Improved local spectral gap thresholds for lattices of finite size

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Knabe's theorem lower bounds the spectral gap of a one-dimensional frustration-free local Hamiltonian in terms of the local spectral gaps of finite regions. It also provides a local spectral gap threshold for Hamiltonians that are gapless in the thermodynamic limit, showing that the local spectral gap must scale inverse linearly with the length of the region for such systems. Recent works have further improved upon this threshold, tightening it in the one-dimensional case and extending it to higher dimensions. Here, we show a local spectral gap threshold for frustration-free Hamiltonians on a finite-dimensional lattice that is optimal up to a constant factor that depends on the dimension of the lattice. Our proof is based on the detectability lemma framework and uses the notion of a coarse-grained Hamiltonian (introduced in [Anshu *et al.*, Phys. Rev. B **93**, 205142]) as a link connecting the (global) spectral gap and the local spectral gap.

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I. INTRODUCTION

A central problem in condensed-matter physics is to understand the properties of the ground states of spin systems. While finding a complete description of the ground states can be a daunting task, many important ground state properties (notably the area laws [1–3] and the decay of correlation [4–6]) are intertwined with the spectral gap of the associated Hamiltonian. Thus, understanding the spectral gap of a local Hamiltonian takes central stage in the mathematical physics of spin systems.

Recent results on the undecidability of the spectral gap [7,8] show that there is no general scheme for computing the spectral gap of an arbitrary local Hamiltonian (even under the assumption of translation invariance). But for a large and important family known as the frustration-free Hamiltonians (to be defined shortly), there are two powerful methods that provide criteria for system-size-independent lower bounds on the spectral gap. First is the martingale method of Nachtergaele [9] that guarantees a large spectral gap whenever a certain product of the local ground space projectors is close to the global ground space projector. The second method, introduced by Knabe [10], bounds the spectral gap whenever the "local" spectral gap in a finite region is large enough. These tools have found several applications in recent years, such as in the classification of gapped phases for qubits [11], the gap of generic translationally invariant Hamiltonians [12], properties of random quantum circuits [13], etc. Through the successful application of Knabe's method, [14] showed that the Affleck-Lieb-Kennedy-Tasaki (AKLT) model [15] on a hexagonal lattice is gapped, solving a long-standing open problem.

A standard way to represent a local Hamiltonian is via a Hermitian matrix H, which admits the decomposition $H = \sum_\alpha P_\alpha$ as a sum of local terms that act on a small number of spins. By a simple rescaling of energy, which does not change the physical properties of the system, we can assume that $P_\alpha \succeq 0$ for all α. The Hamiltonian H is said to be frustration free if its ground space G satisfies $P_\alpha G = 0$ for all α. Frustration-free Hamiltonians are very well studied in the literature, with some widely known examples being the AKLT model [15], parent Hamiltonians of tensor networks [16,17], instances of the Heisenberg model, and quantum satisfiability instances [18,19].

A useful simplifying assumption for the frustration-free Hamiltonians is that P_α can be considered a projector (that is, $P_\alpha^2 = P_\alpha$). This changes the spectral gap of H by just a constant factor, more precisely the smallest nonzero eigenvalue of P_α (minimized over all α). Knabe's method, which is the central focus of the present work, applies to translationally invariant frustration-free Hamiltonians on a periodic chain of spins. It states the following.

Knabe's theorem. Let $H = \sum_{i=1}^n P_{i,i+1}$ be a translationally invariant nearest-neighbor Hamiltonian on a periodic chain of n spins, with spectral gap γ. Let $h_{k,t} = \sum_{i=k+1}^{k+t-1} P_{i,i+1}$ be the Hamiltonian restricted to the spins $\{k+1, k+2, \dots, k+t\}$. Let $\gamma(t)$ be the spectral gap of $h_{k,t}$, which does not depend on k due to translation invariance. Then $\gamma + \frac{1}{t-2} \geqslant \frac{t-1}{t-2}\gamma(t)$.

We provide a sketch of the argument to help compare with our techniques. The spectral gap γ of H is the largest number that satisfies $H^2 \succeq \gamma H$. In order to lower bound γ, we may expand $H^2 = \sum_{i,i'} P_{i,i+1}P_{i',i'+1}$ and use the fact that $P_{i,i+1}$ are projectors to simplify $H^2 = H + \sum_{i\neq i'} P_{i,i+1}P_{i',i'+1}$. If all the terms $P_{i,i+1}P_{i',i'+1}$ were positive semidefinite, we would obtain $H^2 \succeq H$, leading to $\gamma \geqslant 1$. But this is not the case, as overlapping local terms $P_{i,i+1}$ and $P_{i+1,i+2}$ need not commute. To handle such terms, Knabe [10] invokes the Hamiltonians $h_{k,t}$ and makes use of the operator inequality $h_{k,t}^2 \succeq \gamma(t)h_{k,t}$. This helps in lower bounding sums of the form $\sum_{i,i'} P_{i,i+1}P_{i',i'+1}$ in terms of $\gamma(t)$.

An important consequence of Knabe's theorem is that if H is gapless in the thermodynamic limit (that is, $\gamma \to 0$ as $n \to \infty$), then the local gap $\gamma(t)$ must decay at least as fast as $\frac{1}{t-1}$. This is often termed the "local gap threshold."

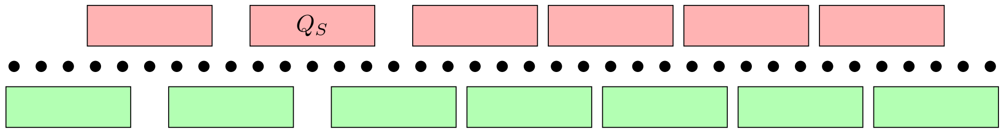

FIG. 1. For a subset S of spins, Q_S denotes a projector orthogonal to the local ground space on spins in S. A coarse-grained Hamiltonian is obtained by summing over a collection of such projectors, for example, the green and red projectors.

The additive term that captures the local gap threshold was improved in a recent work [20] which shows the inequality $\gamma + \frac{5}{t^2-4} \geqslant \frac{5}{6}\gamma(t)$. This inequality is tight up to constants, as witnessed by the Heisenberg ferromagnet (see Sec. 2 in [20] for details). The authors also consider the problem on a two-dimensional periodic square lattice $\mathcal{L}$, with a nearest-neighbor translationally invariant Hamiltonian $H = \sum_e P_e$. Here, the index e runs over the edges of the lattice. In the same spirit as above, they obtain the inequality $\gamma + \frac{6}{t^2} \geqslant \gamma(t)$, where γ is the spectral gap of H and $\gamma(t)$ is the spectral gap of the Hamiltonian h_S restricted over a square region S of side length t.

Subsequent works have made further progress in this direction. The results of [20] have been extended to a two-dimensional lattice with open boundary conditions in [21], with the additive term scaling as $t^{-3/2}$. The work [22] shows that for a gapless Hamiltonian on a lattice $\mathcal{L}$ of finite dimension, the local gap threshold scales as $O(\frac{\ln^2(t)}{t})$. Remarkably, it builds upon the Martingale method [9] and the detectability lemma [23], rather than the techniques in [10,20,21] sketched earlier. More recently, [24] improved this to an upper bound of $\frac{3}{t}$ (on a finite-dimensional lattice) for the hypercubic regions of side length t.

II. OUR RESULT

We prove a nearly optimal local gap threshold of $O(\frac{1}{t^2})$ on any finite-dimensional regular lattice. More precisely, let us consider a D-dimensional regular lattice $\mathcal{L}$ with unit cells as hypercubes and spins situated on the vertices. Suppose the local Hamiltonian H is defined as $H = \sum_e P_e$, where e runs over the unit cells of $\mathcal{L}$ and P_e is supported only on the 2^D vertices of the corresponding unit cell. This particular setup is chosen for convenience, and our results can be generalized to other lattices as long as the interactions P_e are local. As before, let γ be the spectral gap of H. For a tuple of integers $(t_1, \ldots, t_D)$, we let $\gamma(t_1, t_2, \ldots, t_D)$ denote the minimum spectral gap over all Hamiltonians h_S restricted to hyperrectangles S of size $t_1 \times t_2 \times \ldots \times t_D$ (where t_i is the side length along the ith axis). We show that

$$\gamma(t_1, t_2, \ldots, t_D) = O\left(\gamma + \frac{1}{\min_q t_q^2}\right), \qquad (1)$$

where the notation $O(\cdot)$ hides the factors that depend on D (see the formal statement in Theorem 2 in the Appendix). Note that we do not require H to be translationally invariant. The statement applies to both the open and periodic boundary conditions on the Hamiltonian H. For hypercubic regions with $t_1 = t_2 = \cdots = t_D = t$, the additive term scales as $O(\frac{1}{t^2})$, improving upon prior works for t larger than a constant that depends on D.

As discussed after the proof of Theorem 2 in the Appendix, the additive term of $\frac{1}{\min_q t_q^2}$ cannot be improved even in the translationally invariant case (as witnessed by many parallel copies of a chain of the Heisenberg ferromagnet), except potentially for the constant that depends on D. Further, Eq. (1) would be false if $\gamma(t_1, t_2, \ldots, t_D)$ were defined as an average (instead of a minimum) over hyperrectangles of size $t_1 \times t_2 \times \ldots \times t_D$.

III. PROOF OUTLINE

It suffices to consider the one-dimensional case to discuss the proof technique. We will explain later that the higher-dimensional case is a simple recursive application of this one-dimensional argument. Consider the one-dimensional nearest-neighbor Hamiltonian $H = \sum_i P_{i,i+1}$ on an open chain of spins, with spectral gap γ and ground space G. Let $\gamma(t)$ be the minimum spectral gap over all Hamiltonians $\sum_{i=k+1}^{k+t-1} P_{i,i+1}$, where $k \in \{0, 1, \ldots, n-t\}$. Central to our argument is the coarse-grained Hamiltonian $\bar{H}(t) = \sum_S Q_S$ from [25], which has the same ground space G. Here, S are some sets of t consecutive spins (see Fig. 1), and Q_S project onto the nonzero eigenstates of $\sum_{i,i+1\in S} P_{i,i+1}$. Let $\gamma(\bar{H}(t))$ be the spectral gap of $\bar{H}(t)$. The coarse-grained Hamiltonian provides a link between γ and $\gamma(t)$, as made precise in the following observation [26]:

$$\gamma(\bar{H}(t)) \leqslant \frac{2\gamma}{\gamma(t)}. \qquad (2)$$

Its formal proof (in slight generality incorporating the higher-dimensional lattices) will be given in the Appendix. It was shown in [25] that for $t = \Omega(\frac{1}{\sqrt{\gamma}})$, $\gamma(\bar{H}(t)) = \Omega(1)$. This immediately says that $\gamma(t) = O(\gamma)$ for this choice of t. An extension of this result to all t relies on an estimate of the "shrinking ability" of low-degree Chebyshev polynomials, which is shown in Claim 1 in the Appendix (see also Theorem 42 in [27] for a similar estimate). It shows that $\gamma(\bar{H}(t)) = \Omega(\frac{t^2\gamma}{1+t^2\gamma})$, using the converse of the detectability lemma [25]. Plugging in Eq. (2), we find that $\gamma(t) = O(\gamma + \frac{1}{t^2})$. Note that we did not require translation invariance and the argument can easily be modified for the periodic chain by considering a similar coarse-grained Hamiltonian.

To explain the argument for higher-dimensional lattices, consider a Hamiltonian $H = \sum_{i=1}^{n_1-1}\sum_{j=1}^{n_2-1} P_{i,j}$ on a two-dimensional square lattice $\{1, 2, \ldots, n_1\} \times \{1, 2, \ldots, n_2\}$, where $P_{i,j}$ is supported on the spins $\{(i, j), (i+1, j), (i, j+1), (i+1, j+1)\}$. Following [28], we can view this Hamiltonian as a one-dimensional Hamiltonian $H = \sum_{i=1}^{n_1-1} H_i$, where $H_i = \sum_{j=1}^{n_2-1} P_{i,j}$ is the "column" Hamiltonian acting on two columns of spins, that is, $\{\forall j : (i, j)\}$ and $\{\forall j : (i+1, j)\}$

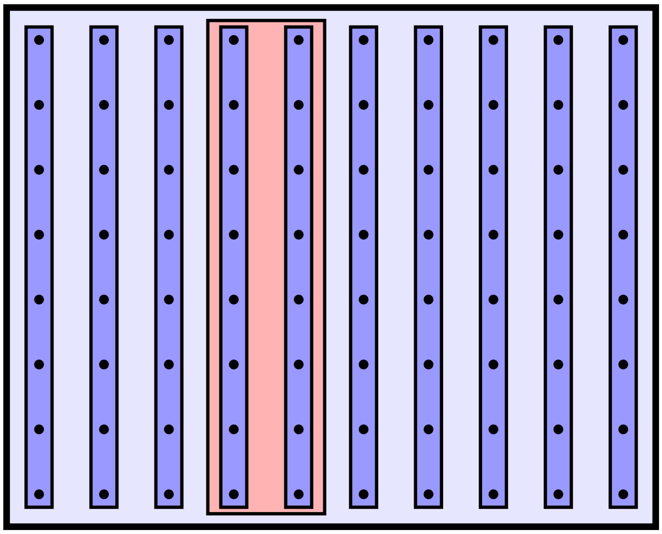

FIG. 2. We can view the Hamiltonian on a two-dimensional lattice as a Hamiltonian on a one-dimensional chain of a column of spins (dark blue rectangles). The interaction H_4 between the fourth and fifth columns is shown as the red rectangle, which decomposes as $H_4 = \sum_{j=1}^{n_2-1} P_{4,j}$.

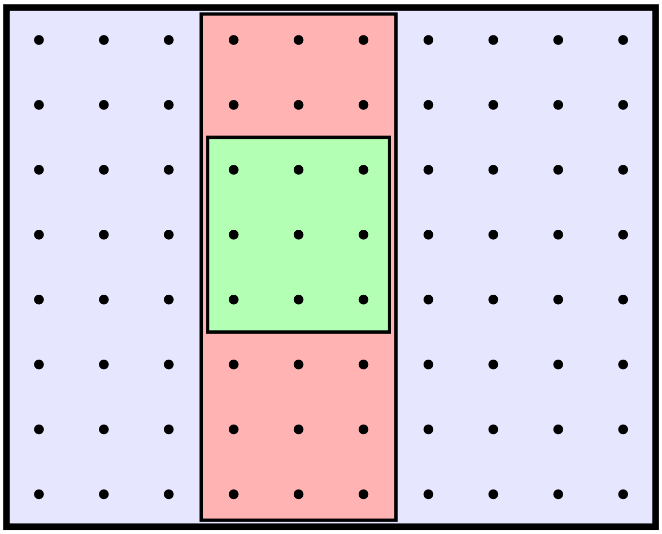

FIG. 3. Our strategy is to lower bound the spectral gap of H with the spectral gap of the Hamiltonian h_S supported on the red region $S \times \{1, 2, \ldots, n_2\}$. The spectral gap of h_S can, in turn, be lower bounded by the spectral gap of the Hamiltonian $h_{S,S'}$ supported on the green region $S \times S'$.

(see Fig. 2). Such a one-dimensional view is not helpful for the technique used in [10,20,21,24], as the column Hamiltonians H_i are not projectors (recall the sketch of the proof given in the Introduction, which crucially uses the fact that $P_{i,i+1}$ are projectors). But our method can be applied to a sum of column Hamiltonians in the same manner as the one-dimensional case. We relate the spectral gap of H to the spectral gap of $h_S \stackrel{\text{def}}{=} \sum_{i\in S} H_i$, where S is some continuous subset of $\{1, 2, \ldots, n_1\}$ of size t_1, up to the additive factor of $O(\frac{1}{t_1^2})$. Now, h_S is a local Hamiltonian on $t \times n_2$ spins (red region in Fig. 3) and can also be viewed as a sum $\sum_{j=1}^{n_2-1} H'_j$ of "row" Hamiltonians $H'_j \stackrel{\text{def}}{=} \sum_{i\in S} P_{i,j}$ acting on rows of spins $\{\forall\, i \in S : (i, j)\}$ and $\{\forall\, i \in S : (i, j+1)\}$. Thus, we can apply the same argument to h_S, relating its spectral gap to the spectral gap of some local Hamiltonian $h_{S,S'} \stackrel{\text{def}}{=} \sum_{i\in S, j\in S'} P_{i,j}$ (green region in Fig. 3). Here, S' is a set of size t_2, implying that $h_{S,S'}$ is supported on a square region of size $t_1 \times t_2$. The overall additive factor is $O(\frac{1}{t_1^2} + \frac{1}{t_1^2}) = O(\frac{1}{\min_{q\in\{1,2\}} t_q^2})$. The same recursive argument applies to higher dimensions.

IV. COMPARISON TO PRIOR WORK

As already mentioned, our tools significantly differ from those employed in [10,20,21,24]. Similar to us, Ref. [22] employs the detectability lemma and its converse to obtain the local gap threshold. But it does not use the coarse-grained Hamiltonians and builds upon the Martingale method. We remark that it may be possible to improve their local gap threshold from $O(\frac{\ln^2(t)}{t})$ to $O(\frac{\text{poly}[\ln(t)]}{t^2})$. This is because the statement in Theorem 11 in [22] can be improved using the ideas presented in [29]. Such an improvement would still be slightly weaker than our bound in Eq. (1), which does not contain the poly$[\ln(t)]$ factor.

V. CONCLUSION

In this work, we have derived a relation between the (global) spectral gap and the local spectral gap of frustration-free local Hamiltonians on a lattice, along the lines of Knabe [10]. The relation is optimal up to factors that depend on the dimension of the lattice. It may be potentially improved if the Hamiltonian has further symmetry. For concreteness, consider a local Hamiltonian $H = \sum_e P_e$, where e runs over the edges of the lattice and P_e is the same interaction across every edge (in other words, the Hamiltonian is isotropic and translationally invariant). In this case, we conjecture that the additive term in Eq. (1) can be improved to $\frac{1}{\sum_q t_q^2}$, which is the inverse-squared diameter of the hyperrectangles.

Our proof is based on the technique of the coarse-grained Hamiltonian introduced in [25] and shows how the detectability lemma [23] can be used to capture yet another feature of the frustration-free systems. Given the recent success of Knabe's method for bounding the spectral gap of the AKLT model on a hexagonal lattice [14], it would be interesting to apply our method to bound the spectral gaps of a larger class of frustration-free Hamiltonians on a two-dimensional lattice (see [30,31] for more applications of prior techniques). It would also be interesting to find implications of our results for the existence or absence of chiral edge modes in three or more dimensions (cf. [21]).

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APPENDIX

1. Low-degree behavior of Chebyshev polynomials

A Chebyshev polynomial of degree m is defined as

$$T_m(x) = \begin{cases} \cos[m \arccos(x)] \text{ if } |x| < 1, \\ \cosh[m \cosh^{-1}(x)] \text{ if } |x| \geqslant 1. \end{cases}$$

It has found applications in area laws [2,3], the subvolume law [28], and the decay of correlation [29]. We have the following claim (see also Theorem 42 in [27]).

Claim 1. Fix $\nu \in (0, \frac{1}{4})$ and a real number $m > 0$. Consider the polynomial

$$\mathrm{Step}_{m,\nu}(x) = \frac{T_{\lceil m \rceil}\left(-1 + \frac{2x}{1-\nu}\right)}{T_{\lceil m \rceil}\left(\frac{1+\nu}{1-\nu}\right)}.$$

It holds that $\mathrm{Step}_{m,\nu}(1) = 1$ and

$$|\mathrm{Step}_{m,\nu}(x)| \leqslant \frac{1}{1 + \frac{m^2\nu}{2(1-\nu)}}$$

for $x \in (0, 1-\nu)$.

Proof. The relation $\mathrm{Step}_{m,\nu}(1) = 1$ trivially holds. Since $T_{\lceil m \rceil}(-1 + \frac{2x}{1-\nu}) \in \{-1, 1\}$ for $x \in (0, 1-\nu)$, we have $|\mathrm{Step}_{m,\nu}(x)| \leqslant \frac{1}{T_{\lceil m \rceil}(\frac{1+\nu}{1-\nu})}$ for $x \in (0, 1-\nu)$. We wish to upper bound $\frac{1}{T_{\lceil m \rceil}(\frac{1+\nu}{1-\nu})}$. Let w be such that $\cosh(w) = \frac{1+\nu}{1-\nu}$. Then

$$\begin{aligned} T_{\lceil m \rceil}\left(\frac{1+\nu}{1-\nu}\right) &= \cosh(\lceil m \rceil w) \geqslant 1 + \frac{\lceil m \rceil^2 w^2}{2} \\ &\geqslant 1 + \frac{m^2 w^2}{2}. \end{aligned} \tag{A1}$$

Now,

$$\begin{aligned} \frac{1+\nu}{1-\nu} &= \cosh(w) = \frac{e^w + e^{-w}}{2} \\ \Rightarrow \frac{2\nu}{1-\nu} &= \frac{e^w + e^{-w} - 2}{2} = \frac{\left(e^{\frac{w}{2}} - e^{-\frac{w}{2}}\right)^2}{2}. \end{aligned}$$

This implies

$$e^{\frac{w}{2}} - e^{-\frac{w}{2}} = 2\sqrt{\frac{\nu}{1-\nu}}.$$

Solving the quadratic equation for $e^{\frac{w}{2}}$, we find

$$e^{\frac{w}{2}} = \sqrt{1 + \frac{\nu}{1-\nu}} + \sqrt{\frac{\nu}{1-\nu}} \geqslant 1 + \sqrt{\frac{\nu}{1-\nu}}.$$

Thus,

$$w \geqslant 2\ln\left(1 + \sqrt{\frac{\nu}{1-\nu}}\right) \geqslant \sqrt{\frac{\nu}{1-\nu}},$$

for $\nu \leqslant \frac{1}{4}$. Equation (A1) now implies

$$T_{\lceil m \rceil}\left(\frac{1+\nu}{1-\nu}\right) \geqslant 1 + \frac{m^2 w^2}{2} \geqslant 1 + \frac{m^2\nu}{2(1-\nu)},$$

which leads to

$$|\mathrm{Step}_{m,\nu}(x)| \leqslant \frac{1}{T_{\lceil m \rceil}\left(\frac{1+\nu}{1-\nu}\right)} \leqslant \frac{1}{1 + \frac{m^2\nu}{2(1-\nu)}}$$

for $x \in (0, 1-\nu)$.

2. Formal setup and the main result

Here, we introduce notations to analyze both the open chain and closed chain of qudits [32]. Let $[a:b]$ denote the set $\{a, a+1, \ldots, b\}$. Consider a one-dimensional closed chain of n qudits, indexed by integers $\{1, 2, \ldots, n\}$, of potentially varying dimensions. The indices of the qudits are taken in a manner in which the $(n+k)$th index is the same as the kth index. Introduce the nearest-neighbor local Hamiltonian

$$H = \sum_{i=1}^{n} H_i, \tag{A2}$$

where H_i is a Hermitian operator which acts nontrivially only on qudits $i, i+1$. Further assume that H_i admits the decomposition

$$H_i = \sum_j P_{ij}, \tag{A3}$$

where P_{ij} are projectors that act nontrivially only on qudits $i, i+1$. We have the following assumptions on the set of projectors $\{P_{ij}\}_{i,j}$:

(i) Each P_{ij} does not commute with, at most, g other terms from the set $\{P_{ij}\}_{i,j}$.

(ii) The projectors can be divided into L layers $T_1, T_2, \ldots, T_L$, where the terms within each layer mutually commute.

We further assume H is frustration free, which means that the ground energy is zero. Let G be the ground space of H. Note that frustration freeness implies that $P_{ij}G = 0$. We shall write $G_\perp$ for the subspace of states orthogonal to G.

If we are interested in an open chain of qudits, then we simply assume that $H_n = 0$ (note that we are not considering the translationally invariant case). This will lead to some minor changes that we will highlight as the arguments proceed.

For a contiguous subset S of the chain, let $h_S \stackrel{\text{def}}{=} \sum_{i:i,i+1 \in S} H_i$ be the local Hamiltonian made out of terms in Eq. (A2) that are entirely supported in S. Define $\gamma(S)$ as the smallest nonzero eigenvalue of h_S, and let $\gamma \stackrel{\text{def}}{=} \gamma([1:n])$ be the spectral gap of H. Let

$$\gamma(t) \stackrel{\text{def}}{=} \min_a \gamma([a : a+t-1])$$

denote the minimum spectral gap over all continuous segments of length t. Observe that the set of continuous segments is different for the open chain and the closed chain. Thus, the minimization over a in the above expression requires the additional condition that $a \in [1 : n-t+1]$ for the open chain. Our main theorem is as follows, which upper bounds $\gamma(t)$ in terms of the spectral gap γ. The statement remains the same for both the open chain and the closed chain.

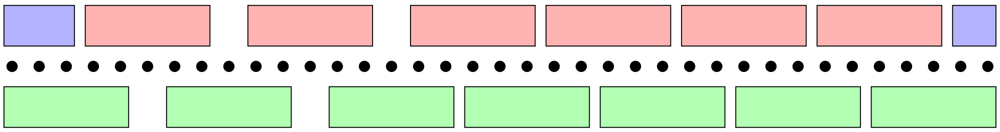

FIG. 4. Dividing the chain into contiguous segments of length t: Here, we assume $n = 37$ and $t = 5$. The remainder when n is divided by t is 2. We set $r_1 = r_2 = 1$ and $r_k = 0$ for $k > 2$. The green rectangles represent the sets S_j. The red and blue rectangles represent the sets T_j. The blue rectangles are to be viewed as a single contiguous region on the closed chain when $H_n \neq 0$ and are assumed to not exist on the open chain when $H_n = 0$. The first three rectangles, both green and red, are separated by one qudit.

Theorem 1. Suppose $\gamma \leqslant \frac{g^2}{4}$. For every integer $8L^2 < t < n/5$, it holds that

$$\gamma(t) \leqslant \frac{10^3 L^2 g^2}{t^2} + 6\gamma.$$

Note that we have not tried to optimize the parameters appearing in the above expression. For specific applications, it may be possible to obtain stronger bounds. The rest of the section is devoted to the proof of Theorem 1.

a. Detectability lemma

The detectability lemma [23] is an important tool for the study of frustration-free systems. Its central object is the detectability lemma operator, defined as a product of projectors $\mathbb{1} - P_{ij}$ taken layer by layer. More precisely, define

$$DL(H) \stackrel{\text{def}}{=} \prod_{\alpha\in[1:L]} \prod_{i,j\in T_\alpha} (\mathbb{1} - P_{ij}).$$

The following lemma holds, the statement of which is taken from Corollary 3 in [25].

Lemma 1. Detectability lemma [23]. For any quantum state $\psi \in G_\perp$, we have

$$\|DL(H)|\psi\rangle\|^2 \leqslant \frac{1}{1+\gamma/g^2}.$$

A converse result stated in Lemma 4 in [25] is a corollary of [33].

Lemma 2. Converse of the detectability lemma [25,33]. For any quantum state ψ,

$$\|DL(H)|\psi\rangle\|^2 \geqslant 1 - 4\langle\psi|H|\psi\rangle.$$

Here, we provide a short proof (with a minor improvement) in the special case of $L = 2$. The proof is deferred to the end of this Appendix.

Lemma 3. Suppose $L = 2$. It holds that

$$\|DL(H)|\psi\rangle\|^2 \geqslant 1 - 3\langle\psi|H|\psi\rangle.$$

b. Coarse-grained Hamiltonian

Another tool that we will use is the notion of a coarse-grained Hamiltonian [2,25]. Let Q_S be the projector orthogonal to the ground space of h_S. By convention, we set $Q_\phi = 0$ for the empty set ϕ. Fix a coarse-graining parameter $8L^2 < t < n/5$, and let $\mathsf{quo} = \lfloor \frac{n}{t} \rfloor$ and $r = n - t \times \mathsf{quo}$ be the quotient and remainder when n is divided by t, respectively . Identify sets $S_1, S_2, \ldots, S_{\mathsf{quo}}$ using the following rules.

(i) $S_k \stackrel{\text{def}}{=} [s_k : s'_k]$, with $1 \leqslant s_1 < s'_1 < s_2 < s'_2 < \cdots < s_{\mathsf{quo}} < s'_{\mathsf{quo}} \leqslant n$. Further, $|S_k| = t$.

(ii) Let $r_k = s_{k+1} - s'_k - 1$ be the number of qudits sandwiched between S_k, S_{k+1} for $k \in [1 : \mathsf{quo} - 1]$. Let $r_{\mathsf{quo}} = n - s'_{\mathsf{quo}} + s_1 - 1$ be the number of qudits sandwiched between S_{quo} and S_1. Observe that $\sum_{k=1}^{\mathsf{quo}} r_k = r$. We require that r_k are not too large. That is, $r_k \leqslant \lceil r/\mathsf{quo} \rceil$ for all k. Since $\frac{n}{t} > 5$, this implies that

$$r_k \leqslant \lceil r/5 \rceil \leqslant t/4. \tag{A4}$$

(iii) For the open chain, with $H_n = 0$, we require $s_1 = 1$ and $s'_{\mathsf{quo}} = n$.

Next, choose another collection of quo continuous sets of size t each, which are placed "half way" between adjacent S's. More precisely, the sets $T_1, T_2, \ldots, T_{\mathsf{quo}}$ have the following properties.

(i) For $k < \mathsf{quo}$, $T_k = [s'_k - \lfloor \frac{t-r_k}{2} \rfloor + 1 : s_{k+1} + \lceil \frac{t-r_k}{2} \rceil - 1]$.

(ii) For the open chain (with $H_n = 0$), let $T_{\mathsf{quo}} = \phi$. For the closed chain, let

$$T_{\mathsf{quo}} = \left[s'_{\mathsf{quo}} - \left\lfloor \frac{t - r_{\mathsf{quo}}}{2} \right\rfloor + 1 : s_1 + \left\lceil \frac{t - r_{\mathsf{quo}}}{2} \right\rceil - 1 \right].$$

Two examples of these sets are depicted in Figs. 4 and 5. Observe that the set T_k has an overlap of at least $\lfloor \frac{t-r_k}{2} \rfloor$ with sets S_k and S_{k+1}. Using $t \geqslant 8L^2 \geqslant 8$ and Eq. (A4), this can be lower bounded by

$$\begin{aligned} \left\lfloor \frac{t - r_k}{2} \right\rfloor &\geqslant \left\lfloor \frac{t - t/4}{2} \right\rfloor \\ &= \left\lfloor \frac{3t}{8} \right\rfloor \geqslant \frac{3t}{8} - 1 = \frac{t}{4} + \frac{t}{8} - 1 \geqslant \left\lfloor \frac{t}{4} \right\rfloor. \end{aligned} \tag{A5}$$

Following [25], we define the coarse-grained Hamiltonian

$$\bar{H}(t) \stackrel{\text{def}}{=} \sum_k (Q_{S_k} + Q_{T_k})$$

and the corresponding detectability operator

$$DL(t) \stackrel{\text{def}}{=} \left(\prod_k (\mathbb{1} - Q_{S_k}) \right) \left(\prod_k (\mathbb{1} - Q_{T_k}) \right).$$

Observe that the ground space of $\bar{H}(t)$ coincides with G. Let the spectral gap of $\bar{H}(t)$ be $\gamma(\bar{H}(t))$. The following lemma was shown in [25]. We provide its proof towards the end of this Appendix for completeness.

Lemma 4. It holds that

$$\begin{aligned} 1 - 3\gamma(\bar{H}(t)) &\leqslant \max_{\psi\in G_\perp} \|DL(t)|\psi\rangle\|^2 \\ &\leqslant \max_{x\in(0,1-\frac{\gamma}{g^2+\gamma})} \mathrm{Step}_{\frac{t}{8L}, \frac{\gamma}{g^2+\gamma}}(x). \end{aligned}$$

Now, we proceed to the proof of our main theorem.

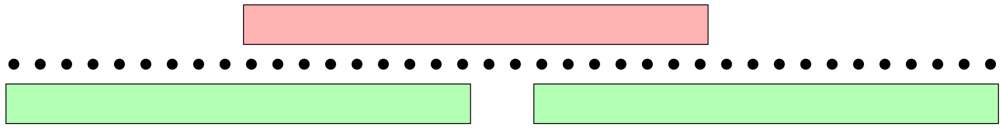

FIG. 5. Assume $n = 38$, $t = 18$, and $H_n = 0$ (open chain). In this case, $r = 2$. There is exactly one set T_1 and two sets S_1, S_2.

c. *Proof of Theorem 1*

We start with the inequality for all $1 \leqslant k \leqslant \mathsf{quo}$,

$$Q_{S_k} + Q_{T_k} \preceq \frac{1}{\gamma(S_k)} h_{S_k} + \frac{1}{\gamma(T_k)} h_{T_k} \preceq \frac{1}{\gamma(t)}\big(h_{S_k} + h_{T_k}\big).$$

Note that the above inequality also holds in the case of the open chain, as $T_{\mathsf{quo}} = \phi$ implies $Q_{T_{\mathsf{quo}}} = 0$ and $h_{T_{\mathsf{quo}}} = 0$. Summing over k and using the definition of $\bar{H}(t)$, this implies that

$$\begin{aligned}\bar{H}(t) &\preceq \frac{1}{\gamma(t)} \sum_k \left(h_{S_k} + h_{T_k}\right)\\ &= \frac{1}{\gamma(t)} \sum_k \left(\sum_{i:\mathrm{Supp}(H_i)\in S_k} H_i + \sum_{i:\mathrm{Supp}(H_i)\in T_k} H_i \right)\\ &\preceq \frac{2}{\gamma(t)} H.\end{aligned}$$

Here, the last inequality holds since each H_i is supported within, at most, one S_k and, at most, one T_k. As a result, we have the following inequality:

$$\begin{aligned}\gamma(\bar{H}(t)) &= \min_{\psi\in G_\perp} \langle\psi|\bar{H}(t)|\psi\rangle\\ &\leqslant \frac{2}{\gamma(t)} \min_{\psi\in G_\perp} \langle\psi|H|\psi\rangle = \frac{2\gamma}{\gamma(t)}.\end{aligned} \tag{A6}$$

Lemma 3 ensures that

$$\gamma(\bar{H}(t)) \geqslant \tfrac{1}{3}\big(1 - \max_{x\in(0,1-\frac{\gamma}{g^2+\gamma})} \mathrm{Step}_{\frac{t}{8L},\frac{\gamma}{g^2+\gamma}}(x)\big).$$

Now we use Claim 1, setting $m = \frac{t}{8L}$ and $\nu = \frac{\gamma}{g^2+\gamma} \leqslant \frac{\gamma}{g^2} \leqslant \frac{1}{4}$. This ensures that $\frac{\nu}{1-\nu} = \frac{\gamma}{g^2}$, and we obtain

$$\begin{aligned}\gamma(\bar{H}(t)) &\geqslant \frac{1}{3}\left(\frac{\frac{m^2\nu}{2(1-\nu)}}{1 + \frac{m^2\nu}{2(1-\nu)}}\right) = \frac{1}{3}\frac{t^2\gamma}{128L^2g^2 + t^2\gamma}\\ &\geqslant \frac{t^2\gamma}{400L^2g^2 + 3t^2\gamma}.\end{aligned}$$

Substituting it in Eq. (A6), we find

$$\frac{2\gamma}{\gamma(t)} \geqslant \frac{t^2\gamma}{400L^2g^2 + 3t^2\gamma} \Rightarrow \gamma(t) \leqslant \frac{10^3L^2g^2}{t^2} + 6\gamma.$$

This concludes the proof.

3. Local versus global spectral gap on *D*-dimensional lattices

Consider a D-dimensional regular lattice $\mathcal{L} = [1:n_1] \times [1:n_2] \times \cdots \times [1:n_d]$, and let

$$H_{\mathcal{L}} = \sum_{\mathbf{i}} P_{\mathbf{i}}$$

be a frustration-free local Hamiltonian, where the index $\mathbf{i}$ enumerates the unit cells of the lattice and $P_{\mathbf{i}}$ acts nontrivially only on the vertices of the $\mathbf{i}$th unit cell. Let γ be the spectral gap of $H_{\mathcal{L}}$. Since Theorem 1 also applies to periodic chains, the results below can similarly be extended to Hamiltonians with periodic boundary conditions on the lattice. We study this model as an illustrative example and highlight that the results below easily generalize for any local Hamiltonian of constant locality on the lattice.

In the above setting, we have $L, g \leqslant (3D)^D$. For a region $R \subseteq \mathcal{L}$, let $\gamma(R)$ be the spectral gap of the Hamiltonian

$$H_R = \sum_{\mathbf{i}:P_{\mathbf{i}}\in\mathrm{supp}(R)} P_{\mathbf{i}}.$$

For integers $t_1, \dots, t_D$, we define $\gamma(t_1, \dots, t_D)$ as the minimum of $\gamma(R)$ over all hyperrectangular regions R of dimension $t_1 \times t_2 \times \dots \times t_D$. Formally,

$$\begin{aligned}&\gamma(t_1, \dots, t_D)\\ &\quad= \min_{a_1,a_2,\dots,a_D:0\leqslant a_i\leqslant n_i-t_i} \gamma([a_1+1 : a_1+t_1]\\ &\qquad\times [a_2+1 : a_2+t_2] \times \cdots \times [a_D+1 : a_D+t_D]).\end{aligned}$$

We show the following theorem.

Theorem 2. Suppose $2^6 4^D L < t_s < n_s/5$ for all $s \in [1:D]$ and $\gamma \leqslant \frac{g^2}{16^D}$. It holds that

$$\gamma(t_1, t_2, \dots, t_D) \leqslant 6^D\gamma + 200L^2g^2 6^D \frac{1}{\min_q t_q^2}.$$

Proof. The proof will follow by inductive application of Theorem 1.

Base case. We view $H_{\mathcal{L}}$ as a Hamiltonian on a one-dimensional chain of large qudits. This is achieved by combining the qudits $\{i\} \times [1:n_2] \times \cdots \times [1:n_D]$ into a single ith qudit of the chain. Defining

$$H_i \stackrel{\mathrm{def}}{=} \sum_{\mathbf{i}:P_{\mathbf{i}}\in\mathrm{supp}(\{i,i+1\}\times[1:n_2]\times\cdots\times[1:n_D])} P_{\mathbf{i}}$$

[see Eq. (A3)], we obtain the identity $H_{\mathcal{L}} = \sum_{i=1}^{n_1-1} H_i$, which is the decomposition given in Eq. (A2). This allows us to conclude, from Theorem 1, that

$$\gamma(t_1, n_2, \dots, n_D) \leqslant \frac{10^3L^2g^2}{t_1^2} + 6\gamma. \tag{A7}$$

Since

$$\gamma \leqslant \frac{1}{16^{D-1}}\frac{g^2}{16}$$

and

$$\frac{10^{\frac{3}{2}}Lg}{t_1} = \frac{1}{4^D}\frac{g}{2}\frac{2\times 10^{\frac{3}{2}}L4^D}{t_1} \leqslant \frac{1}{4^D}\frac{g}{2},$$

Eq. (A7) additionally implies that

$$\gamma(t_1, n_2, \dots, n_D) \leqslant \frac{1}{16^D}\frac{g^2}{4} + \frac{6}{16^{D-1}}\frac{g^2}{16} \leqslant \frac{g^2}{16^{D-1}} < \frac{g^2}{4}, \tag{A8}$$

maintaining the condition on the spectral gap in Theorem 1.

Recursion. Fix an $s \in [2:D]$. Assume

$$\gamma(t_1, t_2, \dots, t_{s-1}, n_s, \dots, n_D) \leqslant \frac{g^2}{16^{D-s+1}} < \frac{g^2}{4}, \tag{A9}$$

which is true for $s = 2$ via Eq. (A8) and for $s > 2$ via Eq. (A9) in the previous recursion. Let

$$R = [a_1+1 : a_1+t_1] \times \dots \times [a_{s-1}+1 : a_{s-1}+t_{s-1}] \times [1 : n_s] \times \dots \times [1 : n_D]$$

be a hyperrectangle that achieves the minimum in the definition of

$$\gamma(t_1, t_2, \dots, t_{s-1}, n_s, \dots, n_D).$$

Defining

$$H'_i \overset{\text{def}}{=} \sum_{\substack{\mathbf{i}:P_{\mathbf{i}}\in \\ \mathrm{supp}([a_1+1:a_1+t_1]\times\cdots\times[a_{s-1}+1:a_{s-1}+t_{s-1}]\{i,i+1\}[1:n_{s+1}]\times\cdots\times[1:n_D])}} P_{\mathbf{i}},$$

we have the decomposition

$$H_R = \sum_{i=1}^{n_s-1} H'_i,$$

which is the same as given in Eq. (A2). Since the values of g and L remain unchanged for H_R, we can apply Theorem 1 [see Eq. (A9)] and obtain the relation

$$\gamma(t_1, t_2, \dots, t_s, n_{s+1}, \dots, n_D) \leqslant \frac{10^3 L^2 g^2}{t_s^2} + 6\gamma(R) = \frac{10^3 L^2 g^2}{t_s^2} + 6\gamma(t_1, t_2, \dots, t_{s-1}, n_s, \dots, n_D). \tag{A10}$$

Using Eqs. (A9) and (A10), we further have

$$\gamma(t_1, t_2, \dots, t_s, n_{s+1}, \dots, n_D) \leqslant \frac{1}{16^D}\frac{g^2}{4} + \frac{8g^2}{16^{D-s+1}} \leqslant \frac{g^2}{16^{D-s}}.$$

This ensures that Eq. (A9) continues to be satisfied as we update $s \to s+1$.

Having obtained Eq. (A10) for all $s \in [2:D]$ and Eq. (A7), we combine them to arrive at the upper bound

$$\gamma(t_1, t_2, \dots, t_D) \leqslant 6^D\gamma + 10^3 L^2 g^2 \left(\sum_{q=1}^{D} \frac{6^{D-q}}{t_q^2}\right) \leqslant 6^D\gamma + 10^3 L^2 g^2 \frac{6^D}{5}\frac{1}{\min_q t_q^2}.$$

This concludes the proof. ■

The dependence on $\min_q t_q^2$ cannot be improved, although the dependence on D might not be optimal. To show this, we provide the following example adapted from [20]. We consider the Heisenberg ferromagnet, which is a one-dimensional chain of qubits with a frustration-free local Hamiltonian defined by the nearest-neighbor interaction $\frac{1}{2}(|01\rangle - |10\rangle)(\langle 01| - \langle 10|)$. The spectral gap of an open chain of length n_1 is $\frac{\pi^2}{2n_1^2}$. We take $n_2 \times n_3 \times \dots \times n_D$ independent copies of this system and arrange them on a D-dimensional lattice, with the chains running in the "first" dimension. That is, for each $i_2, \dots, i_D \in [1:n_2] \times \dots \times [1:n_D]$, the set of qubits $\{(i, i_2, i_3, \dots, i_D)\}_{i=1}^{n_1}$ interacts via the nearest-neighbor Heisenberg ferromagnetic interaction. Consider all hyperrectangles of dimension $t \times n_2 \times \dots \times n_D$. Any such hyperrectangle contains $n_2 \times n_3 \times \dots \times n_D$ independent copies of the Heisenberg ferromagnetic chain of length t, and hence, the local spectral gap in this hyperrectangle is the minimum local spectral gap of each copy, which is $\frac{\pi^2}{2t^2}$. Equivalently, $\gamma(t, n_2, \dots, n_D) = \frac{\pi^2}{2t^2}$. On the other hand, in the limit $n_1, n_2, \dots, n_D \to \infty$, we have $\gamma \to 0$. Since t is the smallest of $\{t, n_2, \dots, n_D\}$, this saturates the bound in Theorem 2 (up to the factors that depend on D).

The definition of $\gamma(t_1, t_2, \dots, t_D)$ takes a minimum over all hyperrectangles of dimension $t_1 \times t_2 \times \dots \times t_D$. To see that this is cannot be improved to an average of the spectral gap over all hyperrectangles, consider the following Hamiltonian for $D = 1$:

$$H = \sum_{i=1}^{k-1} P_{i,i+1} + \sum_{i=k+1}^{n} P'_i,$$

where $P_i = \frac{1}{2}(|01\rangle - |10\rangle)(\langle 01| - \langle 10|)$ and $P'_i = |1|\rangle|\langle 1|$. This is the same Heisenberg ferromagnet on the first k qubits and a trivial Hamiltonian on the rest. For this Hamiltonian, $\gamma = O(\frac{1}{2k^2})$. But the spectral gap, averaged over all Hamiltonians on line segments of length k, is at least $1 - \frac{k}{n}$. This is much larger than $\gamma + \frac{1}{k^2}$.

4. Proof of Lemma 3

Proof. Define two projectors

$$\Pi_1 \overset{\text{def}}{=} \prod_{i,j\in T_1}(\mathbb{1} - P_{ij}), \quad \Pi_2 \overset{\text{def}}{=} \prod_{i,j\in T_2}(\mathbb{1} - P_{ij}).$$

Since P_{ij} mutually commute for all $i, j \in T_\alpha$, we have

$$\Pi_1 \succeq \mathbb{1} - \sum_{i,j\in T_1} P_{ij}, \quad \Pi_2 \succeq \mathbb{1} - \sum_{i,j\in T_2} P_{ij}.$$

Adding both sides, we find

$$\Pi_1 + \Pi_2 \succeq 2\mathbb{1} - \left(\sum_{i,j\in T_1} P_{ij} + \sum_{i,j\in T_2} P_{ij}\right) = 2\mathbb{1} - H. \tag{A11}$$

Next, we apply Jordan's lemma [34], which states that Π_1 and Π_2 can be simultaneously block diagonalized in the following sense. There exist orthogonal projectors $\bar{\Pi}_\beta$ of the dimension of, at most, 2, such that

$$\Pi_\alpha = \sum_\beta \bar{\Pi}_\beta \Pi_\alpha \bar{\Pi}_\beta, \quad \forall \alpha \in \{0, 1\}.$$

Moreover, $|v_{\alpha,\beta}|\rangle\langle v_{\alpha,\beta}| \stackrel{\text{def}}{=} \bar{\Pi}_\beta \Pi_\alpha \bar{\Pi}_\beta$ is either a one-dimensional normalized vector or a null vector. As a consequence, we have the identities

$$\Pi_2\Pi_1\Pi_2 = \sum_\beta |\langle v_{1,\beta}|v_{2,\beta}\rangle|^2 |v_{2,\beta}|\rangle\langle v_{2,\beta}|,$$

$$\Pi_1 + \Pi_2 = \sum_\beta (|v_{1,\beta}|\rangle\langle v_{1,\beta}| + |v_{2,\beta}|\rangle\langle v_{2,\beta}|). \quad \text{(A12)}$$

We will show the following claim.

Claim 2. Let $0 \leqslant \nu \leqslant \frac{3-\sqrt{5}}{2}$. It holds that

$$\begin{aligned}&|v_{1,\beta}|\rangle\langle v_{1,\beta}| + |v_{2,\beta}|\rangle\langle v_{2,\beta}| \\ &\quad \preceq \nu |\langle v_{1,\beta}|v_{2,\beta}\rangle|^2 |v_{2,\beta}|\rangle\langle v_{2,\beta}| + (2-\nu)\bar{\Pi}_\beta.\end{aligned}$$

Before proving the claim, let us show how it implies the lemma. Setting $\nu = \frac{1}{3} < \frac{3-\sqrt{5}}{2}$ and substituting Claim 2 in Eq. (A12), we find that

$$\begin{aligned}\Pi_1 + \Pi_2 &\preceq \tfrac{1}{3}\Pi_2\Pi_1\Pi_2 + \left(2 - \tfrac{1}{3}\right)\mathbb{1} \\ &= \tfrac{1}{3} DL^\dagger(H)DL(H) + \tfrac{5}{3}\mathbb{1}.\end{aligned}$$

Using this in Eq. (A11), we obtain

$$\begin{aligned}2\mathbb{1} - H &\preceq \tfrac{1}{3}DL^\dagger(H)DL(H) + \tfrac{5}{3}\mathbb{1} \\ &\Rightarrow \tfrac{1}{3}\mathbb{1} - H \preceq \tfrac{1}{3}DL^\dagger(H)DL(H).\end{aligned}$$

This proves the lemma after multiplying both sides by $|\psi\rangle$.

Proof of Claim 2. Let $|0\rangle \stackrel{\text{def}}{=} |v_{2,\beta}\rangle$ and $a|0\rangle + b|1\rangle = |v_{1,\beta}\rangle$, where $|a|^2 + |b|^2 = 1$. The claimed inequality is equivalent, in matrix representation, to

$$\begin{aligned}\begin{pmatrix} 1+|a|^2 & ab^* \\ a^*b & |b|^2 \end{pmatrix} &\preceq \nu|a|^2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + (2-\nu)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2-\nu|b|^2 & 0 \\ 0 & 2-\nu \end{pmatrix}.\end{aligned}$$

This can be rewritten as

$$\begin{aligned}0 &\preceq \begin{pmatrix} 1-|a|^2-\nu|b|^2 & -ab^* \\ -a^*b & 2-\nu-|b|^2 \end{pmatrix} \\ &= \begin{pmatrix} (1-\nu)|b|^2 & -ab^* \\ -a^*b & 1+|a|^2-\nu \end{pmatrix}.\end{aligned}$$

Since the trace of the matrix on the right-hand side is positive for $\nu < 1$, the above inequality is satisfied if the determinant is non-negative. The determinant can be computed to be

$$\begin{aligned}&(1+|a|^2-\nu)|b|^2(1-\nu) - |a|^2|b|^2 \\ &= |b|^2[(1-\nu)^2 + |a|^2(1-\nu) - |a|^2] \\ &= |b|^2[(1-\nu)^2 - \nu|a|^2],\end{aligned}$$

which is non-negative for all ν satisfying $(1-\nu)^2 - \nu \geqslant 0$. This is satisfied if $\nu \leqslant \frac{3-\sqrt{5}}{2}$. This completes the proof. ■

5. Proof of Lemma 4

Proof. The lower bound follows from Lemma 3. The upper bound uses the following claim, adapted from [29].

Claim 3. Let F be any polynomial of the degree of, at most, $\lceil \frac{t}{8L} \rceil$ such that $F(1) = 1$ (see Claim B.1 in [28]). It holds that

$$DL(t) = \left(\prod_k (\mathbb{1} - Q_{S_k})\right) F[DL(H)^\dagger DL(H)] \left(\prod_k (\mathbb{1} - Q_{T_k})\right). \quad \text{(A13)}$$

Before we outline the proof of this claim, note that we can set $F = \mathrm{Step}_{\frac{t}{8L}, \frac{\gamma}{g^2+\gamma}}$ to obtain

$$\begin{aligned}&\max_{\psi\in G_\perp} \|DL(t)|\psi\rangle\|^2 \\ &= \max_{\psi\in G_\perp} \left\| \left(\prod_k (\mathbb{1} - Q_{S_k})\right) \right. \\ &\quad \left. \times \mathrm{Step}_{\frac{t}{8L}, \frac{\gamma}{g^2+\gamma}}[DL(H)^\dagger DL(H)] \left(\prod_k (\mathbb{1} - Q_{T_k})\right)|\psi\rangle \right\|^2 \\ &\leqslant \max_{\psi\in G_\perp} \|\mathrm{Step}_{\frac{t}{8L}, \frac{\gamma}{g^2+\gamma}}[DL(H)^\dagger DL(H)]|\psi\rangle\|^2.\end{aligned}$$

In the last inequality, we used the following:

$$\left(\prod_k (\mathbb{1} - Q_{T_k})\right)|\psi\rangle \in G_\perp, \quad \left\|\left(\prod_k (\mathbb{1} - Q_{T_k})\right)|\psi\rangle\right\| \leqslant 1.$$

From Lemma 1, the second largest eigenvalue of $DL(H)^\dagger DL(H)$ is, at most, $\frac{1}{1+\frac{\gamma}{g^2}} = 1 - \frac{\gamma}{g^2+\gamma}$. This concludes the proof of Lemma 4.

Proof outline of Claim 3. Following Claim B.1 in [28], we consider the "layer operators"

$$DL_\alpha \stackrel{\text{def}}{=} \prod_{i,j:P_{ij}\in T_\alpha} (\mathbb{1} - P_{ij}).$$

Observe that $DL(H) = DL_1 DL_2 \cdots DL_L$ and hence

$$DL(H)^\dagger DL(H) = DL_L \cdots DL_2 DL_1 DL_2 \cdots DL_L.$$

This implies that the operator

$$\begin{aligned}[DL(H)^\dagger DL(H)]^q &= (DL_L \cdots DL_2 DL_1 DL_2 \cdots DL_{L-1})^{q-1} \\ &\quad \times DL_L \cdots DL_2 DL_1 DL_2 \cdots DL_L\end{aligned}$$

is a product of $(2L-2)(q-1) + 2L - 1 = q(2L-2) + 1$ operators DL_α. Suppose we have

$$q \leqslant \left\lceil \frac{t}{8L} \right\rceil \Rightarrow q(2L-2) + 1 < \lfloor t/4 \rfloor \quad (\text{using } t \geqslant 8L^2).$$

Since the overlap between an S set and the adjacent T set is at least $\lfloor t/4 \rfloor$ [Eq. (A5)], all the operators can be "absorbed" in either $[\prod_k (\mathbb{1} - Q_{S_k})]$ or $[\prod_k (\mathbb{1} - Q_{T_k})]$. This ensures that

$$\begin{aligned}&\left(\prod_k (\mathbb{1} - Q_{S_k})\right)[DL(H)^\dagger DL(H)]^q \left(\prod_k (\mathbb{1} - Q_{T_k})\right) \\ &= \left(\prod_k (\mathbb{1} - Q_{S_k})\right)\left(\prod_k (\mathbb{1} - Q_{T_k})\right).\end{aligned}$$

This proves the claim if we take the linear combination of the above equation according to the polynomial F. ■

[1] M. B. Hastings, J. Stat. Mech. (2007) P08024.
[2] I. Arad, Z. Landau, and U. Vazirani, Phys. Rev. B **85**, 195145 (2012).
[3] I. Arad, A. Kitaev, Z. Landau, and U. Vazirani, arXiv:1301.1162.
[4] M. B. Hastings, Phys. Rev. B. **69**, 104431 (2004).
[5] M. B. Hastings and T. Koma, Commun. Math. Phys. **265**, 781 (2006).
[6] B. Nachtergaele and R. Sims, Commun. Math. Phys. **265**, 119 (2006).
[7] T. S. Cubitt, D. Perez-Garcia, and M. M. Wolf, Nature (London) **528**, 207 (2015).
[8] J. Bausch, T. Cubitt, A. Lucia, and D. Perez-Garcia, arXiv:1810.01858.
[9] B. Nachtergaele, Commun. Math. Phys. **175**, 565 (1996).
[10] S. Knabe, J. Stat. Phys. **52**, 627 (1988).
[11] S. Bravyi and D. Gosset, J. Math. Phys. **56**, 061902 (2015).
[12] M. Lemm, Phys. Rev. B **100**, 035113 (2019).
[13] F. G. S. L. Brandão, A. W. Harrow, and M. Horodecki, Commun. Math. Phys. **346**, 397 (2016).
[14] M. Lemm, A. W. Sandvik, and L. Wang, arXiv:1910.11810.
[15] I. Affleck, T. Kennedy, E. H. Lieb, and H. Tasaki, Phys. Rev. Lett. **59**, 799 (1987).
[16] M. Fannes, B. Nachtergaele, and R. F. Werner, Commun. Math. Phys. **144**, 443 (1992).
[17] C. Fernández-González, N. Schuch, M. M. Wolf, J. I. Cirac, and D. Pérez-García, Commun. Math. Phys. **333**, 299 (2015).
[18] S. Bravyi, Contemp. Math. **536** (2011).
[19] D. Gosset and D. Nagaj, in *2013 IEEE 54th Annual Symposium on Foundations of Computer Science* (IEEE, Piscataway, NJ, 2013), pp. 756–765.
[20] D. Gosset and E. Mozgunov, J. Math. Phys. **57**, 091901 (2016).
[21] M. Lemm and E. Mozgunov, J. Math. Phys. **60**, 051901 (2019).
[22] M. J. Kastoryano and A. Lucia, J. Stat. Mech. (2018) 033105.
[23] D. Aharonov, I. Arad, Z. Landau, and U. Vazirani, in *Proceedings of the Forty-First Annual ACM Symposium on Theory of Computing, STOC '09* (ACM, New York, 2009), pp. 417–426.
[24] M. Lemm, arXiv:1902.07141.
[25] A. Anshu, I. Arad, and T. Vidick, Phys. Rev. B **93**, 205142 (2016).
[26] D. Gosset (private communication).
[27] L. Eldar and A. W. Harrow, in *2017 IEEE 58th Annual Symposium on Foundations of Computer Science (FOCS)* (IEEE, Piscataway, NJ, 2017), pp. 427–438.
[28] A. Anshu, I. Arad, and D. Gosset, arXiv:1905.11337.
[29] D. Gosset and Y. Huang, Phys. Rev. Lett. **116**, 097202 (2016).
[30] H. Abdul-Rahman, M. Lemm, A. Lucia, B. Nachtergaele, and A. Young, Math. Phys. **741**, 1 (2020).
[31] M. Lemm, A. Sandvik, and S. Yang, J. Stat. Phys. **177**, 1077 (2019).
[32] We have shifted to the terminology “qudits” instead of “spins,” which is more standard in quantum information.
[33] J. Gao, Phys. Rev. A **92**, 052331 (2015).
[34] C. Jordan, Bull. Soc. Math. Fr. **3**, 103 (1875).