4)}$$

$$= \sum_{n\neq 0}\frac{J_n(\gamma)}{in\Omega}e^{in\Omega t} + J_0(\gamma)t, \quad \text{(F5)}$$

$$\int dt\, e^{i\gamma\cos(\Omega t)} = 2\sum_{n=1}^{\infty}\frac{i^n J_n(\gamma)}{n\Omega}\sin(n\Omega t) + J_0(\gamma)t, \quad \text{(F6)}$$

which are based on the Jacobi-Anger expansion [66]. To solve the time-dependent Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi_{\mathbf{k}}(t) = H(t)\psi_{\mathbf{k}}(t) \quad \text{(F7)}$$

we choose the ansatz

$$\psi_{\mathbf{k}}(t) = e^{-(i/\hbar)F(t)} \quad \text{with} \quad F(t) = \int dt\, H(t). \quad \text{(F8)}$$

Integrating the Hamiltonian (F3) yields

$$\begin{aligned} F(t) = &-2g[J_0(a\gamma_x)\cos(k_x a) + J_0(a\gamma_y)\cos(k_y a)]t \\ &- ge^{ik_x a}\sum_{n\neq 0}\frac{J_n(a\gamma_x)}{in\Omega}e^{in\Omega t} - ge^{-ik_x a}\sum_{n\neq 0}\frac{J_n(a\gamma_x)}{-in\Omega}e^{-in\Omega t} \\ &- 2ge^{ik_y a}\sum_{n=1}^{\infty}\frac{i^n J_n(a\gamma_y)}{n\Omega}\sin(n\Omega t) \\ &- 2ge^{-ik_y a}\sum_{n=1}^{\infty}\frac{(-i)^n J_n(a\gamma_y)}{n\Omega}\sin(n\Omega t). \end{aligned} \quad \text{(F9)}$$

The light parameters are defined by $\gamma_i = eA_i/\hbar$. The quasienergy is the nonoscillatory part of $F(t)$, thus

$$\epsilon = -2g[J_0(a\gamma_x)\cos(k_x a) + J_0(a\gamma_y)\cos(k_y a)]. \quad \text{(F10)}$$

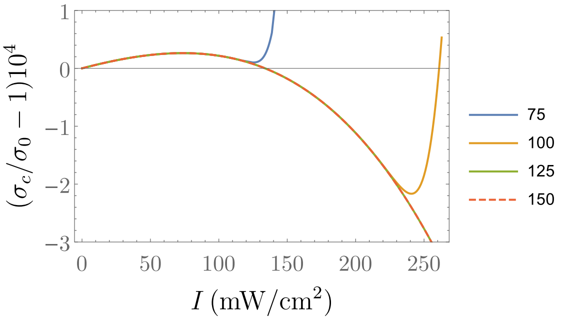

FIG. 7. The conductivity (shifted and rescaled) for a driven square lattice model is shown as a function of intensity I for different numbers of Floquet modes taken into account.

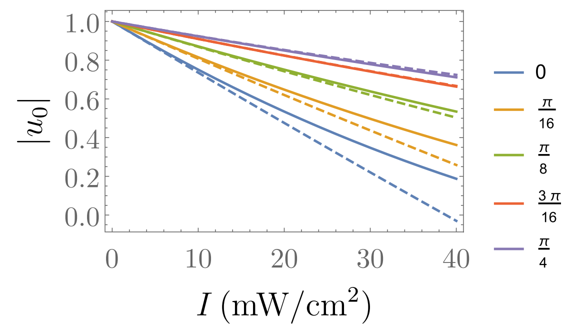

FIG. 8. The absolute value of the Floquet function u_0 plotted as a function of the intensity I for different $\mathbf{k}_F$ directions ϕ. The dashed lines correspond to the approximation Eq. (F17) [only the term proportional to $(k_F\gamma)^2$ is taken into account]. Parameters: $E_F = 10$ meV, $\Omega = 0.3$ THz.

The Floquet function can be, under the use of the Jacobi-Anger expansion [66], expressed as

$$u(t, \gamma_x, \gamma_y) = \prod_{n\neq 0} \sum_{\ell\ell'=-\infty}^{\infty} J_\ell\left(-\frac{2g}{\hbar}\frac{J_n(a\gamma_y)}{n\Omega}\right) \times J_{\ell'}\left(-\frac{2g}{\hbar}\frac{(-1)^n J_n(a\gamma_x)}{n\Omega}\right) \times e^{i\ell(k_y a + n\Omega t)} e^{i\ell'(k_x a + n\Omega t - \frac{n\pi}{4})} \tag{F11}$$

$$= \prod_{n\neq 0} \sum_{\ell\ell'=-\infty}^{\infty} c_{n,\ell,\ell'}\, e^{i(\ell+\ell')n\Omega t} \tag{F12}$$

$$= \sum_{n=-\infty}^{\infty} u_n^s(\gamma_x, \gamma_y)\, e^{-in\Omega t} \tag{F13}$$

together with

$$c_{n,\ell,\ell'} := J_\ell\left(-\frac{2g}{\hbar}\frac{J_n(a\gamma_y)}{n\Omega}\right) J_{\ell'}\left(-\frac{2g}{\hbar}\frac{(-1)^n J_n(a\gamma_x)}{n\Omega}\right) \times e^{i\ell k_y a} e^{i\ell'(k_x a - n\pi/4)}. \tag{F14}$$

The closed expression for the Floquet functions has been used to calculate the inset of Fig. 3. A substantial number of Floquet modes have to be taken into account to calculate the conductivity. Figure 7 shows the convergence of the numerical calculations for a different number of Floquet modes taken into account.

2. Comparison with the continuum model

As promised in Sec. VII, we would like to clarify the difference between the results for the conductivity in the continuum case and the tight-binding model.

Corresponding to Eq. (62), features related to the driving are visible if

$$z_{\varepsilon_F} = \frac{\hbar^2\gamma k_F}{m}\Big/\hbar\Omega, \quad \text{with} \quad \gamma = \frac{eA}{\hbar}, \tag{F15}$$

is of the order of 1, since the first zero of the Bessel function J_0 is at $\approx$2.4. Now let us consider the tight-binding model. In contrast to the parabolic spectrum the Floquet functions of the circularly driven square lattice are rather complicated as can be seen by examining Eq. (F13). However, we can approximate them for small γ_i and $\mathbf{k}$ and focus on $|u_0|$ to get a grasp on how they depend on the intensity. For simplification we set $\gamma_x = \gamma_y$ and choose spherical coordinates for the wave vectors, $k_x = k_F\cos(\phi)$, $k_y = k_F\sin(\phi)$. A lengthy calculation yields

$$u_0^s(\gamma,\gamma) = \frac{\Omega}{2\pi}\int_0^T dt\, u(t,\gamma,\gamma) \tag{F16}$$

$$= 1 - \frac{2-\sqrt{2}\sin(2\phi)}{8}\left(\frac{\alpha a^2 k_F\gamma}{\Omega}\right)^2 - \frac{3(2\sqrt{2}-3)}{256}\left(\frac{\alpha a\gamma}{\Omega}\right)^4 - \frac{\alpha^2}{256}\frac{a^4\gamma^4}{\Omega^2} + \mathcal{O}\left(k_F^{4-p}\gamma^p\right), \tag{F17}$$

with $\alpha = 2g/\hbar$ and $p \in \{0,\ldots,4\}$. The corresponding u_0^{p} for the parabolic case can be deduced from Ref. [20]:

$$u_0^{\mathrm{p}} = J_0(z_{\varepsilon_F}) \tag{F18}$$

$$= 1 - \frac{\hbar^2 k_F^2}{2m}\frac{\gamma^2}{2m\Omega^2} + \mathcal{O}\left(z_{\varepsilon_F}^3\right). \tag{F19}$$

A comparison of both cases results in

$$\frac{u_0^{\mathrm{p}}}{u_0^{\mathrm{s}}} \approx \frac{1}{1-\frac{\sin(2\phi)}{\sqrt{2}}}, \tag{F20}$$

which gives on average $\langle u_0^{\mathrm{p}}/u_0^{\mathrm{s}}\rangle_\phi = \sqrt{2}$. Although this is a rough estimate (taking into account all Floquet functions the difference in conductivity is even more significant, as shown in Fig. 3) it shows that already at small intensities $I \sim \gamma^2$ one can expect a difference between the result for the conductivity depending on whether the parabolic approximation is used or the tight-binding model. For moderate values of $E_F = 10$ meV and a low driving frequency $\Omega = 0.3$ THz we also evaluate the full expression for u_0^{s}, Eq. (F13). The result is plotted in Fig. 8 where it is also compared to the approximation Eq. (F17). A convergence test, shown in Fig. 9, shows that a considerable number of Bessel functions have to be taken into account in Eq. (F13) to reach convergence at

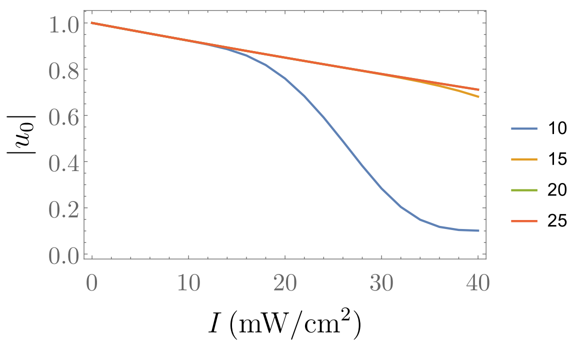

FIG. 9. Check of convergence of the Floquet function u_0 with the number of maximal n, l, and l' taken into account in Eq. (F14), here, as an example, for a wave-vector direction of $\phi = \pi/4$. Parameters: $E_F = 10$ meV, $\Omega = 0.3$ THz.

$I > 1$ mW/cm^2. In Fig. 10 the same result is compared to the continuum approximation result.

APPENDIX G: CURRENT OPERATOR FOR THE SQUARE LATTICE

Following Ref. [67], the current density $\mathbf{j}$ is given by the continuity equation

$$\frac{\partial \rho(\mathbf{r}, t)}{\partial t} = -\nabla \cdot \mathbf{j}(\mathbf{r}, t), \tag{G1}$$

where ρ describes the density of particles. Introducing the polarization operator $\mathbf{P}(t) = \int d^3r\, \mathbf{r}\rho(\mathbf{r}, t)$, it can be shown that

$$\frac{\partial \mathbf{P}(t)}{\partial t} = \int d^3r\, \mathbf{r}\frac{\partial \rho(\mathbf{r}, t)}{\partial t} = -\int d^3r\, \mathbf{r}\nabla \cdot \mathbf{j}(\mathbf{r}, t) \tag{G2}$$

$$= \int d^3r\, \mathbf{j}(\mathbf{r}, t). \tag{G3}$$

Now, assume a hopping Hamiltonian in second quantization. The polarization operator is the product of position operator $\mathbf{r}_m$ and particle number operator $n_m = a_m^\dagger a_m$,

$$\mathbf{P} = e\sum_m \mathbf{r}_m n_m. \tag{G4}$$

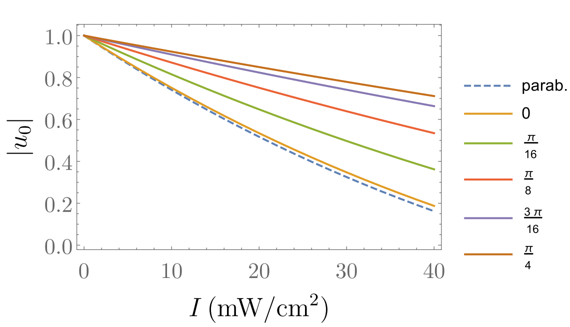

FIG. 10. The absolute value of the Floquet function u_0 plotted as a function of the intensity I. The dashed line corresponds to the Floquet function u_0^{p} of the continuum model, Eq. (F18). Parameters: $E_F = 10$ meV, $\Omega = 0.3$ THz.

Using the Heisenberg equation of motion the current operator is

$$\mathbf{J}(t) = \frac{\partial \mathbf{P}}{\partial t} = \frac{i}{\hbar}[H, \mathbf{P}] \tag{G5}$$

$$= i\frac{e}{\hbar}\sum_{n,m}(\mathbf{r}_n - \mathbf{r}_m)g_{n,m}(t)a_n^\dagger a_m. \tag{G6}$$

First, we assume that only hopping to nearest neighbors is present. The nearest-neighbor vectors are denoted as $\boldsymbol{\delta}$ such that the current operator becomes

$$\mathbf{J}(t) = i\frac{e}{\hbar}\sum_{n,j}\boldsymbol{\delta}_j g_{\boldsymbol{\delta}_j}(t)a_n^\dagger a_{n+\boldsymbol{\delta}_j}. \tag{G7}$$

In the last step we assumed that the hopping amplitude is only a function of the difference of the positions. Expanding the creation and annihilation operators in momentum space

$$a_n^{(\dagger)} = \sum_{\mathbf{k}} e^{(-)i\mathbf{k}\cdot\mathbf{r}_n}a_{\mathbf{k}}^{(\dagger)} \tag{G8}$$

and performing the sum over all positions yields for the current operator

$$\mathbf{J}(t) = i\frac{eg}{\hbar}\sum_{j,\mathbf{k}}\boldsymbol{\delta}_j e^{i(e/\hbar)\mathbf{A}(t)\cdot\boldsymbol{\delta}_j}\, e^{i\mathbf{k}\cdot\boldsymbol{\delta}_j}a_{\mathbf{k}}^\dagger a_{\mathbf{k}} \tag{G9}$$

$$\equiv \sum_{\mathbf{k}}\mathbf{J}(\mathbf{k}, t)\, a_{\mathbf{k}}^\dagger a_{\mathbf{k}}. \tag{G10}$$

Now focus on the square lattice with nearest-neighbor vectors $\boldsymbol{\delta}_1 = a(1, 0)^{\mathrm{T}}$ and $\boldsymbol{\delta}_1 = a(0, 1)^{\mathrm{T}}$ and circularly polarized driving $\mathbf{A}(t) = A(\cos(\Omega t), \sin(\Omega t))^{\mathrm{T}}$. The current coefficient becomes

$$\mathbf{J}(\mathbf{k}, t) = i\frac{ega}{\hbar}\begin{pmatrix} e^{i\mathbf{k}\cdot\boldsymbol{\delta}_1}\, e^{i(e/\hbar)\mathbf{A}(t)\cdot\boldsymbol{\delta}_1} - e^{-i\mathbf{k}\cdot\boldsymbol{\delta}_1}\, e^{-i(e/\hbar)\mathbf{A}(t)\cdot\boldsymbol{\delta}_1} \\ e^{i\mathbf{k}\cdot\boldsymbol{\delta}_2}\, e^{i(e/\hbar)\mathbf{A}(t)\cdot\boldsymbol{\delta}_2} - e^{-i\mathbf{k}\cdot\boldsymbol{\delta}_2}\, e^{-i(e/\hbar)\mathbf{A}(t)\cdot\boldsymbol{\delta}_2} \end{pmatrix} \tag{G11}$$

$$= i\frac{ega}{\hbar}\sum_{n=-\infty}^{\infty}\begin{pmatrix} e^{i\mathbf{k}\cdot\boldsymbol{\delta}_1}\, i^n J_n(\gamma)e^{in\Omega t} - e^{-i\mathbf{k}\cdot\boldsymbol{\delta}_1}\, (-i)^n J_n(\gamma)e^{-in\Omega t} \\ e^{i\mathbf{k}\cdot\boldsymbol{\delta}_2}\, J_n(\gamma)e^{in\Omega t} - e^{-i\mathbf{k}\cdot\boldsymbol{\delta}_2}\, J_n(\gamma)e^{-in\Omega t} \end{pmatrix}. \tag{G12}$$

Now focus on the $n = 0$ component, i.e., average over one driving period,

$$\bar{\mathbf{J}}(\mathbf{k}) \equiv \frac{1}{T}\int_0^{2\pi/\Omega} dt\, \mathbf{J}(\mathbf{k}, t) \tag{G13}$$

$$= i\frac{ega}{\hbar}J_0(\gamma)\begin{pmatrix} e^{i\mathbf{k}\cdot\boldsymbol{\delta}_1} - e^{-i\mathbf{k}\cdot\boldsymbol{\delta}_1} \\ e^{i\mathbf{k}\cdot\boldsymbol{\delta}_2} - e^{-i\mathbf{k}\cdot\boldsymbol{\delta}_2} \end{pmatrix} \tag{G14}$$

$$= -2\frac{ega}{\hbar}J_0(\gamma)\begin{pmatrix} \sin(k_x a) \\ \sin(k_y a) \end{pmatrix}. \tag{G15}$$

Comparison with the quasienergy of the circularly driven square lattice $\epsilon(\mathbf{k}) = -2gJ_0(\gamma)[\cos(k_x a) + \cos(k_y a)]$ yields

$$\bar{\mathbf{J}}(\mathbf{k}) = -\frac{e}{\hbar}\nabla_{\mathbf{k}}\epsilon(\mathbf{k}). \tag{G16}$$

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