Weak localization corrections to the thermal conductivity in *s*-wave superconductors

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We study the thermal conductivity in disordered *s*-wave superconductors. Expanding on previous works for normal metals, we develop a formalism that tackles particle diffusion as well as the weak localization (WL) and weak antilocalization (WAL) effects focusing on the two-dimensional case. Using a Green's functions diagrammatic technique, which takes into account the superconducting nature of the system by working in Nambu space, we identify the system's low-energy modes, the diffuson and the cooperon. The timescales that characterize the diffusive regime are energy dependent; this is in contrast to the normal state, where the relevant timescale is the impurity scattering time τ_e, independent of energy. The energy dependence introduces a novel energy scale ε_*, which for dirty superconductors ($\tau_e\Delta \ll 1$, with Δ the gap) is given by $\varepsilon_* = \sqrt{\Delta/\tau_e}$. From the diffusive behavior of the low-energy modes, we obtain the WL correction to the thermal conductivity. We give explicit expressions in two dimensions. We determine the regimes in which the correction depends manifestly on ε_* and propose an optimal regime to verify our results in an experiment. In particular, we find a parametrically large reduction of the weak localization correction in a dirty superconductor, in comparison with its value in the normal state, when the temperature is lowered by 10% below the transition temperature.

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I. INTRODUCTION

The study of quantum effects in the transport properties of disordered conductors has a long history. For thermal conductivity in normal metals, a fundamental question was whether such corrections obey the Wiedemann-Franz (WF) law relating the electrical conductivity σ to thermal conductivity K [1]. For noninteracting electrons, the WF law is expected to hold with the inclusion of quantum corrections in the weak localization (WL) regime but the numerical coefficient known as the Lorenz number $L_0 = K/\sigma T$, with T the temperature, is reduced when approaching the Anderson localization transition [2]. Away from the transition, deviations have been calculated due to electron-electron interactions [3]. Mesoscopic fluctuations can also lead to violations of the WF law [4]. In the superconducting state the dc electrical resistance vanishes and hence there is no WF law; in fact, approaching the critical temperature from the normal state, superconducting fluctuations lead to a divergent electrical conductivity, whereas they only constitute a finite correction to K [5,6]. Sufficiently far below the critical temperature, fluctuations are negligible and the leading-order expression for the thermal conductivity of a BCS superconductor has been obtained in the early work of Ref. [7]. Further extensions to this result include the effects of electron-phonon scattering [8,9], strong coupling [10], and paramagnetic impurities [11]. However, to the best of our knowledge, the question of the fate of the weak localization correction to the thermal conductivity in the superconducting state has so far only been addressed for normal/superconductor/normal (SNS) junctions [12] and not for the bulk.

In this paper, we analytically calculate the weak localization correction to the thermal conductivity in *s*-wave superconductors, including weak antilocalization (WAL) in a system with spin-orbit scattering, focusing on the two-dimensional case. To that end, we extend the formalism used to study diffusion in normal metals (see, e.g., Ref. [13]), so that it can be used for superconductors as well. Technically, we work with matrix Green's functions in Nambu space. In the next section, we introduce the model for disordered superconductors to establish our notation. In Sec. III, we study diffusion in disordered superconductors in depth by generalizing the ladder approximation. Using the classification of Ref. [14], we focus on the *A*-type diffusons and cooperons since in a time-reversal-invariant system, the *D*-type ones do not contribute to thermal transport [10]. In contrast to the normal state, the diffusion constant in the superconducting state depends on energy (measured from the Fermi energy). This energy dependence manifests itself in the condition defining the diffusive regime in the time domain, which is now not simply given by the requirement of time being long compared to the impurity scattering time τ_e. We find that the corresponding timescale in the superconducting state is different for energies below or above an energy scale ε_* which is a function of the superconducting gap Δ and the scattering time; for disordered superconductors with $\tau_e\Delta \ll 1$, we find $\varepsilon_* = \sqrt{\Delta/\tau_e}$.

In Sec. IV, we make use of the results of the preceding section to calculate the thermal conductivity from the Kubo formula. We recover previous results [7,10] for the Drude-Boltzmann contribution to the thermal conductivity, which, because of the opening of the superconducting gap, is suppressed as temperature is reduced. As the diffusion constant is energy dependent, we have to specify whether the phase-coherence length or the phase-coherence time is constant in a material in order to evaluate the weak localization

correction. We obtain results for both scenarios; in general, the WL correction is temperature dependent. Interestingly, the suppression of the WL correction with decreasing temperature is generally stronger than that of the Drude-Boltzmann term. Of possible experimental interest is the temperature region of order $T_\Delta \approx 0.9T_c$, where T_c is the critical temperature of the superconductor and T_Δ is defined by $k_B T_\Delta = \Delta(T)$. On one hand, this temperature is sufficiently high that the strong (exponential) suppression of the (Drude-Boltzmann) thermal conductivity has not yet taken place. On the other hand, for disordered superconductors, this temperature is low enough that most of the weak localization correction is already suppressed. This temperature is therefore optimal in order to observe the deviation of the WL correction in the superconducting state from its normal-state value, as we predict the thermal conductivity to be larger than expected from its value just above T_c. We summarize our findings in Sec. V.

A number of details can be found in Appendices A–E. In Appendix A, we discuss the normalization of the diffusion probability and relate it to particle conservation. Appendix B presents the calculations of Sec. III B in momentum space instead of in real space. The explicit expression for the cooperon and its derivation are given in Appendix C. In Appendix D we show the explicit calculation of the WL correction to the thermal conductivity. Finally, in Appendix E, we treat a superconducting system with spin-orbit scattering and derive the superconducting WAL correction.

II. MODEL

The (mean field) Hamiltonian for a superconductor with s-wave pairing can be expressed in the Bogoliubov–de Gennes (BdG) form as [15]

$$H = \sum_{\boldsymbol{k}} \boldsymbol{\Psi}_{\boldsymbol{k}}^\dagger \hat{H}_{\mathrm{BdG}}(k) \boldsymbol{\Psi}_{\boldsymbol{k}} \tag{1}$$

with the Nambu vector

$$\boldsymbol{\Psi}_{\boldsymbol{k}} = \begin{pmatrix} c_{\boldsymbol{k}\uparrow} \\ c_{-\boldsymbol{k}\downarrow}^\dagger \end{pmatrix}, \tag{2}$$

where $c_{\boldsymbol{k}\sigma}^\dagger$ and $c_{\boldsymbol{k}\sigma}$ are creation and annihilation operators for electrons with momentum $\boldsymbol{k}$ and spin σ, respectively. The BdG Hamiltonian is given by

$$\hat{H}_{\mathrm{BdG}}(k) = \epsilon_k \tau_3 - \Delta \tau_1, \tag{3}$$

where the hat denotes matrices in the Nambu space. Here, $\epsilon_k = k^2/2m - \mu$, m is the electron mass, $\mu = k_F^2/2m$ the Fermi energy with k_F the Fermi momentum, Δ the superconducting order parameter, and τ_i the Pauli matrices in Nambu space (we omit hats on these matrices for notational simplicity). For later use, we introduce the basis $\{|\mathrm{e}\rangle, |\mathrm{h}\rangle\}$ in Nambu space, where the states $|\mathrm{e}\rangle$ and $|\mathrm{h}\rangle$ stand for electron and hole, respectively. The Bogoliubov–de Gennes Hamiltonian (3) includes the noninteracting electron and hole Hamiltonians in its diagonal terms as well as the pairing term, given by Δ, in its off-diagonal terms. The retarded and advanced Green's functions are then solutions of

$$(E - \hat{H}_{\mathrm{BdG}} \pm i0^+)\hat{G}_E^{R,A} = 1. \tag{4}$$

We distinguish the four different elements of the matrix Green's function as follows:

$$\hat{G}_E^{R,A} = \begin{pmatrix} G_E^{R,A} & F_E^{R,A} \\ \bar{F}_E^{R,A} & \bar{G}_E^{R,A} \end{pmatrix}. \tag{5}$$

The diagonal terms, that is, the electron and hole Green's functions, describe electron and hole propagation, respectively. The off-diagonal terms, known as anomalous Green's functions, account for particle-hole conversion, i.e., Andreev reflection.

So far, we have considered a clean superconductor. To treat the elastic scattering of electrons off impurities we introduce a random disorder potential. The disorder potential $\hat{V}(\boldsymbol{r}) = V(r)\tau_3$ is taken to be Gaussian distributed with $\overline{V(r)} = 0$, where the overline $(\overline{\cdots})$ denotes the disorder average. We work in the weak disorder limit $k_F l_e \gg 1$, where l_e is the mean-free path, which allows for the perturbative treatment of impurity scattering. We define the disorder parameter γ_e by relating it to the disorder average of the variance of the potential such that

$$\overline{\hat{V}(\boldsymbol{r}) \otimes \hat{V}(\boldsymbol{r}')} = \gamma_e \delta^{(d)}(\boldsymbol{r} - \boldsymbol{r}')\hat{U}_v, \tag{6}$$

where $\hat{U}_v = \tau_3 \otimes \tau_3$. The disorder parameter is related to the scattering time $\tau_e = l_e/v_F$ in the normal state and to the normal-state density of states per spin ρ_0 as $\gamma_e = 1/2\pi\rho_0\tau_e$ with $v_F = k_F/m$ the Fermi velocity.

In a normal metal, both electrical and thermal conductivity are attributed to free conduction electrons, and both phenomena can be understood by studying electron diffusion. In superconductors, the (super)current is carried by Cooper pairs; the thermal conductivity, however, is still related to particle diffusion. In order to study diffusion in the superconducting state, in the next section we develop a matrix formalism in Nambu space that enables us to generalize the diagrammatic approach well established in the study of diffusion in the normal state.

III. PARTICLE DIFFUSION AND WEAK LOCALIZATION

In this section, we study the propagation of particles in disordered conventional superconductors in the weak disorder limit $k_F l_e \gg 1$. In this limit, localization affects the transport coefficients, but Anderson localization [16] does not yet take place. Throughout this section, we expand to the superconducting state the diagrammatic treatment of particle propagation in a normal metal presented in Ref. [13]. The main technical change involves modifying the Feynman diagrams to include all the four components of the superconducting Green's function defined in Eq. (5) [17] (see Fig. 1). We define the quantum diffusion probability matrix $\hat{P}_\omega(\boldsymbol{r}, \boldsymbol{r}')$ as

$$\hat{P}_\omega(\boldsymbol{r}, \boldsymbol{r}') = \overline{\hat{G}_{E+\omega}^R(\boldsymbol{r}, \boldsymbol{r}') \otimes \hat{G}_E^A(\boldsymbol{r}', \boldsymbol{r})^T}, \tag{7}$$

where the retarded Green's functions in real space are given by $\hat{G}_E^R(\boldsymbol{r}, \boldsymbol{r}') = \langle \boldsymbol{r}'|\hat{G}_E^R|\boldsymbol{r}\rangle$ and

$$\hat{G}_E^A(\boldsymbol{r}', \boldsymbol{r})^T = \hat{G}_E^R(\boldsymbol{r}, \boldsymbol{r}')^*. \tag{8}$$

The matrix $\hat{P}_\omega(\boldsymbol{r}, \boldsymbol{r}')$ acts on the space spanned by $|i, j\rangle = |i\rangle \otimes |j\rangle$ with $i, j \in \{\mathrm{e}, \mathrm{h}\}$; that is, $|i\rangle$ and $|j\rangle$ are basis states in the Nambu spaces pertaining to the retarded and advanced

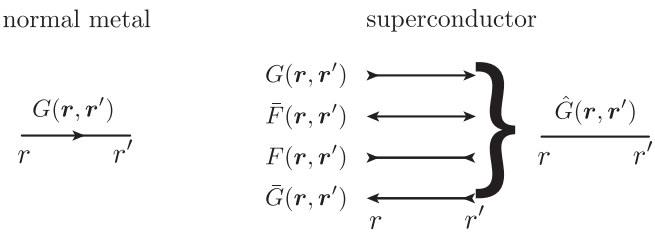

FIG. 1. Feynman diagrams for the time-ordered Green's functions in a normal metal and a superconductor. Time evolution occurs from left to right. The four components of the Green's function for a superconductor are distinguished by the arrows at the ends. The matrix formulation in the Nambu formalism is represented by an arrowless line.

Green's functions, respectively. We discuss the proper normalization of this probability in Appendix A. We stress that $\hat{P}_\omega(\boldsymbol{r},\boldsymbol{r}')$ depends on the energy argument E appearing in the Green's functions, although we do not highlight this in the notation for simplicity: scattering off impurities being elastic, the energy argument can be treated as a parameter that is constant during diffusion. The diagrammatic expression for $\hat{P}_\omega(\boldsymbol{r},\boldsymbol{r}')$ in the ladder approximation is shown in Fig. 2. In each diagram shown in the figure, the upper line represents the retarded Green's function in Nambu space from point $\boldsymbol{r}$ to point $\boldsymbol{r}'$, and the lower one represents its complex conjugate, given by Eq. (8). We calculate three main contributions to particle propagation, starting with the Drude-Boltzmann contribution $\hat{P}_{0,\omega}(\boldsymbol{r},\boldsymbol{r}')$. This contribution accounts for the probability of propagation in a disordered medium without colliding with any impurities. Subsequently, we include classical scattering events and calculate the diffuson $\hat{P}_{d,\omega}(\boldsymbol{r},\boldsymbol{r}')$. We show that in the superconducting state the so-called diffusive or hydrodynamic approximation is applicable beyond a timescale that differs from that of the normal state and depends on energy E. We define the total probability of diffusion as the sum of

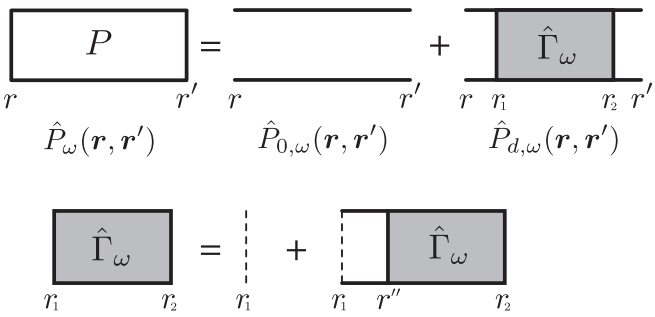

FIG. 2. Representation of the ladder approximation for the diffusion probability matrix $\hat{P}_\omega(\boldsymbol{r},\boldsymbol{r}')$ in a superconductor as defined in Eq. (7). The upper arrowless line represents the disorder-averaged retarded superconducting Green's function expressed in matrix form in the Nambu formalism. The lower line represents its complex conjugate. The dashed lines represent impurity scattering. The total probability of diffusion $\hat{P}_\omega(\boldsymbol{r},\boldsymbol{r}')$ is composed by the Drude-Boltzmann contribution $\hat{P}_{0,\omega}(\boldsymbol{r},\boldsymbol{r}')$ and the diffuson $\hat{P}_{d,\omega}(\boldsymbol{r},\boldsymbol{r}')$. The latter includes the structure factor $\hat{\Gamma}_\omega(\boldsymbol{r}_1,\boldsymbol{r}_2)$ which accounts for elastic scattering with static impurities.

these two contributions

$$\hat{P}_\omega(\boldsymbol{r},\boldsymbol{r}') = \hat{P}_{0,\omega}(\boldsymbol{r},\boldsymbol{r}') + \hat{P}_{d,\omega}(\boldsymbol{r},\boldsymbol{r}'). \tag{9}$$

In the last part of the section, we consider the effect of coherent backscattering and derive the weak localization correction to particle diffusion $\hat{P}_{c,\omega}(\boldsymbol{r},\boldsymbol{r}')$, that is the cooperon contribution. In this way, we generalize previous studies of weak localization in superconductors, which considered the effect on the density of superconducting electrons [18] and on nonlocal transport in normal/superconductor/normal structures [19,20].

A. Drude-Boltzmann contribution

The Drude-Boltzmann contribution $\hat{P}_{0,\omega}(\boldsymbol{r},\boldsymbol{r}')$ is given by

$$\hat{P}_{0,\omega}(\boldsymbol{r},\boldsymbol{r}') = \overline{\hat{G}^R_{E+\omega}(\boldsymbol{r},\boldsymbol{r}')} \otimes \overline{\hat{G}^A_E(\boldsymbol{r}',\boldsymbol{r})^T}. \tag{10}$$

The disorder-averaged superconducting retarded Green's function can be explicitly calculated in momentum space, where it is given by [17]

$$\overline{\hat{G}^R_E(k)} = \frac{\bar{E}\tau_0 + \epsilon_k\tau_3 - \bar{\Delta}\tau_1}{\bar{E}^2 - \epsilon_k^2 - \bar{\Delta}^2} \tag{11}$$

with

$$\bar{E} = E\left[1 + i\frac{1}{2\tau_e}\frac{\mathrm{sgn}(E)}{\sqrt{E^2-\Delta^2}}\right] \tag{12}$$

and

$$\bar{\Delta} = \Delta\left[1 + i\frac{1}{2\tau_e}\frac{\mathrm{sgn}(E)}{\sqrt{E^2-\Delta^2}}\right]. \tag{13}$$

The Fourier transform of Eq. (11) into real space can then be calculated in the limit $\mu \gg \epsilon, \Delta$, with $\epsilon = \sqrt{E^2-\Delta^2}$, by linearizing the spectrum around k_F. We provide the explicit result in the two-dimensional case since it will be of particular interest for the weak localization correction. In the limit $k_F R \gg 1$ with $\boldsymbol{R} = \boldsymbol{r}' - \boldsymbol{r}$, we have ($E > 0$)

$$\overline{\hat{G}^R_E(\boldsymbol{r},\boldsymbol{r}')} = \frac{m}{\sqrt{2\pi k_F R}} e^{i\frac{\epsilon R}{v_F} - \frac{R}{2l_e}}\left[i\frac{E}{\epsilon}\cos\left(k_F R + \frac{3\pi}{4}\right)\tau_0 - \sin\left(k_F R + \frac{3\pi}{4}\right)\tau_3 - i\frac{\Delta}{\epsilon}\cos\left(k_F R + \frac{3\pi}{4}\right)\tau_1\right]. \tag{14}$$

The advanced Green's function in real space can be arrived at using Eq. (8). Having obtained the disorder-averaged superconducting Green's functions in real space, $\hat{P}_{0,\omega}(\boldsymbol{r},\boldsymbol{r}')$ can be found from Eq. (10). In the next section we use $\hat{P}_{0,\omega}(\boldsymbol{r},\boldsymbol{r}')$ to calculate the diffuson.

B. Diffusion in disordered superconductors: The diffuson

The diffuson $\hat{P}_{d,\omega}(\boldsymbol{r},\boldsymbol{r}')$ is the classical probability of propagation from $\boldsymbol{r}$ to $\boldsymbol{r}'$ accounting for all paths including at least one scattering event. Summation over these paths is performed in the ladder approximation, as sketched in Fig. 2, giving the equation

$$\hat{P}_{d,\omega}(\boldsymbol{r},\boldsymbol{r}') = \int d^d r_1 \int d^d r_2 \hat{P}_{0,\omega}(\boldsymbol{r},\boldsymbol{r}_1)\hat{\Gamma}_\omega(\boldsymbol{r}_1,\boldsymbol{r}_2)\hat{P}_{0,\omega}(\boldsymbol{r}_2,\boldsymbol{r}'). \tag{15}$$

The Drude-Boltzmann factors account for the trajectory before the first scattering event and after the last one, at $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, respectively. The structure factor $\hat{\Gamma}_\omega(\boldsymbol{r}_1, \boldsymbol{r}_2)$ includes all scattering events. In our formalism, it is a 4×4 matrix defined self-consistently by

$$\hat{\Gamma}_\omega(\boldsymbol{r}_1, \boldsymbol{r}_2) = \gamma_e \hat{U}_v \left[\delta^{(d)}(\boldsymbol{r}_1 - \boldsymbol{r}_2) + \int d^d r'' \hat{P}_{0,\omega}(\boldsymbol{r}_1, \boldsymbol{r}'') \hat{\Gamma}_\omega(\boldsymbol{r}'', \boldsymbol{r}_2) \right] \tag{16}$$

(see the bottom half of Fig. 2). The Drude-Boltzmann contribution decays on a length scale of the order of the mean-free path l_e [cf. Eq. (14)]. Here we are interested in the diffusive regime, where the length scale λ over which the structure factor varies is much longer than the mean-free path $\lambda \gg l_e$. We can then approximate $\hat{\Gamma}_\omega(\boldsymbol{r}_1, \boldsymbol{r}_2) \approx \hat{\Gamma}_\omega(\boldsymbol{r}, \boldsymbol{r}')$. In this limit, Eq. (15) can be approximately rewritten as

$$\hat{P}_{d,\omega}(\boldsymbol{r}, \boldsymbol{r}') = \langle \hat{P}_0 \rangle_{\boldsymbol{r}} \hat{\Gamma}_\omega(\boldsymbol{r}, \boldsymbol{r}') \langle \hat{P}_0 \rangle_{\boldsymbol{r}}, \tag{17}$$

with $\langle \hat{P}_0 \rangle_{\boldsymbol{r}} \equiv \langle \hat{P}_{0,\omega=0} \rangle_{\boldsymbol{r}}$ and

$$\langle \hat{P}_{0,\omega} \rangle_{\boldsymbol{r}} \equiv \int d^d r' \hat{P}_{0,\omega}(\boldsymbol{r}, \boldsymbol{r}'). \tag{18}$$

Diffusion takes place at sufficiently long times beyond the scale $\tau_{\min}$ so that terms of the order $(\omega\tau_{\min})^2$ and $(\omega\tau_{\min})(l_e/\lambda)^2$ can be neglected in comparison to those of order $(\omega\tau_{\min})$ and $(l_e/\lambda)^2$, respectively. For the diffusion, in a normal metal the scale $\tau_{\min}$ is simply given by $\tau_e = l_e/v_F$. Analogously, for a superconductor, we obtain the scattering time

$$\tau_s = \frac{l_e}{v_g}, \quad v_g = v_F \frac{\epsilon}{|E|} \tag{19}$$

with v_g the group velocity of the quasiparticles. However, we find that diffusion only sets in after the longer time

$$\tau_{\min} = \max\left\{ \tau_s, \frac{\Delta}{\epsilon^2} \right\}. \tag{20}$$

The second scale Δ/ϵ^2 appears in order that the diffusive modes [first two entries of Eq. (25) below] are decoupled from the massive modes (last two entries). The definition in Eq. (20) reduces to $\tau_{\min} = \tau_e$ in the normal state, while in the superconducting one we find

$$\tau_{\min} = \begin{cases} \frac{\Delta}{\epsilon^2}, & E < E_*, \\ \tau_s, & E > E_*, \end{cases} \tag{21}$$

where E_* is defined as the energy at which $\tau_s = \Delta/\epsilon^2$. The magnitude of E_* is sensitive to the disorder strength in the superconductor. Writing $E_* = \Delta + \varepsilon_*$, we obtain

$$\varepsilon_* \simeq \begin{cases} \sqrt{\frac{\Delta}{\tau_e}}, & \tau_e\Delta \ll 1 \\ \frac{1}{2\Delta\tau_e^2}, & \tau_e\Delta \gg 1 \end{cases}, \tag{22}$$

where the condition $\tau_e\Delta \ll 1$ identifies the dirty regime, in which $\varepsilon_* \gg \Delta$, and $\tau_e\Delta \gg 1$ the clean case, where $\varepsilon_* \ll \Delta$. We will discuss in Sec. IV the relevance of this and other energy scales to the thermal conductivity.

To obtain the diffusion equation for the structure factor $\hat{\Gamma}_\omega$, we expand the latter up to second order in $\boldsymbol{r}'' - \boldsymbol{r}_1$ around $\boldsymbol{r}'' = \boldsymbol{r}_1$ in the right-hand side of Eq. (16). That equation can then be cast in the form

$$\hat{M}_\omega(\boldsymbol{r}) \hat{\Gamma}_\omega(\boldsymbol{r}, \boldsymbol{r}') = \gamma_e \delta^{(d)}(\boldsymbol{r}' - \boldsymbol{r}), \tag{23}$$

with the matrix operator

$$\hat{M}_\omega(\boldsymbol{r}) = \hat{U}_v^{-1} - \gamma_e \langle \hat{P}_{0,\omega} \rangle_{\boldsymbol{r}} - \frac{\gamma_e}{2d} \langle r^2 \hat{P}_0 \rangle_{\boldsymbol{r}} \nabla_{\boldsymbol{r}}^2. \tag{24}$$

We have again neglected terms of order $(\omega\tau_{\min})(l_e/\lambda)^2$ by evaluating $\langle r^2 \hat{P}_{0,\omega} \rangle_{\boldsymbol{r}}$ at $\omega = 0$, which we denote by removing the ω subscript. The integration over space of the Drude-Boltzmann contribution $\langle \hat{P}_{0,\omega} \rangle_{\boldsymbol{r}}$ can be performed directly using Eq. (10) and the Green's function in real space [we remind that in the diffusive regime we only need to keep terms of order $(\omega\tau_{\min})^0$ and $(\omega\tau_{\min})^1$]. Using the relation $\langle r^2 \hat{P}_0 \rangle_{\boldsymbol{r}} = 2l_e^2 \langle \hat{P}_0 \rangle_{\boldsymbol{r}}$ and the definition of the potential matrix $\hat{U}_v$ [see the text after Eq. (6)], the matrix operator $\hat{M}_\omega(\boldsymbol{r})$ is obtained straightforwardly.

We wish to study the structure of $\hat{M}_\omega(\boldsymbol{r})$ to understand the diffusive modes of $\hat{\Gamma}_\omega(\boldsymbol{r}, \boldsymbol{r}')$. It is convenient to introduce the states $|a_\pm\rangle = \frac{1}{\sqrt{2}}(|\mathrm{e}, \mathrm{e}\rangle \pm |\mathrm{h}, \mathrm{h}\rangle)$ and $|b_\pm\rangle = \frac{1}{\sqrt{2}}(|\mathrm{e}, \mathrm{h}\rangle \pm |\mathrm{h}, \mathrm{e}\rangle)$. We then work in the basis $B = \{|a_-\rangle, \cos(\theta)|a_+\rangle + \sin(\theta)|b_+\rangle, \cos(\theta)|b_+\rangle - \sin(\theta)|a_+\rangle, |b_-\rangle\}$, where $\Delta/E = \tan(\theta)$. In this basis, the structure of $\hat{M}_\omega(\boldsymbol{r})$ simplifies and the behavior of the diffusive modes can be singled out. Indeed, we find in the diffusive regime the result

$$\hat{M}_\omega(\boldsymbol{r}) = \mathrm{diag}\left(\tau_s \mathcal{D}, \tau_s \frac{\epsilon^2}{E^2 + \Delta^2} \mathcal{D}, -\frac{E^2 + \Delta^2}{\epsilon^2}, -1 \right), \tag{25}$$

with $\mathcal{D} = -i\omega - D_s \nabla_{\boldsymbol{r}}^2$. Here D_s is the superconducting diffusion constant

$$D_s = \frac{v_g l_e}{d} \tag{26}$$

which, similarly to the scattering rate above, is energy dependent. On the other hand, the mean-free path, proportional to $\sqrt{D_s \tau_s}$, remains independent of energy and equal to that in the normal state. These findings are in agreement with those in Ref. [7].

Equation (25) shows that in the diffusive limit $\hat{M}_\omega(\boldsymbol{r})$ is a diagonal matrix with two diffusive and two fast modes. We will neglect the fast modes and focus on the diffusive ones. To this end, we define $\hat{\mathbb{M}}_\omega(\boldsymbol{r})$ as the 2×2 matrix obtained by projecting $\hat{M}_\omega(\boldsymbol{r})$ into the subspace spanned by $\{|a_-\rangle, \cos(\theta)|a_+\rangle + \sin(\theta)|b_+\rangle\}$. According to Eq. (23), the structure factor $\hat{\Gamma}_\omega(\boldsymbol{r}, \boldsymbol{r}')$ in this subspace satisfies the equation

$$\left[\tau_s \begin{pmatrix} 1 & 0 \\ 0 & \frac{\epsilon^2}{E^2 + \Delta^2} \end{pmatrix} \mathcal{D} \right] \hat{\Gamma}_\omega(\boldsymbol{r}, \boldsymbol{r}') = \gamma_e \delta^{(d)}(\boldsymbol{r}' - \boldsymbol{r}), \tag{27}$$

where the terms in square brackets are the matrix $\hat{\mathbb{M}}_\omega(\boldsymbol{r})$. We can rewrite Eq. (17) using $\hat{\Gamma}_\omega(\boldsymbol{r}, \boldsymbol{r}')$ as

$$\hat{P}_{d,\omega}(\boldsymbol{r}, \boldsymbol{r}') = \hat{P}_v \hat{\Gamma}_\omega(\boldsymbol{r}, \boldsymbol{r}') \hat{P}_v^T, \tag{28}$$

where $\hat{P}_v$ is the matrix with dimension 4×2 that encompasses the first two columns of $\langle \hat{P}_0 \rangle_{\boldsymbol{r}}$ in the previously introduced basis B; it has the useful property $\gamma_e^2 \hat{P}_v^T \hat{P}_v = 1$. The diffuson thus found is a rank-two matrix that takes a diagonal form in

the basis $\tilde{B} = \{|a_-\rangle, \cos(\theta)|a_+\rangle - \sin(\theta)|b_+\rangle, \cos(\theta)|b_+\rangle + \sin(\theta)|a_+\rangle, |b_-\rangle\}$, which is also the eigenbasis of $\langle \hat{P}_{0,\omega}\rangle_{\boldsymbol{r}}$. Its nonzero terms follow a diffusion equation in the subspace $\tilde{B}_2 = \{|a_-\rangle, \cos(\theta)|a_+\rangle - \sin(\theta)|b_+\rangle\}$ given by

$$\frac{v_F}{2\pi\rho_0 v_g}\begin{pmatrix}1 & 0\\ 0 & \frac{\epsilon^2}{E^2+\Delta^2}\end{pmatrix}\mathcal{D}\,\hat{\mathsf{P}}_{d,\omega}(\boldsymbol{r},\boldsymbol{r}') = \delta^{(d)}(\boldsymbol{r}'-\boldsymbol{r}). \tag{29}$$

We emphasize here the first diffusive mode shown in Eq. (29): it relates to quasiparticle conservation [21] and, as we will see later on, to the thermal conductivity. We thus define for later use the quasiparticle diffusion probability

$$\begin{aligned} P^{\mathrm{qp}}_{\omega}(\boldsymbol{r},\boldsymbol{r}') &\equiv \langle a_-|\hat{P}_{\omega}(\boldsymbol{r},\boldsymbol{r}')|a_-\rangle \\ &= \tfrac{1}{2}\mathrm{Tr}\big[\overline{\tau_3\hat{G}^R_E(\boldsymbol{r},\boldsymbol{r}')\tau_3\hat{G}^A_E(\boldsymbol{r}',\boldsymbol{r})}\big] \end{aligned} \tag{30}$$

and, correspondingly, the Drude-Boltzmann, quasiparticle diffuson, and cooperon contributions $P^{\mathrm{qp}}_{i,\omega} = \langle a_-|\hat{P}_{i,\omega}(\boldsymbol{r},\boldsymbol{r}')|a_-\rangle$, with $i \in \{0, d, c\}$.

The two diffusons shown in Eq. (29) resemble the diffuson in the normal metal, but in the superconductive state the diffusion constant and the scattering time depend on the group velocity v_g which is no longer equal to the Fermi velocity [Eq. (29) can also be reformulated to include the energy scaling v_g/v_F in the frequency component rather than in the diffusion constant]. After applying the temporal Fourier transform, we obtain a direct relation between the probabilities of diffusion in the superconducting and normal states

$$\begin{aligned} &\langle i', j'|\hat{P}_d(\boldsymbol{r},\boldsymbol{r}';t)|i,j\rangle \\ &= P_{\mathrm{n}}(\boldsymbol{r},\boldsymbol{r}';tv_g/v_F)\frac{\pi\rho_0 v_g}{(E^2-\Delta^2)v_F}((2E^2-\Delta^2)\delta_{i,j}\delta_{i,i'}\delta_{j,j'} \\ &\quad + \Delta^2(1-\delta_{i,i'})(1-\delta_{j,j'}) + \Delta^2\delta_{i,i'}\delta_{j,j'}(1-\delta_{i,j}) \\ &\quad - \Delta E[\delta_{i,i'}(1-\delta_{j,j'}) + \delta_{j,j'}(1-\delta_{i,i'})]), \end{aligned} \tag{31}$$

where $i, j, i', j' \in \{\mathrm{e}, \mathrm{h}\}$ and the normal-state diffusion probability satisfies the equation

$$\left(-D\nabla^2_{\boldsymbol{r}} + \frac{\partial}{\partial t}\right)P_{\mathrm{n}}(\boldsymbol{r},\boldsymbol{r}';t) = \delta^{(d)}(\boldsymbol{r}'-\boldsymbol{r})\delta(t), \tag{32}$$

where the diffusion constant D coincides with the limit of zero order parameter for D_s of Eq. (26) (in which case $v_g \to v_F$). We note that the factor $v_F/2\pi\rho_0 v_g$ appearing in Eqs. (29) and (31) is due to the unconventional normalization used for the probability. Since our main interest is the calculation of the thermal conductivity, it is more convenient to directly calculate the disorder-average product of Green's function which does not correspond to the normalized probability of diffusion. More details on the normalization are given in Appendix A. Equations (27) and (29) can also be obtained in momentum space: by inverting matrix $\hat{M}_{\omega}(\boldsymbol{q})$, the calculation of both $\hat{\Gamma}_{\omega}(\boldsymbol{q})$ and $\hat{P}_{d,\omega}(\boldsymbol{q})$ is straightforward (see Appendix B). Relations between diffusion in the superconducting and normal states similar to Eq. (31) have been recently obtained for energies below the gap and at $\omega = 0$ [22]. We note that such subgap (virtual) diffusion can mediate the exchange interaction between two spin qubits tunnel coupled to a superconductor [23].

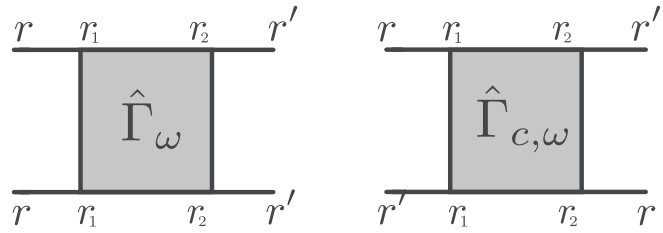

FIG. 3. Diagrammatic representation of the diffuson (left) and the cooperon (right). The cooperon corresponds to reversing one of the trajectories that take part in the probability so that the impurities are encountered in inverse order. The upper Green's functions are retarded while the lower ones are advanced.

C. Weak localization: The cooperon

After studying classical diffusion within the ladder approximation, we now focus on the first quantum correction to the probability of diffusion arising from localization effects (the cooperon contribution). The cooperon matrix $\hat{P}_{c,\omega}(\boldsymbol{r},\boldsymbol{r}')$, shown schematically next to the diffuson in Fig. 3, corresponds to the quantum interference between two trajectories covering the exact same path but in opposite directions. This interference effect is reflected in the structure of the expression

$$\begin{aligned} \hat{P}_{c,\omega}(\boldsymbol{r},\boldsymbol{r}') = \int d^d r_1 \int d^d r_2 \big(\overline{\hat{G}^R_{E+\omega}(\boldsymbol{r},\boldsymbol{r}_1)} \otimes \overline{\hat{G}^A_E(\boldsymbol{r}',\boldsymbol{r}_1)^T}\big) \\ \times \hat{\Gamma}_{c,\omega}(\boldsymbol{r}_1,\boldsymbol{r}_2)\big(\overline{\hat{G}^R_{E+\omega}(\boldsymbol{r}_2,\boldsymbol{r}')} \otimes \overline{\hat{G}^A_E(\boldsymbol{r}_2,\boldsymbol{r})^T}\big), \end{aligned} \tag{33}$$

$$\begin{aligned} \hat{\Gamma}_{c,\omega}(\boldsymbol{r}_1,\boldsymbol{r}_2) = \gamma_e\hat{U}_v\Big[\delta^{(d)}(\boldsymbol{r}_1-\boldsymbol{r}_2) + \int d^d r''\big(\overline{\hat{G}^R_{E+\omega}(\boldsymbol{r}_1,\boldsymbol{r}'')} \\ \otimes \overline{\hat{G}^A_E(\boldsymbol{r}_1,\boldsymbol{r}'')^T}\big)\hat{\Gamma}_{c,\omega}(\boldsymbol{r}'',\boldsymbol{r}_2)\Big]. \end{aligned} \tag{34}$$

Since the disorder-averaged Green's functions decay exponentially in real space [cf. Eq. (14)], it can already be seen above that the cooperon is exponentially suppressed in $|\boldsymbol{r}'-\boldsymbol{r}|/l_e$. We can simplify Eq. (34) by noting that for a time-reversal-invariant system, i.e., for $\hat{G}^R_E(\boldsymbol{r},\boldsymbol{r}') = \hat{G}^R_E(\boldsymbol{r}',\boldsymbol{r})$, it is identical to Eq. (16) and thus $\hat{\Gamma}_{c,\omega}(\boldsymbol{r}_1,\boldsymbol{r}_2) = \hat{\Gamma}_{\omega}(\boldsymbol{r}_1,\boldsymbol{r}_2)$. We now again assume the latter to vary slowly on the scale of the mean-free path and thus make the approximation $\hat{\Gamma}_{c,\omega}(\boldsymbol{r}_1,\boldsymbol{r}_2) \approx \hat{\Gamma}_{\omega}(\boldsymbol{r},\boldsymbol{r})$ in Eq. (33) which, neglecting terms of order $(\omega\tau_{\min})^2$, $(\omega\tau_{\min})(l_e/\lambda)^2$, and higher, becomes approximately

$$\hat{P}_{c,\omega}(\boldsymbol{r},\boldsymbol{r}') = \hat{F}(\boldsymbol{R})\hat{\Gamma}_{\omega}(\boldsymbol{r},\boldsymbol{r})\hat{F}(\boldsymbol{R}). \tag{35}$$

Here, we define $\boldsymbol{R} = \boldsymbol{r}' - \boldsymbol{r}$ and

$$\begin{aligned} \hat{F}(\boldsymbol{R}) &= \int d^d r_1\, \overline{\hat{G}^R_E(\boldsymbol{r},\boldsymbol{r}_1)} \otimes \overline{\hat{G}^A_E(\boldsymbol{r}',\boldsymbol{r}_1)^T} \\ &= \int \frac{d^d k}{(2\pi)^d}\, e^{i\boldsymbol{k}\cdot\boldsymbol{R}}\overline{\hat{G}^R_E(k)} \otimes \overline{\hat{G}^A_E(-k)^T}, \end{aligned} \tag{36}$$

which can be calculated by direct integration using Eq. (11). Note the similarity between Eqs. (17) and (35), which become equivalent for $\boldsymbol{r} = \boldsymbol{r}'$, since $\langle \hat{P}_0\rangle_{\boldsymbol{r}} = \hat{F}(0)$. In fact, there exists a general relation between the cooperon and the diffuson of

the form

$$\hat{P}_{c,\omega}(\boldsymbol{r},\boldsymbol{r}') = \hat{P}_{d,\omega}(\boldsymbol{r},\boldsymbol{r})\hat{f}(\boldsymbol{R}), \tag{37}$$

with $\hat{f}(\boldsymbol{R})$ given explicitly in Appendix C.

In the following, we focus on $P^{\rm qp}_{c,\omega}(\boldsymbol{r},\boldsymbol{r}') = \langle a_-|\hat{P}_{c,\omega}(\boldsymbol{r},\boldsymbol{r}')|a_-\rangle$, which will later be related to the thermal conductivity. We find

$$P^{\rm qp}_{c,\omega}(\boldsymbol{r},\boldsymbol{r}') = P^{\rm qp}_{d,\omega}(\boldsymbol{r},\boldsymbol{r})f^{\rm qp}(\boldsymbol{R}), \tag{38}$$

with

$$f^{\rm qp}(\boldsymbol{R}) \approx \begin{cases} \frac{1}{2}e^{-\frac{R}{l_e}} & \text{(1D)}, \\ \frac{1}{\pi k_F R}e^{-\frac{R}{l_e}} & \text{(2D)}, \\ \frac{1}{2k_F^2R^2}e^{-\frac{R}{l_e}} & \text{(3D)}, \end{cases} \tag{39}$$

where we have assumed $k_F R \gg 1$ and $\mu \gg \epsilon$, and we have averaged fast oscillations over a spatial region of extension large compared to the Fermi wavelength $1/k_F$ but small compared to the mean-free path l_e. The weak localization correction (38) is a positive contribution to the probability of diffusion that is negligible when $R \gg l_e$. Consequentially, particles have an enhanced probability of returning to the origin. Due to conservation of the total probability, this implies a reduced probability of diffusion over long distances. This effect will be seen as a decrease of the thermal conductivity in Sec. IV and is qualitatively the same effect that the WL correction has on the transport coefficients of a normal metal.

The condition for the validity of the diffusive approximation $\omega\tau_{\rm min} \ll 1$ affects the return probability $\hat{P}_{d,\omega}(\boldsymbol{r},\boldsymbol{r})$. Based on that condition, the diffusive behavior of the system breaks down when considering very short timescales. On the other hand, on long timescales diffusion is limited by the phase-coherence time τ_ϕ (which can be related to the phase-coherence length via $L_\phi = \sqrt{D_s\tau_\phi}$). The return probability at zero frequency is then given by

$$\hat{P}_d(\boldsymbol{r},\boldsymbol{r}) = \int_{\tau_{\rm min}}^{\tau_\phi} dt\, \hat{P}_d(\boldsymbol{r},\boldsymbol{r};t), \tag{40}$$

where $\hat{P}_d(\boldsymbol{r},\boldsymbol{r}';t)$ is given in Eq. (31). Solving the diffusion equation (32) in d-dimensional free space we obtain

$$P^{\rm qp}_d(\boldsymbol{r},\boldsymbol{r}';t) = \frac{2\pi\rho_0 v_g}{v_F}\frac{1}{(4\pi D_s t)^{d/2}}e^{-R^2/(4D_s t)}, \tag{41}$$

where, as mentioned above, we focus for later use on the element $P^{\rm qp}_d(\boldsymbol{r},\boldsymbol{r}';t) = \langle a_-|\hat{P}_d(\boldsymbol{r},\boldsymbol{r}';t)|a_-\rangle$. Inserting this result into Eq. (40) and performing the integral yields the return probability at zero frequency

$$P^{\rm qp}_d(\boldsymbol{r},\boldsymbol{r}) = \frac{4\pi\rho_0}{D(4\pi)^{d/2}}D_s^{1-d/2}\begin{cases} \sqrt{\tau_\phi} - \sqrt{\tau_{\rm min}} & \text{(1D)}, \\ \ln\left(\frac{\sqrt{\tau_\phi}}{\sqrt{\tau_{\rm min}}}\right) & \text{(2D)}, \\ \frac{1}{\sqrt{\tau_{\rm min}}} - \frac{1}{\sqrt{\tau_\phi}} & \text{(3D)}. \end{cases} \tag{42}$$

This result is similar to the one for normal metals [13], but there is a crucial difference: unlike in the normal state where $\tau_{\rm min} = \tau_e$, in the superconducting state $\tau_{\rm min}$ of Eq. (20) is an energy-dependent quantity (this holds also for D_s). The energy dependence is qualitatively different in the two regimes separated by the energy E_* [see Eq. (22)], and the energy E_* itself takes different values in the clean and dirty regimes.

IV. THERMAL CONDUCTIVITY

In this section, we connect the results of the previous section concerning particle propagation to the thermal conductivity which is a physical observable. We obtain quantum corrections to the known results for the Drude-Boltzmann contribution [10]. In particular, we derive explicit results for the weak localization correction to the thermal conductivity in two dimensions. Interestingly, in the superconducting state this correction displays a temperature dependence that differs from that in the normal state [or its simple extension to be discussed after Eq. (54)]. Different regimes arise depending on the relations between temperature T, order parameter $\Delta(T)$, and the energy scale $\varepsilon_* = E_* - \Delta$.

Our starting point is Kubo's formula for the thermal conductivity $K(T)$ [24]; it can be written in terms of a product of Green's functions [10]

$$K = \frac{1}{4\pi k_B T^2 m^2}\int_\Delta^\infty dE\, \frac{E^2}{\cosh^2\left(\frac{E}{2k_BT}\right)}I, \tag{43}$$

with[1]

$$I = \int \frac{d^dk}{(2\pi)^d}\frac{d^dk'}{(2\pi)^d}k_xk_x'\mathrm{Tr}\left[\overline{\tau_3\mathrm{Im}\,\hat{G}^R_E(\boldsymbol{k},\boldsymbol{k}')\tau_3\mathrm{Im}\,\hat{G}^R_E(\boldsymbol{k}',\boldsymbol{k})}\right], \tag{44}$$

where k_B is the Boltzmann constant and we take x as the direction of the temperature gradient (and hence of heat propagation in an isotropic material, to which we restrict our attention). Its diagrammatic representation can be seen in Fig. 4.

Using the above expression, we rewrite $I = I_A - I_D$ as the difference between two integrals with

$$I_A = \mathrm{Re}\int \frac{d^dk}{(2\pi)^d}\frac{d^dk'}{(2\pi)^d}\frac{k_xk_x'}{2}\mathrm{Tr}\left[\overline{\tau_3\hat{G}^R_E(\boldsymbol{k},\boldsymbol{k}')\tau_3\hat{G}^A_E(\boldsymbol{k}',\boldsymbol{k})}\right], \tag{45}$$

$$I_D = \mathrm{Re}\int \frac{d^dk}{(2\pi)^d}\frac{d^dk'}{(2\pi)^d}\frac{k_xk_x'}{2}\mathrm{Tr}\left[\overline{\tau_3\hat{G}^R_E(\boldsymbol{k},\boldsymbol{k}')\tau_3\hat{G}^R_E(\boldsymbol{k}',\boldsymbol{k})}\right]. \tag{46}$$

In the regime $\mu \gg \epsilon, \Delta$ discussed after Eq. (13), we can approximate $k_x \approx k_F u_x$, where $\boldsymbol{u}$ is the unit vector on the Fermi surface, and similarly for k_x'. Therefore, only the relative angle between the two momenta $\boldsymbol{k},\boldsymbol{k}'$ matters. Indeed, we discuss below the dependence of the disorder-averaged product of Green's functions on the relative orientation of $\boldsymbol{k}$ and $\boldsymbol{k}'$. Once this dependence is known, the integrals I_A and I_D can then be related to the so called A-type and D-type diffusive modes, respectively [14]. The A-type modes contribution I_A is proportional to the probability of diffusion $\hat{P}_\omega(\boldsymbol{r},\boldsymbol{r}')$ studied in Sec. III, and the D-type modes one to $\hat{P}^D_\omega(\boldsymbol{r},\boldsymbol{r}') = \overline{\hat{G}^R_{E+\omega}(\boldsymbol{r},\boldsymbol{r}')\otimes\hat{G}^R_E(\boldsymbol{r}',\boldsymbol{r})}$. In systems with time-reversal symmetry it can be shown that the D-type modes do not contribute to the thermal conductivity, i.e, $I_D = 0$ [10].

[1]Note that the general expression for the Green's functions in momentum space depends on both the initial and final momentum $\boldsymbol{k}$ and $\boldsymbol{k}'$. The simplified expression given in 11 assumes that after disorder averaging the Green's functions are $\propto\delta_{\boldsymbol{k},\boldsymbol{k}'}$.

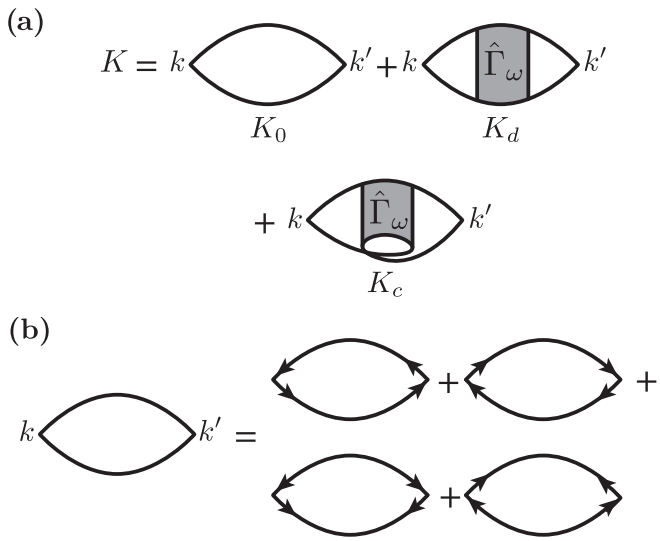

FIG. 4. (a) Diagrammatic representation of the Drude-Boltzmann, diffuson, and cooperon contributions to the thermal conductivity in a disordered system (K_0, K_d, and K_c, respectively). The arrowless lines represent superconducting Green's functions in Nambu space. The vertex contractions differentiate K from the probability of diffusion (see Fig. 2). (b) Explicit representation of the four different components arising from the vertex contractions. These four terms correspond to the four superconducting Green's functions pairs obtained from the trace seen in Eq. (44).

With $I = I_A$ we can calculate the thermal conductivity using the results from the previous section. We only briefly sketch how to use those results here; more details on how to relate transport coefficients to the propagation probability can be found in Ref. [13].

As done for the total probability of diffusion, we distinguish three contributions to the thermal conductivity, K_0, K_d, and K_c, which can be seen in Fig. 4(a). We define as such the integrals I_0, I_d, and I_c such that, for $i \in \{0, d, c\}$ we have

$$K_i = \frac{1}{4\pi k_B T^2 m^2} \int_\Delta^\infty dE \, \frac{E^2}{\cosh^2\left(\frac{E}{2k_BT}\right)} I_i. \tag{47}$$

The Drude-Boltzmann integral I_0 represents propagation in a disordered medium without any scattering event taking place. In the absence of scattering, the initial and final momenta of the Green's functions are the same, with $\overline{\hat{G}^R_E(\boldsymbol{k}, \boldsymbol{k}')} \otimes \overline{\hat{G}^A_E(\boldsymbol{k}', \boldsymbol{k})^T} \propto \delta^{(d)}(\boldsymbol{k} - \boldsymbol{k}')$. Then, the angular integration in momentum space is equivalent to taking the product of momenta out of the integral as k_F^2/d, and the relation between I_0 and $P_0^{\rm qp}(\boldsymbol{r}, \boldsymbol{r}')$ can be obtained by going into real space, using the Fourier transform for a translational-invariant system

$$\int \frac{d^d k}{(2\pi)^d} \overline{\hat{G}^R_E(k)} \otimes \overline{\hat{G}^A_E(k)^T} = \int d^d r' \hat{P}_0(\boldsymbol{r}, \boldsymbol{r}'). \tag{48}$$

The Drude-Boltzmann integral is then

$$I_0 = \frac{k_F^2}{d} \mathrm{Re} \int d^d r' P_0^{\rm qp}(\boldsymbol{r}, \boldsymbol{r}'), \tag{49}$$

and using Eq. (18) [see also Eq. (B6)] we obtain $I = k_F^2/d\gamma_e$. Inserting the result into Eq. (43) we obtain the Drude-Boltzmann contribution to the thermal conductivity

$$K_0 = \frac{D\rho_0}{2k_BT^2} \int_\Delta^\infty dE \, \frac{E^2}{\cosh^2\left(\frac{E}{2k_BT}\right)}, \tag{50}$$

where $D = v_F l_e/d$ is the diffusion constant in the normal state. This formula agrees with previous calculations [10]. It is equivalent to the result in the normal state with the sole difference that only states with energy $E > \Delta$ contribute. The absence of states below the gap is reflected in the lower limit of the integral and leads to the exponential suppression of K_0 at temperatures $k_BT \ll \Delta$.

For the diffuson integral I_d, we find simply $I_d = 0$. This result is valid for isotropic scattering by impurities: the initial and final momenta of the Green's functions ($\boldsymbol{k}$ and $\boldsymbol{k}'$, respectively) have uncorrelated directions after a large number of scattering events, which leads to the vanishing of the angular integration in Eq. (45). Anisotropic scattering would result in the substitution of the scattering time τ_e with the transport time in Eq. (50) [13] (τ_e enters that equation via the mean-free path in the diffusion constant).

Similar considerations to those above make it possible to relate I_c to $P_c^{\rm qp}(\boldsymbol{r}, \boldsymbol{r}')$. The cooperon accounts for an enhanced probability of a particle to return to its initial point; therefore, its initial and final momenta will be approximately opposite to each other. The integrand of I_c is then sharply peaked around $\boldsymbol{k} = -\boldsymbol{k}'$, and can be approximated to be proportional to $\delta^{(d)}(\boldsymbol{k} + \boldsymbol{k}')$.[2] We again take the product of momenta out of the integral, and going over to real space yields

$$I_c = -\frac{k_F^2}{d} \mathrm{Re} \int d^d r' P_c^{\rm qp}(\boldsymbol{r}, \boldsymbol{r}'). \tag{51}$$

After substituting Eq. (38) into the above expression we have

$$I_c = -\frac{k_F^2}{d} \mathrm{Re}\, P_d^{\rm qp}(\boldsymbol{r}, \boldsymbol{r}) \int d^d R \, f^{\rm qp}(\boldsymbol{R}). \tag{52}$$

Using the expressions for $f^{\rm qp}(\boldsymbol{R})$ given in Eq. (39), we find that for all dimensions

$$\int d^d R \, f^{\rm qp}(\boldsymbol{R}) = \frac{\tau_e}{\pi \rho_0}, \tag{53}$$

and inserting these results into Eq. (43), we arrive at

$$K_c = -\frac{D}{4\pi^2 k_B T^2 \rho_0} \int_\Delta^\infty dE \, \frac{E^2}{\cosh^2\left(\frac{E}{2k_BT}\right)} P_d^{\rm qp}(\boldsymbol{r}, \boldsymbol{r}), \tag{54}$$

where the return probability $P_d^{\rm qp}(\boldsymbol{r}, \boldsymbol{r})$, given by Eq. (42), is a function of energy E. This energy dependence leads to a temperature dependence of K_c which we study in the following for two dimensions. We note that it is crucial to retain this energy dependence. Neglecting the energy dependence of the return probability, we would find the incorrect result $K_c/K_0 = -P_d^{\rm qp}(\boldsymbol{r}, \boldsymbol{r})/2\pi^2\rho_0^2$ and the temperature dependence of K_c would simply follow from the one in the normal state.

[2]This result can be obtained mathematically by calculating the structure factor for the cooperon in momentum space. It has a peak at $\boldsymbol{k} + \boldsymbol{k}' = 0$, with $\hat{\Gamma}_{c,\omega}(\boldsymbol{k} + \boldsymbol{k}') = \hat{\Gamma}_\omega(\boldsymbol{k} + \boldsymbol{k}')$, where $\hat{\Gamma}_\omega(\boldsymbol{q})$ is defined in Eq. (B2); see also Ref. [13].

A. Regimes for the WL correction to the thermal conductivity

As remarked above, the dependence of the return probability $P_d^{\rm qp}(\boldsymbol{r},\boldsymbol{r})$ on energy makes it possible for the WL correction K_c to the thermal conductivity to have a temperature dependence that differs from that of the main (Drude-Boltzmann) contribution K_0. Here, we explore when such a deviation takes place and under which conditions it could be observable. To this aim, let us introduce the temperature T_Δ defined by $k_B T_\Delta = \Delta(T_\Delta)$; for our purposes, the temperature dependence of the gap on temperature is approximately captured by the interpolation formula [25]

$$\Delta(T) \approx 1.76 k_B T_c \tanh\left(1.74\sqrt{\frac{T_c}{T}-1}\right), \tag{55}$$

with T_c the critical temperature of the superconductor. From this expression we find $T_\Delta \approx 0.9 T_c$. Clearly, both K_0 and K_c are exponentially suppressed in the low-temperature regime $T \ll T_\Delta$ [see Eqs. (50) and (54)], making their accurate measurement challenging. Therefore, the high-temperature regime $T_\Delta \lesssim T < T_c$ is most relevant in order to observe the effects of weak localization. For completeness, we consider both regimes below (details of the calculations are presented in Appendix D).

A second relevant temperature scale, denoted by T_*, can be defined via the equation $k_B T_* = \varepsilon_*(T_*)$, where ε_* depends on temperature through the gap $\Delta(T)$ [see Eq. (22)]. For dirty superconductors, $\tau_e \Delta(0) \ll 1$, we have $T_* \simeq T_c$, while for clean ones, $\tau_e \Delta(0) \gg 1$, we find $T_* \ll T_c$, indicating that qualitatively different behaviors can be expected in the two cases. Finally, with regard to the effect of phase coherence on K_c, we consider two possibilities, namely, an energy-independent coherence time τ_ϕ or an energy-independent coherence length $L_\phi = \sqrt{D_S \tau_\phi}$. These two possibilities are equivalent in the normal state, but in the superconducting one they are not, due to the energy dependence of the diffusion constant D_S [Eq. (26)].

1. High-temperature regime

In the high-temperature regime $T \gtrsim T_\Delta$, in order to find the leading contributions to the heat conductivity, we approximate $k_B T \gg \Delta$. Then, the WL correction in this regime does not depend on the gap Δ. Moreover, for a superconductor in the clean limit, since $\varepsilon_* \ll \Delta$, the relative correction coincides with the one in the normal state[3]

$$\frac{K_c}{K_0} = \frac{-1}{\pi k_F l_e}\begin{cases}\ln\left(\frac{\tau_\phi}{\tau_e}\right) & (\tau_\phi \text{ fixed}),\\ 2\ln\left(\frac{L_\phi}{l_e}\right) & (L_\phi \text{ fixed}).\end{cases} \tag{56}$$

The same expressions hold for a dirty superconductor sufficiently close to T_c, so that $\varepsilon_* \ll k_B T$, but since, as mentioned above, $T_* \simeq T_c$, this result has very limited applicability. More interestingly, there exists an intermediate regime $T_\Delta \lesssim T \lesssim T_*$, in which the WL correction depends on the ratio $k_B T/\varepsilon_*$:

$$\frac{K_c}{K_0} = \frac{-1}{\pi k_F l_e}\begin{cases}\ln\left(\frac{\tau_\phi}{\tau_e}\right) - 2\ln\left(\frac{\varepsilon_*}{k_B T}\right) & (\tau_\phi \text{ fixed}),\\ 2\ln\left(\frac{L_\phi}{l_e}\right) - 2\ln\left(\frac{\varepsilon_*}{k_B T}\right) & (L_\phi \text{ fixed}).\end{cases} \tag{57}$$

Note that, in the high-temperature regime, the temperature dependence of the WL correction is insensitive to the assumption of energy-independent dephasing time vs length; this can be traced back to the fact that at the relevant energy scale (given by temperature), we have for the group velocity $v_g \approx v_F$ [see Eq. (19)].

2. Low-temperature regime

In the low-temperature regime $T \ll T_\Delta$ we have $k_B T \ll \Delta$, which results in the exponential suppression of both K_0 and K_c discussed above. Their ratio, however, is not exponentially suppressed. Indeed, the WL correction for a dirty superconductor is given by

$$\frac{K_c}{K_0} = \frac{-1}{\pi k_F l_e}\begin{cases}\ln\left(\frac{\tau_\phi}{\tau_e}\right) - \ln\left(\frac{\varepsilon_*^2}{\Delta k_B T}\right) & (\tau_\phi \text{ fixed}),\\ 2\ln\left(\frac{L_\phi}{l_e}\right) - \ln\left(\frac{\varepsilon_*^2}{\Delta^{3/2}\sqrt{k_B T}}\right) & (L_\phi \text{ fixed}).\end{cases} \tag{58}$$

In both cases, at the crossover temperature T_Δ the correction agrees with that found in the high-temperature regime. However, the temperature dependence is now sensitive to the assumption of energy-independent dephasing time/length.

For a clean superconductor in the regime $T_* < T < T_\Delta$, the normalized WL correction is

$$\frac{K_c}{K_0} = \frac{-1}{\pi k_F l_e}\begin{cases}\ln\left(\frac{\tau_\phi}{\tau_e}\right) - \frac{1}{2}\ln\left(\frac{\Delta}{k_B T}\right) & (\tau_\phi \text{ fixed}),\\ 2\ln\left(\frac{L_\phi}{l_e}\right) & (L_\phi \text{ fixed}).\end{cases} \tag{59}$$

We note that, according to Eqs. (56) and (59), for $T > T_*$ and assuming energy-independent dephasing length, the WL correction in the clean case coincides with that in the normal state. This finding resembles that for the WL correction to the heat conductance of superconductor/normal/superconductor junctions with short (shorter than dephasing length) normal part in the absence of phase gradient and gap differences [12]. In that case, the latter two assumptions ensure that the transmission probability of quasiparticles excitations through the junction is independent of energy. Similarly here, the assumptions of energy-independent dephasing length and sufficiently high temperature ensure that the return probability of Eq. (42) is energy independent over the relevant energy range. For a clean superconductor, there exists also a regime where $k_B T \ll \varepsilon_*$, where this energy independence does not hold. This regime is calculated in Appendix D, but we do not discuss it here further as it has a limited validity at temperatures where the thermal conductivity is strongly suppressed.[4]

V. SUMMARY AND DISCUSSION

In this work, we have calculated the weak localization correction to the thermal conductivity in conventional disordered superconductors. As our starting point, we have studied

[3]We note that sufficiently close to T_c, a crossover from clean to dirty takes place when $\Delta\tau_e \sim 1$. However, the condition $\varepsilon_* \ll T$ is met, which ensures the validity of our results (see also the discussion for the dirty case).

[4]We stress that all the results of this section are valid only under certain conditions on τ_ϕ or L_ϕ, explained in Appendix D, which ensure that the sum of the logarithms is positive.

diffusion with the help of a general formalism based on semiclassical Green's functions and their corresponding matrix expressions in Nambu space (see Sec. III). The formalism can be straightforwardly expanded to tackle systems with different symmetries; as an example, in Appendix E we investigate diffusion in the presence of weak spin-orbit scattering.

The thermal conductivity K can be obtained from the probability of diffusion and, similarly to the calculation of electrical conductivity in the normal state, the weak localization correction can be related to the cooperon $P_c^{\rm qp}(\boldsymbol{r},\boldsymbol{r}')$ (see Sec. III C). In fact, the correction always reduces the thermal conductivity which is consistent with the results for electrical conductivity in normal metals [13,26]. Our calculations in Sec. III show that diffusion is reduced as the probability of return to the origin is increased due to WL.

As the temperature decreases below the critical temperature, the thermal conductivity is suppressed due to the opening of the gap Δ in the density of states; this leads to the well-known exponential suppression of K at temperatures $T \ll T_\Delta \approx 0.9T_c$. Interestingly, we find that the WL correction is affected not only by the gap, but also by a second energy scale ε_* related to both the gap and the impurity scattering time τ_e [see Eq. (22)]. This energy scale encodes the fact that the onset of diffusion takes longer and longer times as the energy approaches the gap (while being limited only by the scattering time in the normal state); similarly, the diffusion constant decreases as energy decreases toward the gap [Eq. (26)]. As a consequence, by lowering temperature the probability of return to the origin is decreased compared to the normal state, and the magnitude of the WL correction decreases.

For both clean ($\tau_e\Delta \gg 1$) and dirty ($\tau_e\Delta \ll 1$) superconductors, we have considered the high- ($T > T_\Delta$) and low- ($T < T_\Delta$) temperature regimes, as summarized in Fig. 5 for two dimensions. We highlight the regime $T_\Delta \approx 0.9T_c < T < T_* \approx T_c$, which exists only in dirty superconductors, as the most interesting for the experimental verification of our results. In this temperature range, the thermal conductivity is not yet exponentially suppressed but, at the same time most of the decrease in the magnitude of the WL correction has taken place [see Fig. 5(a) and Eq. (57)]. This result holds independently of the assumptions made about the dephasing mechanism which we have only treated at a phenomenological level. At the temperature T_Δ the reduction of the weak localization correction is proportional to $\ln(1/\tau_e\Delta)$ and is therefore parametrically large; such a reduction is measurable on top of other contributions to the thermal conductivity (for example, due to phonons) which are smooth in this temperature window. Moreover, such a temperature dependence of the WL correction is peculiar to the superconducting state and differentiates it from the normal state, in which another quantity (usually, magnetic field) needs to be tuned in order to measure the WL correction.

An interesting question for future research is the generalization of the approach presented here to calculate transport properties in disordered d-wave superconductors [27–29], for which the weak localization correction to thermal conductivity has so far been considered only in the mixed state [30]. For both s- and d-wave superconductors, calculating the effect of Zeeman splitting on the WL correction could also afford another avenue to experimentally check our theory.

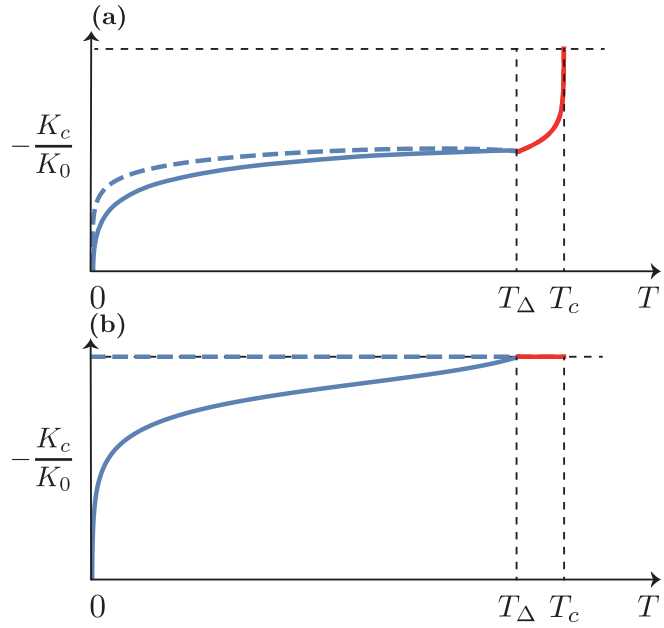

FIG. 5. Schematic representation of the normalized weak localization correction $-K_c/K_0$ as a function of temperature for an s-wave superconductor in the (a) dirty limit, where $T_* \approx T_c$ and (b) clean limit, where $T_* \ll T_c$. The blue color highlights the behavior in the low-temperature regime $T < T_\Delta$, and the red color in the high-temperature one. The solid lines represent the results for energy-independent coherence time (fixed τ_ϕ) and the dashed lines (in red and blue color) for energy-independent coherence length (L_ϕ fixed). The horizontal dashed black line represents the normalized weak localization correction in the normal state [see Eq. (56)].

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APPENDIX A: NORMALIZATION OF THE DIFFUSION PROBABILITY

To discuss the normalization of the probability we consider particle conservation: in a superconducting system, the number of electrons plus the number of holes must be conserved. Let us define the two probabilities

$$P_{e,\omega}(\boldsymbol{r},\boldsymbol{r}') = \overline{G^R_{E+\omega}(\boldsymbol{r},\boldsymbol{r}')G^A_E(\boldsymbol{r}',\boldsymbol{r})} - \overline{F^R_{E+\omega}(\boldsymbol{r},\boldsymbol{r}')\bar{F}^A_E(\boldsymbol{r}',\boldsymbol{r})}, \tag{A1}$$

$$P_{h,\omega}(\boldsymbol{r},\boldsymbol{r}') = \overline{\bar{G}^R_{E+\omega}(\boldsymbol{r},\boldsymbol{r}')\bar{G}^A_E(\boldsymbol{r}',\boldsymbol{r})} - \overline{\bar{F}^R_{E+\omega}(\boldsymbol{r},\boldsymbol{r}')F^A_E(\boldsymbol{r}',\boldsymbol{r})}. \tag{A2}$$

Here, $P_{e,\omega}(\boldsymbol{r},\boldsymbol{r}')$ is the probability that an electron propagates from $\boldsymbol{r}$ to $\boldsymbol{r}'$ plus the probability that said electron converts into a hole at some point during the trajectory. $P_{h,\omega}(\boldsymbol{r},\boldsymbol{r}')$ is the equivalent for holes. These two quantities are related to $\hat{P}_\omega(\boldsymbol{r},\boldsymbol{r}')$ by

$$\langle a_-|\hat{P}_\omega(\boldsymbol{r},\boldsymbol{r}')|a_-\rangle = \tfrac{1}{2}[P_{e,\omega}(\boldsymbol{r},\boldsymbol{r}') + P_{h,\omega}(\boldsymbol{r},\boldsymbol{r}')]. \tag{A3}$$

We can define the normalized probabilities $\mathcal{P}_{e,\omega}(\boldsymbol{r},\boldsymbol{r}') = AP_{e,\omega}(\boldsymbol{r},\boldsymbol{r}')$ and $\mathcal{P}_{h,\omega}(\boldsymbol{r},\boldsymbol{r}') = AP_{h,\omega}(\boldsymbol{r},\boldsymbol{r}')$ such that

$$\langle \mathcal{P}_{e,\omega}\rangle_{\boldsymbol{r}} = \langle \mathcal{P}_{h,\omega}\rangle_{\boldsymbol{r}} = \frac{i}{\omega}, \tag{A4}$$

which is the Fourier transform into frequency space of the normalization condition

$$\int d^d r' \mathcal{P}_{e,\omega}(\boldsymbol{r},\boldsymbol{r}';t) = \int d^d r' \mathcal{P}_{h,\omega}(\boldsymbol{r},\boldsymbol{r}';t) = 1. \tag{A5}$$

Let us now consider the diffusion equation followed by $P^{\mathrm{qp}}_{d,\omega}(\boldsymbol{r},\boldsymbol{r}') \equiv \langle a_-|\hat{P}_{d,\omega}(\boldsymbol{r},\boldsymbol{r}')|a_-\rangle$, given by the first element of Eq. (29):

$$\frac{v_F}{2\pi\rho_0 v_g}\left(-D_s\nabla^2_{\boldsymbol{r}} - i\omega\right)P^{\mathrm{qp}}_{d,\omega}(\boldsymbol{r},\boldsymbol{r}') = \delta^{(d)}(\boldsymbol{r}'-\boldsymbol{r}). \tag{A6}$$

After spatial integration, we find the normalization factor $A = v_F/2\pi\rho_0 v_g$.

APPENDIX B: SUPERCONDUCTING DIFFUSON IN MOMENTUM SPACE

To work in momentum space, we start by taking the Fourier transform (23). The Laplace operator $\nabla^2_{\boldsymbol{r}}$ becomes the relative momentum squared $\boldsymbol{q}^2$, and $\hat{M}_\omega(\boldsymbol{q})$ can be inverted to obtain

$$\hat{\Gamma}_\omega(\boldsymbol{q}) = \gamma_e \hat{M}_\omega(\boldsymbol{q})^{-1}. \tag{B1}$$

After calculating the inverse of $\hat{M}_\omega(\boldsymbol{q})$ explicitly, we can simplify it in the diffusive regime discussed in Sec. III B, and $\hat{\Gamma}_\omega(\boldsymbol{q})$ is reduced to a rank-two matrix whose nonzero elements correspond to $\hat{\Gamma}_\omega(\boldsymbol{q}) = \gamma_e \hat{\mathsf{M}}_\omega(\boldsymbol{q})^{-1}$, given in the basis $B_2 = \{|a_-\rangle, \cos(\theta)|a_+\rangle + \sin(\theta)|b_+\rangle\}$ by

$$\hat{\Gamma}_\omega(\boldsymbol{q}) = \frac{\gamma_e}{\tau_s}\begin{pmatrix} \frac{1}{D_s q^2 - i\omega} & 0 \\ 0 & \frac{E^2+\Delta^2}{\epsilon^2}\frac{1}{D_s q^2 - i\omega} \end{pmatrix}. \tag{B2}$$

The diffuson, given by

$$\hat{P}_{d,\omega}(\boldsymbol{q}) = \hat{P}_{0,\omega}(\boldsymbol{q})\hat{\Gamma}_\omega(\boldsymbol{q})\hat{P}_{0,\omega}(\boldsymbol{q}), \tag{B3}$$

can be approximated in the limit of small relative momentum $\boldsymbol{q}$ and relative frequency ω as

$$\hat{P}_{d,\omega}(\boldsymbol{q}) = \hat{P}_0(0)\hat{\Gamma}_\omega(\boldsymbol{q})\hat{P}_0(0), \tag{B4}$$

where $\hat{P}_0(\boldsymbol{q}=0) = \langle \hat{P}_0\rangle_{\boldsymbol{r}}$, given in the original Nambu basis [defined after Eq. (8)] by

$$\langle \hat{P}_0\rangle_{\boldsymbol{r}} = \frac{1}{2\gamma_e\epsilon^2}\begin{pmatrix} 2E^2-\Delta^2 & -\Delta E & -\Delta E & \Delta^2 \\ -\Delta E & \Delta^2 & \Delta^2 & -\Delta E \\ -\Delta E & \Delta^2 & \Delta^2 & -\Delta E \\ \Delta^2 & -\Delta E & -\Delta E & 2E^2-\Delta^2 \end{pmatrix}, \tag{B5}$$

and in its eigenbasis $\tilde{B}$ by

$$\langle \hat{P}_0\rangle_{\boldsymbol{r}} = \frac{1}{\gamma_e}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{E^2+\Delta^2}{\epsilon^2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{B6}$$

The diffuson can then be calculated by direct matrix multiplication. We obtain

$$\hat{P}_d(\boldsymbol{q}) = \frac{1}{\tau_s}\frac{1}{D_s q^2 - i\omega}\langle \hat{P}_0\rangle_{\boldsymbol{r}}, \tag{B7}$$

which corresponds to a rank-two matrix that can be written as

$$\hat{\mathsf{P}}_{d,\omega}(\boldsymbol{q}) = \frac{2\pi\rho_0 v_g}{v_F}\begin{pmatrix} \frac{1}{D_s q^2 - i\omega} & 0 \\ 0 & \frac{E^2+\Delta^2}{\epsilon^2}\frac{1}{D_s q^2 - i\omega} \end{pmatrix} \tag{B8}$$

in the basis $\tilde{B}_2 = \{|a_-\rangle, \cos(\theta)|a_+\rangle - \sin(\theta)|b_+\rangle)\}$. This is again equivalent to the result obtained by solving Eq. (29) after performing a Fourier transform into momentum space.

APPENDIX C: SUPERCONDUCTING COOPERON

Here, we work out explicitly the relation between cooperon and diffusion in the superconducting state. Since we are interested in the diffusive regime, the matrix $\hat{\Gamma}_{c,\omega}(\boldsymbol{r},\boldsymbol{r}) = \hat{\Gamma}_\omega(\boldsymbol{r},\boldsymbol{r})$ can be simplified as a 2×2 matrix that follows Eq. (27) in the subspace spanned by $B_2 = \{|a_-\rangle, \cos(\theta)|a_+\rangle + \sin(\theta)|b_+\rangle\}$. As done in Eq. (28), we rewrite Eq. (35) as

$$\hat{P}_{c,\omega}(\boldsymbol{r},\boldsymbol{r}') = \hat{F}_v(\boldsymbol{R})\hat{\Gamma}_\omega(\boldsymbol{r},\boldsymbol{r})\hat{F}^T_v(\boldsymbol{R}), \tag{C1}$$

where $\hat{F}_v(\boldsymbol{R})$ is defined, similarly to $\hat{P}_v$, as the matrix containing the first two columns of $\hat{F}(\boldsymbol{R})$ in the B basis. By substituting the expression for $\hat{\Gamma}_\omega(\boldsymbol{r},\boldsymbol{r})$ as a function of $\hat{P}_{d,\omega}(\boldsymbol{r},\boldsymbol{r})$ obtained from Eq. (28), we find

$$\hat{P}_{c,\omega}(\boldsymbol{r},\boldsymbol{r}') = \hat{A}(\boldsymbol{R})\hat{P}_{d,\omega}(\boldsymbol{r},\boldsymbol{r})\hat{A}(\boldsymbol{R})^T, \tag{C2}$$

where

$$\hat{A}(\boldsymbol{R}) = \gamma_e^2 \hat{F}_v(\boldsymbol{R})\hat{P}_v^T. \tag{C3}$$

The matrix $\hat{A}(\boldsymbol{R})$ is, like $\hat{P}_{d,\omega}(\boldsymbol{r},\boldsymbol{r})$, a rank-two matrix whose only nonzero terms exist in the subspace spanned by the basis $\tilde{B}_2 = \{|a_-\rangle, \cos(\theta)|a_+\rangle - \sin(\theta)|b_+\rangle)\}$. The cooperon $\hat{P}_{c,\omega}(\boldsymbol{r},\boldsymbol{r}')$ will therefore also share this property, and we can work with Eq. (C2) in the $\tilde{B}_2$ basis subspace to ensure the invertibility of all terms involved and simplify the calculation. We write this as

$$\hat{\mathsf{P}}_{c,\omega}(\boldsymbol{r},\boldsymbol{r}') = \hat{\mathsf{A}}(\boldsymbol{R})\hat{\mathsf{P}}_{d,\omega}(\boldsymbol{r},\boldsymbol{r})\hat{\mathsf{A}}(\boldsymbol{R})^T, \tag{C4}$$

where use of the sans serif fonts denotes the projection into the 2×2 subspace. We deduce from Eq. (29) that $\hat{\mathsf{P}}_{d,\omega}(\boldsymbol{r},\boldsymbol{r})$ is diagonal and proportional to the matrix diag$[1, \epsilon^2/(E^2+\Delta^2)]$ in the $\tilde{B}_2$ basis. We can then write in this basis

$$\hat{\mathsf{P}}_{c,\omega}(\boldsymbol{r},\boldsymbol{r}') = \hat{\mathsf{P}}_{d,\omega}(\boldsymbol{r},\boldsymbol{r})\hat{\mathsf{f}}(\boldsymbol{R}), \tag{C5}$$

where

$$\hat{\mathsf{f}}(\boldsymbol{R}) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{E^2+\Delta^2}{\epsilon^2} \end{pmatrix}\hat{\mathsf{A}}(\boldsymbol{R})\begin{pmatrix} 1 & 0 \\ 0 & \frac{\epsilon^2}{E^2+\Delta^2} \end{pmatrix}\hat{\mathsf{A}}(\boldsymbol{R})^T. \tag{C6}$$

The equation in the full 4×4 space can be obtained by expanding every matrix into the full $\tilde{B}$ basis by filling in zeros in all the other elements of the matrix to obtain

$$\hat{P}_{c,\omega}(\boldsymbol{r},\boldsymbol{r}') = \hat{P}_{d,\omega}(\boldsymbol{r},\boldsymbol{r})\hat{f}(\boldsymbol{R}). \tag{C7}$$

In the two-dimensional case, we have, in the $\tilde{B}_2$ basis

$$\hat{\mathrm{f}}(\boldsymbol{R}) = e^{-R/l_e}\frac{1}{\pi k_F R}\begin{pmatrix} \cos^2\left(k_h R-\frac{\pi}{4}\right)+\cos^2\left(k_e R-\frac{\pi}{4}\right) & \frac{\sqrt{E^2+\Delta^2}}{\epsilon}\left[\cos^2\left(k_h R-\frac{\pi}{4}\right)-\cos^2\left(k_e R-\frac{\pi}{4}\right)\right] \\ \frac{\epsilon}{\sqrt{E^2+\Delta^2}}\left[\cos^2\left(k_h R-\frac{\pi}{4}\right)-\cos^2\left(k_e R-\frac{\pi}{4}\right)\right] & \cos^2\left(k_h R-\frac{\pi}{4}\right)+\cos^2\left(k_e R-\frac{\pi}{4}\right) \end{pmatrix}, \tag{C8}$$

where $k_e = k_F + \epsilon/v_F$ and $k_h = k_F - \epsilon/v_F$. We note that in contrast to $\hat{P}_{d,\omega}(\boldsymbol{r},\boldsymbol{r}')$, this matrix is not diagonal in the $\tilde{B}$ basis; that is, we have not fully separated the two low-energy modes. However, we work in the limit $\mu \gg \epsilon$, where the small difference in the frequency of oscillation between electrons and holes is negligible. The fast oscillations average out when integrating over a length long compared to the Fermi wavelength but small compared to the mean-free path, so that we can obtain an approximate formula for $\hat{\mathrm{f}}(\boldsymbol{R})$ by replacing $\cos^2(k_e R - \pi/4) \approx \cos^2(k_h R - \pi/4) \approx 1/2$. In this approximation, the proportionality factor between the cooperon and the return probability $\hat{P}_{d,\omega}(\boldsymbol{r},\boldsymbol{r})$ [see Eq. (38)] is the same as in the normal state.

APPENDIX D: EVALUATION OF THE WEAK LOCALIZATION CORRECTION

The energy-dependent return probability $P_d^{\mathrm{qp}}(\boldsymbol{r},\boldsymbol{r})$, given in Eq. (42), has different behaviors below and above E_* [see the definition of $\tau_{\min}$ in Eq. (21)]. Accordingly, the energy integral for the WL correction to the thermal conductivity, Eq. (54), is split into two parts,

$$K_c = -\frac{1}{8\pi^2 k_B T^2}(I_1 + I_2), \tag{D1}$$

which in two dimensions are explicitly

$$I_1 = \int_\Delta^{E_*} dE \frac{E^2}{\cosh^2\left(\frac{E}{2k_BT}\right)} \ln\left[\frac{\tau_\phi(E^2-\Delta^2)}{\Delta}\right] \tag{D2}$$

and

$$I_2 = \int_{E_*}^{\infty} dE \frac{E^2}{\cosh^2\left(\frac{E}{2k_BT}\right)} \ln\left(\frac{\tau_\phi\sqrt{E^2-\Delta^2}}{\tau_e E}\right). \tag{D3}$$

Below, we consider two situations: energy-independent phase-coherence time τ_ϕ and energy-independent phase-coherence length $L_\phi = \sqrt{D_s\tau_\phi}$. These two scenarios are equivalent in the normal state, but yield different results in the superconducting one. We note that, strictly speaking, the lower integration limit of I_1 is not Δ but, in the diffusion approximation, the quantity Δ_* defined by requiring that, for the left-hand side of Eq. (40) to be nonzero, $\tau_\phi > \tau_{\min}$. For energy-independent phase time, under the usual assumption that $\tau_\phi \gg \tau_e$ (needed for the general applicability of the diffusive approximation [31]), we find for Δ_* the equation $\Delta_*^2 = \Delta(1/\tau_\phi + \Delta)$; thus, for $\tau_\phi \gg 1/\Delta$, we have $\Delta_* \approx \Delta$, an approximation that is valid for temperature not too close to absolute zero, $k_BT \gg 1/\tau_\phi$ (at lower temperatures, the WL correction is, with logarithmic accuracy, absent, since the modes with energy between Δ and Δ^* are not diffusive). The same approximation is valid in the case of energy-independent phase length (assumed to be long compared to the mean-free path l_e) under the condition $L_\phi \gg \xi$, with $\xi = \sqrt{l_e\xi_\Delta}$, where $\xi_\Delta = v_F/\Delta$ is the BCS coherence length for a clean superconductor.

1. Energy independent τ_ϕ

It is convenient to rewrite $I_1 + I_2 = I_{\mathrm{n}} + I_{\varepsilon_*} + I_3$ with

$$I_{\mathrm{n}} = \int_\Delta^\infty dE \frac{E^2}{\cosh^2\left(\frac{E}{2k_BT}\right)} \ln\left(\frac{\tau_\phi}{\tau_e}\right), \tag{D4}$$

$$I_{\varepsilon_*} = \int_\Delta^{E_*} dE \frac{E^2}{\cosh^2\left(\frac{E}{2k_BT}\right)} \ln\left(\frac{E\sqrt{E^2-\Delta^2}}{E_*\sqrt{E_*^2-\Delta^2}}\right), \tag{D5}$$

$$I_3 = \int_\Delta^\infty dE \frac{E^2}{\cosh^2\left(\frac{E}{2k_BT}\right)} \ln\left(\frac{\sqrt{E^2-\Delta^2}}{E}\right), \tag{D6}$$

where we have used the identity

$$\tau_e = \frac{\Delta}{E_*(E_*^2-\Delta^2)^{1/2}}, \tag{D7}$$

which follows from the definition of E_* [see Eq. (22)].

The integral in Eq. (D4) is defined such that its contribution to the relative correction to the thermal conductivity K_c/K_0 coincides with that in the normal state [see Eq. (56)]. The other two integrals are then responsible for the temperature-dependent deviations from the normal-state expression. We compute I_3 and I_{ε_*} for different temperature regimes with logarithmic accuracy; note that only I_{ε_*} depends on the disorder strength. We first consider the low-temperature regime $T \ll T_\Delta$ for both the dirty and the clean case, and later the high-temperature regime $T \gtrsim T_\Delta$.

a. Low-temperature regime

In the low-temperature regime, since we have $k_BT \ll \Delta$ the hyperbolic cosine can then be approximated as $1/\cosh^2(E/2k_BT) \approx 4e^{-E/k_BT}$. Introducing the dimensionless integration variable $\alpha = (E-\Delta)/k_BT$ and keeping only the leading term in the small parameter k_BT/Δ, we find

$$I_3 \approx \frac{C}{2}\int_0^\infty d\alpha\, e^{-\alpha} \ln\left(\frac{2k_BT}{\Delta}\alpha\right) = \frac{C}{2}\ln\left(2e^{-\gamma_E}\frac{k_BT}{\Delta}\right) \tag{D8}$$

with $C = 4k_BT\Delta^2 e^{-\Delta/k_BT}$ and $\gamma_E \simeq 0.5772\ldots$ the Euler-Mascheroni constant.

For the integral I_{ε_*} we can proceed with the same approximation for the hyperbolic cosine and the same change of integration variable to get

$$I_{\varepsilon_*} \approx \frac{C}{2}\int_0^{\alpha_*} d\alpha\, e^{-\alpha} \ln\frac{\alpha}{\left(1+\frac{k_BT}{\Delta}\alpha_*\right)^2\left(1+\frac{k_BT}{2\Delta}\alpha_*\right)\alpha_*}, \tag{D9}$$

where $\alpha_* = \varepsilon_*/k_BT$. We must now treat separately the disordered ($\tau_e\Delta \ll 1$) and clean ($\tau_e\Delta \gg 1$) cases. In the disordered

case we have $\alpha_* \gg \Delta/k_BT \gg 1$ and we obtain

$$I_{\varepsilon_*} \approx \frac{C}{2}\int_0 d\alpha\, e^{-\alpha} \ln\frac{2\Delta^3\alpha}{(k_BT)^3\alpha_*^4} = \frac{C}{2}\ln\left(\frac{2e^{-\gamma_E}\Delta^3 k_BT}{\varepsilon_*^4}\right). \tag{D10}$$

The sum of Eqs. (D8) and (D10) leads with logarithmic accuracy to the last term in the top line of Eq. (58).

In the clean case, since $\alpha_* k_BT/\Delta \ll 1$, the integral simplifies to

$$I_{\varepsilon_*} \approx \frac{C}{2}\int_0^{\alpha_*} d\alpha\, e^{-\alpha}\ln\frac{\alpha}{\alpha_*}. \tag{D11}$$

At very low temperatures such that $k_BT \ll \varepsilon_*$ we can extend the upper integration limit to infinity and thus find a logarithmic contribution of the form $I_{\varepsilon_*} \approx C\ln(e^{-\gamma_E}k_BT/\varepsilon_*)/2$; we also note here that for this contribution to be present the condition $\tau_\phi \gg 1/\Delta$ mentioned above is not sufficient, and a more stringent one ($\tau_\phi \gg \tau_e^2\Delta$), obtained from demanding $E_* \gg \Delta_*$, is needed. At intermediate temperatures $\varepsilon_* \ll k_BT \ll k_BT_\Delta$, on the other hand, there is no logarithmic contribution from I_{ε_*} and hence the last term in the top line of Eq. (59) is determined solely by Eq. (D8).

b. High-temperature regime

In the high-temperature regime $T \gtrsim T_\Delta$, we can approximate $k_BT \gg \Delta$. The integral $I_3 \sim k_BT\Delta^2$ has then no logarithmic parameter dependence and can be neglected in comparison to $I_n \sim (k_BT)^3\ln(\tau_\phi/\tau_e)$. For I_{ε_*} we must again consider the various regimes separately. However, for k_BT large compared to E_* (which is always true in the clean case at high temperatures, while it would require T in the narrow range between T_* and T_c for the dirty case), we can approximate the hyperbolic cosine with unity; then I_{ε_*} becomes independent of temperature and displays no logarithmic parameter dependence; thus, as I_3 above, I_{ε_*} can be neglected in comparison to I_n and we arrive at the result in the top line of Eq. (56).

We are left with the dirty case in the regime $T_\Delta \lesssim T \lesssim T_*$. Then, Δ is small compared to both E_* and the typical energy $E \sim T$, so that we can write

$$I_{\varepsilon_*} \approx \int_\Delta^{E_*} dE\,\frac{E^2}{\cosh^2\left(\frac{E}{2k_BT}\right)}\,2\ln\left(\frac{E}{E_*}\right) \tag{D12}$$

which, with logarithmic accuracy, is

$$I_{\varepsilon_*} = I_{K_0}2\ln\left(\frac{k_BT}{\varepsilon_*}\right) \tag{D13}$$

with

$$I_{K_0} = \int_\Delta^\infty dE\,\frac{E^2}{\cosh^2\left(\frac{E}{2k_BT}\right)}. \tag{D14}$$

Since we can also write $I_n = I_{K_0}\ln(\tau_\phi/\tau_e)$, the sum $I_n + I_{\varepsilon_*}$ leads to the top line in Eq. (57).

2. Energy independent L_ϕ

In the previous section, we assumed the phase-coherence time to be independent of energy. Since the group velocity v_g [Eq. (19)] in a superconductor and hence the diffusion constant D_s [Eq. (26)] are energy dependent, such a choice for the phase-coherence time leads to an energy-dependent phase-coherence length. As an alternative scenario, we consider here a constant phase-coherence length, expressed in terms of the dephasing time and diffusion constant as $L_\phi = \sqrt{D_s\tau_\phi}$. This choice now leads to an energy-dependent phase-coherence time $\tau_\phi = L_\phi^2/l_e v_g$. We substitute this expression for τ_ϕ together with $\tau_e = l_e/v_F$ in Eqs. (D2) and (D3) to rewrite the integrals in terms of length scales rather than timescales. We obtain $I_1 + I_2 = I_n + I_{\varepsilon_*}$, with

$$I_n = \int_{\Delta_*}^\infty dE\,\frac{E^2}{\cosh^2\left(\frac{E}{2k_BT}\right)}\ln\left(\frac{L_\phi^2}{l_e^2}\right) \tag{D15}$$

and I_{ε_*} as defined in Eq. (D5). The expressions for the different regimes can then be easily obtained using the results for I_{ε_*} in the preceding part of the Appendix. Here, we only note that the condition for the presence of the I_{ε_*} contribution in the clean case for the lowest-temperature regime $T \ll T_*$ [see discussion after Eq. (D11)] can be written as $L_\phi \gg l_e$.

APPENDIX E: WEAK ANTILOCALIZATION: SPIN-ORBIT SCATTERING

In this Appendix, we study weak antilocalization [32] in the presence of spin-orbit scattering in disordered s-wave superconductors. To properly account for spin, we now define the Nambu vector as [cf. Eq. (2)]

$$\Psi_{\boldsymbol{k}} = \begin{pmatrix} c_{\boldsymbol{k}\uparrow} \\ c_{\boldsymbol{k}\downarrow} \\ T\begin{bmatrix} c_{\boldsymbol{k}\uparrow} \\ c_{\boldsymbol{k}\downarrow}\end{bmatrix}\end{pmatrix} = \begin{pmatrix} c_{\boldsymbol{k}\uparrow} \\ c_{\boldsymbol{k}\downarrow} \\ c^\dagger_{-\boldsymbol{k}\downarrow} \\ -c^\dagger_{-\boldsymbol{k}\uparrow}\end{pmatrix}. \tag{E1}$$

The full Nambu space is then the product between the space spanned by $\{|e\rangle, |h\rangle\}$ (the basis used in the main text) and the spin space spanned by $\{|\uparrow\rangle, |\downarrow\rangle\}$; the Pauli matrices τ_i and σ_i act, respectively, on these two subspaces. The spin-orbit scattering can be expressed as an additional term in the Hamiltonian in the form [13]

$$\hat{H}^{\mathrm{SO}}_{\alpha\alpha'}(\boldsymbol{k},\boldsymbol{k}') = iV^{\mathrm{SO}}\boldsymbol{\sigma}_{\alpha\alpha'}\cdot(\boldsymbol{u}_{\boldsymbol{k}}\times\boldsymbol{u}_{\boldsymbol{k}'})\otimes\tau_3, \tag{E2}$$

where V^{SO} is the strength of the spin-orbit scattering potential, $\boldsymbol{u}_{\boldsymbol{k}} = \boldsymbol{k}/k$, the components of the operator $\boldsymbol{\sigma}$ are the Pauli matrices $\{\sigma_x, \sigma_y, \sigma_z\}$ and $\boldsymbol{\sigma}_{\alpha\alpha'} = \langle\alpha'|\boldsymbol{\sigma}|\alpha\rangle$ with $\alpha, \alpha' \in \{\uparrow, \downarrow\}$. The full disorder potential now takes the form $\hat{V}_{\alpha\alpha'}(\boldsymbol{k},\boldsymbol{k}') = V_{\alpha\alpha'}(\boldsymbol{k},\boldsymbol{k}')\otimes\tau_3$ with

$$V_{\alpha\alpha'}(\boldsymbol{k},\boldsymbol{k}') = V_0\delta_{\alpha\alpha'} + iV^{\mathrm{SO}}\boldsymbol{\sigma}_{\alpha\alpha'}\cdot(\boldsymbol{u}_{\boldsymbol{k}}\times\boldsymbol{u}_{\boldsymbol{k}'}). \tag{E3}$$

This leads to a new disorder parameter $\gamma_{\mathrm{tot}} = \overline{\langle|V_{\alpha\alpha'}(\boldsymbol{k},\boldsymbol{k}')|^2\rangle_{\boldsymbol{k}'}} = \gamma_e + \gamma_{\mathrm{SO}}$, with $\gamma_{\mathrm{SO}} = 1/2\pi\rho_0\tau_{\mathrm{SO}}$, where τ_{SO} is the spin-orbit scattering time and γ_e has been defined at the end of Sec. II.

The disorder-averaged superconducting Green's function can be generalized to the full Nambu space as

$$\overline{\hat{G}^{R,A}_E} = \begin{pmatrix} \overline{G^{R,A}_E} & \overline{F^{R,A}_E} \\ \overline{\bar{F}^{R,A}_E} & \overline{\bar{G}^{R,A}_E}\end{pmatrix}\otimes\sigma_0, \tag{E4}$$

and the diffuson and the cooperon can be calculated following a procedure similar to the one used in Sec. III. We define

$$\hat{p}^{\mathrm{SO}}_{d,\omega}(\boldsymbol{r},\boldsymbol{r}') = \langle \hat{P}^{\mathrm{SO}}_0 \rangle_{\boldsymbol{r}} \hat{\Gamma}^{\mathrm{SO}}_{\omega}(\boldsymbol{r},\boldsymbol{r}') \langle \hat{P}^{\mathrm{SO}}_0 \rangle_{\boldsymbol{r}}, \tag{E5}$$

$$\hat{p}^{\mathrm{SO}}_{c,\omega}(\boldsymbol{r},\boldsymbol{r}') = \hat{F}^{\mathrm{SO}}(\boldsymbol{R}) \hat{\Gamma}^{\mathrm{SO}}_{c,\omega}(\boldsymbol{r},\boldsymbol{r}) \hat{F}^{\mathrm{SO}}(\boldsymbol{R}), \tag{E6}$$

which generalize Eqs. (17) and (35), respectively. We use lower case ps to emphasize that not all elements of these matrices correspond to diffusons and cooperons, as we will later see. The terms that do not take collisions into account, i.e., $\langle \hat{P}^{\mathrm{SO}}_0 \rangle_{\boldsymbol{r}}$ and $\hat{F}^{\mathrm{SO}}(\boldsymbol{R})$, are related to those in the absence of spin-orbit scattering by $\langle \hat{P}^{\mathrm{SO}}_0 \rangle_{\boldsymbol{r}} = \langle \hat{P}_0 \rangle_{\boldsymbol{r}} \otimes \sigma_0 \otimes \sigma_0$ and $\hat{F}^{\mathrm{SO}}(\boldsymbol{R}) = \hat{F}(\boldsymbol{R}) \otimes \sigma_0 \otimes \sigma_0$; here $\langle \hat{P}_0 \rangle_{\boldsymbol{r}}$ and $\hat{F}(\boldsymbol{R})$ are as those defined in Eqs. (18) and (36), respectively, but with γ_{tot} replacing γ_e. The equations followed by the structure factors are now given by

$$\hat{M}^{\mathrm{SO}}_{\omega}(\boldsymbol{r}) \hat{\Gamma}^{\mathrm{SO}}_{\omega}(\boldsymbol{r},\boldsymbol{r}') = \gamma_e \delta^{(d)}(\boldsymbol{r}' - \boldsymbol{r}), \tag{E7}$$

$$\hat{M}^{\mathrm{SO}}_{c,\omega}(\boldsymbol{r}) \hat{\Gamma}^{\mathrm{SO}}_{c,\omega}(\boldsymbol{r},\boldsymbol{r}') = \gamma_e \delta^{(d)}(\boldsymbol{r}' - \boldsymbol{r}). \tag{E8}$$

The diffusion matrices $\hat{M}^{\mathrm{SO}}_{\omega}(\boldsymbol{r})$ and $\hat{M}^{\mathrm{SO}}_{c,\omega}(\boldsymbol{r})$ are each defined by an equation similar to Eq. (24), but substituting $\langle \hat{P}_0 \rangle_{\boldsymbol{r}}$ by $\langle \hat{P}^{\mathrm{SO}}_0 \rangle_{\boldsymbol{r}}$ and $\hat{U}_v$ by the potential matrices $\hat{U}^{\mathrm{SO}}_v$ and $\hat{U}^{\mathrm{SO}}_{c,v}$. The potential matrices are no longer equivalent for the diffuson and the cooperon due to the different spin and momenta relations between the retarded and advanced Green's functions in the two cases. They are given by

$$\hat{U}^{\mathrm{SO}}_v = \hat{U}_v \otimes \hat{u}^{\mathrm{SO}}, \tag{E9}$$

$$\hat{U}^{\mathrm{SO}}_{c,v} = \hat{U}_v \otimes \hat{u}^{\mathrm{SO}}_c, \tag{E10}$$

with the (normal metal [13]) matrices $\hat{u}^{\mathrm{SO}}$ and $\hat{u}^{\mathrm{SO}}_c$ given in the basis $\{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$ by

$$\hat{u}^{\mathrm{SO}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \frac{\gamma_{\mathrm{SO}}}{3\gamma_e} \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix}, \tag{E11}$$

$$\hat{u}^{\mathrm{SO}}_c = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \frac{\gamma_{\mathrm{SO}}}{3\gamma_e} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{E12}$$

Here, each element $\langle \gamma\delta|\hat{u}^{\mathrm{SO}}|\alpha\beta\rangle$ and $\langle \gamma\delta|\hat{u}^{\mathrm{SO}}_c|\alpha\beta\rangle$ with $\alpha, \beta, \gamma, \delta \in \{\uparrow, \downarrow\}$ relates the spins of the Green's functions before and after interacting with an impurity, as depicted in Fig. 6. After obtaining $\hat{\Gamma}^{\mathrm{SO}}_{\omega}(\boldsymbol{r},\boldsymbol{r}')$ and $\hat{\Gamma}^{\mathrm{SO}}_{c,\omega}(\boldsymbol{r},\boldsymbol{r})$ from Eqs. (E7) and (E8), the matrices $\hat{p}^{\mathrm{SO}}_{d,\omega}(\boldsymbol{r},\boldsymbol{r}')$ and $\hat{p}^{\mathrm{SO}}_{c,\omega}(\boldsymbol{r},\boldsymbol{r}')$ can be calculated using Eqs. (E5) and (E6).

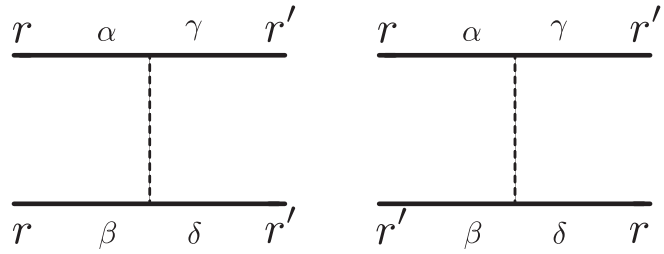

FIG. 6. Elementary vertex with spin-orbit impurity scattering for the diffuson (left) and Cooperon (right).

Not all terms in $\hat{p}^{\mathrm{SO}}_{d,\omega}(\boldsymbol{r},\boldsymbol{r}')$ and $\hat{p}^{\mathrm{SO}}_{c,\omega}(\boldsymbol{r},\boldsymbol{r}')$ represent diffusons or cooperons. The trajectories represented by the retarded and advanced Green's functions in the diffuson and the cooperon are not independent and their spin configurations are related. The diffuson, for instance, is composed by a time-reversed pair of trajectories; this implies that $\alpha = \beta$ and $\gamma = \delta$. We can obtain the diffuson by summing over the final spin configuration while taking this constraint into account. In this way we recover a 4×4 matrix in Nambu space, similar to $\hat{P}_{d,\omega}$ of Sec. III, where each element now accounts for the probability of propagation with and without spin flip. The diffuson for a particle with initial spin α is given by

$$\langle i', j'|\hat{P}^{\mathrm{SO}}_{d,\omega}(\boldsymbol{r},\boldsymbol{r}')|i, j\rangle = \sum_{\beta} \langle i'_{\beta}, j'_{\beta}|\hat{p}^{\mathrm{SO}}_{d,\omega}(\boldsymbol{r},\boldsymbol{r}')|i_{\alpha}, j_{\alpha}\rangle, \tag{E13}$$

where $|i\rangle, |j\rangle \in \{|\mathrm{e}\rangle, |\mathrm{h}\rangle\}$ and $|i_{\alpha}\rangle, |j_{\alpha}\rangle \in \{|\mathrm{e}\rangle \otimes |\alpha\rangle, |\mathrm{h}\rangle \otimes |\bar{\alpha}\rangle\}$ with $\alpha \in \{\uparrow, \downarrow\}$ and $\bar{\alpha} \neq \alpha$. The cooperon also accounts for the probability of propagation with and without spin flip; however, the conditions on the spins are different since the advanced Green's function (lower line in Fig. 6) now covers the trajectory in the opposite direction. It is now necessary that $\alpha = \delta$ and $\gamma = \beta$, and the cooperon contribution for a particle with initial spin α is given by

$$\langle i', j'|\hat{P}^{\mathrm{SO}}_{c,\omega}(\boldsymbol{r},\boldsymbol{r}')|i, j\rangle = \sum_{\beta} \langle i'_{\alpha}, j'_{\beta}|\hat{p}^{\mathrm{SO}}_{c,\omega}(\boldsymbol{r},\boldsymbol{r}')|i_{\beta}, j_{\alpha}\rangle. \tag{E14}$$

Direct calculation (cf. Ref. [13]) shows that spin-orbit scattering does not affect the diffuson, $\hat{P}^{\mathrm{SO}}_{d,\omega}(\boldsymbol{r},\boldsymbol{r}') = \hat{P}_{d,\omega}(\boldsymbol{r},\boldsymbol{r}')$, while the cooperon is now qualitatively different, with

$$\hat{P}^{\mathrm{SO}}_{c,\omega}(\boldsymbol{r},\boldsymbol{r}') = -\tfrac{1}{2}\hat{P}_{c,\omega}(\boldsymbol{r},\boldsymbol{r}'). \tag{E15}$$

As a consequence, in the presence of spin-orbit scattering the quantum correction to the thermal conductivity is

$$\frac{K^{\mathrm{SO}}_c}{K^{\mathrm{SO}}_0} = -\frac{1}{2}\frac{K_c}{K_0}, \tag{E16}$$

where K_c/K_0 is the correction calculated in Sec. IV. This correction, known as WAL effect, increases the total thermal conductivity and is due to destructive interference between self-crossing paths.

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