Supersymmetric Hamiltonian solutions simulated by Andreev bound states

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(Received 29 November 2019; revised manuscript received 25 February 2020; accepted 17 March 2020; published 3 April 2020)

It is argued that an analog of a supersymmetric quantum mechanical spectrum can be realized in a Josephson junction formed by a topological insulator. These Andreev bound states can be studied by means of a circuit quantum electrodynamics setup. At the same time the equilibrium Josephson current has a more complicated analytical structure and is not expressed in terms of the Andreev states by a standard formula.

DOI: 10.1103/PhysRevB.101.134501

I. INTRODUCTION

The current experimental advancements enable studying the Josephson junctions with transmission coefficients close to unity. Among such systems there are junctions based on carbon nanotubes [1–3], aluminium atomic break junction [4], graphene [5,6], InAs nanowires [7–9], two-dimensional (2D) [10] and 3D [11,12] topological insulators. The physical mechanism underlying the high transmission coefficient in the topological insulators is Klein tunneling, see, e.g., Ref. [13], caused by the linearity of the spectrum of the surface states. The high transmission coefficient implies the possibility of the Andreev bound states going deep into the subgap region.

One of the developed effective and economic tools for the description of the superconductivity and superfluidity is represented by the quasiclassical Eilenberger equations, see review [14]. In the case of conventional superconductors this approach relies on the possibility to linearize the electron spectrum near the Fermi surface. Since the quasiclassical function corresponds to the envelope of the microscopic wave function, the corresponding boundary condition has in a general case a nontrivial and nonlinear form, see Ref. [15]. The approach of the quasiclassical equations has recently been adapted to study of the topological insulators in Refs. [16,17].

In this paper I discuss the possibility of formation of the spectrum of Andreev bound states corresponding to the supersymmetric quantum Hamiltonian spectrum, see Ref. [18]. This suggestion relies on the spatial dispersion of the tunneling between superconducting electrode and the topological insulator, leading to a spatially modulated effective gap. The simplest way to obtain Andreev bound states is to use Bogoliubov-de Gennes equation, that can be transformed to the typical structure of the supersymmetric quantum mechanics, resulting in the corresponding series of eigenstates.

The intricate point concerning these Andreev bound states is that the Josephson current is not defined by the standard expression in their terms [see Eq. (18) below]. This expression for the case under consideration is rather leading to the inaccurate result. The origin of the discrepancy can be understood as follows. The standard mean-field description of the superconductivity relies on Gorkov equations written in terms of normal and anomalous Green's functions (these can be simplified to quasiclassical equations). In the equilibrium case the Josephson current is obtained from the expression for the normal Green's function, summed over Matsubara frequencies. The sum is transformed by means of complex analysis into integral, which has contributions both from poles and the branch cuts. The contribution from the poles is identical to the contribution from the Andreev levels, while the contribution from the branch cuts can also matter. In other language, see, e.g., Ref. [19], this contribution corresponds to the scattering continuum. The outlined procedure is performed in the Appendix, where an explicit analytic expression for the Josephson current in a small conventional junction is presented. It is demonstrated that in the asymmetric case (when the gaps on the sides of the junction are different), the Andreev states can provide the subleading contribution to the Josephson current.

Of course, the calculation of the Josephson current via Gorkov equations in the Matsubara technique is just a matter of convenience. The use of Bogoliubov-de Gennes equations with the full description of the wave functions both of the discreet and the continuum spectrum would result in the same expression. This can be exemplified by the comparison of Eq. (A14) and results of Ref. [19], where calculations were performed in the Bogoliubov-de Gennes scheme. The latter scheme was recently employed in Refs. [20,21] for quite complicated Hamiltonians. The switch of the Josephson current between zero and π states and the deviations of response to the external magnetic field from the standard Fraunhofer pattern were studied in these papers.

Similarly, the calculations within the Green's functions approach do not modify the spectrum of the Andreev bound states obtained in this paper via Bogoliubov-de Gennes equation. It is worth noting that Andreev bound states can now be experimentally studied using cQED setup [9,22] (circuit quantum electrodynamics). The predictions of this paper, e.g., appearance of new Andreev bound states at specific values of superconducting phase jump, can be experimentally tested.

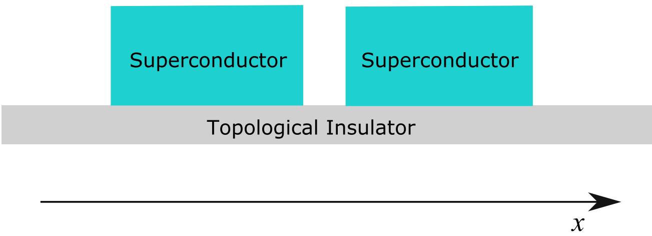

FIG. 1. Schematic realization of a Josephson junction through a topological insulator.

II. EQUATIONS DESCRIBING TOPOLOGICAL INSULATOR INTERFACE

There is a growing interest in study of topological insulators (TI), compounds characterized by an insulating bulk but conducting edges or surfaces. The Josephson junction appearing as a resulting of depositing of two spatially separated superconducting electrodes on the surface of a 3D TI shown in Fig. 1, is described by the following Bogoliubov-de Gennes equation, see Eq. (5.60) in Ref. [13]:

$$\hat{Q}\begin{pmatrix} u_\uparrow(\boldsymbol{r}) \\ u_\downarrow(\boldsymbol{r}) \\ v_\uparrow(\boldsymbol{r}) \\ v_\downarrow(\boldsymbol{r}) \end{pmatrix} = 0, \quad \hat{Q} = \begin{pmatrix} E+\mu-U(x) & \hbar v(i\partial_x+\partial_y) & 0 & -\Delta(x) \\ \hbar v(i\partial_x-\partial_y) & E+\mu-U(x) & \Delta(x) & 0 \\ 0 & \Delta^*(x) & E-\mu+U(x) & \hbar v(i\partial_x-\partial_y) \\ -\Delta^*(x) & 0 & \hbar v(i\partial_x+\partial_y) & E-\mu+U(x) \end{pmatrix}. \tag{1}$$

This equation is acting on the surface of the TI, and the function $\Delta(x)$ originates from the tunneling Hamiltonian approach of Ref. [23]. It should not be confused with the superconducting order parameter of the intrinsic superconductor. Since the coupling constant is assumed to be positive in TI, the pair potential is identically zero there. However, there appears the proximity induced nonzero anomalous function in TI, modeled by the term $\Delta(x)$ in Eq. (1).

While the proximity induced effective gap in Eq. (1) is momentum independent, it acquires such dependence (p-wave type), if one switches to the basis of eigenstates of the Hamiltonian in the normal state of TI. This property is discussed in Ref. [24] with the starting point equivalent to Eq. (1). This interesting effect has been first predicted by Fu and Kane in Ref. [25].

The Planck constant is later set to be unity $\hbar = 1$ and the case of normal incidence $\partial_y = 0$ is considered. The normally incident modes are characterized by the transmission coefficient equal to 1 (Klein tunneling) and mainly govern the current-phase relationship of Josephson current in such junctions, see Ref. [13]. Or one can have in mind the situation with few conducting channels. Thus effectively one-dimensional problem will be considered. When $\partial_y = 0$, we see that $\hat{Q}$ commutes with the matrix $\hat{C}$

$$\hat{C} = \begin{pmatrix} \hat{\sigma}_x & 0 \\ 0 & -\hat{\sigma}_x \end{pmatrix}. \tag{2}$$

So the solutions of the equation $\hat{Q}\psi = 0$ can simultaneously be eigenfunctions of the matrix $\hat{C}$ with eigenvalues $\lambda = \pm 1$ (helicity). We will treat the case $\lambda = 1$ (forward motion), the case $\lambda = -1$ corresponds to the backward motion. The spectrum of the backward motion is related to the spectrum of the forward motion. More precisely, E^2 eigenvalues for the given phase jump χ will be the same for the forward and backward motion. So for the given χ, if there is E eigenvalue corresponding to the forward motion, there is also $-E$ eigenvalue corresponding to the backward motion. This situation corresponds to the one studied in Ref. [13].

So we shall write the forward solution of Eq. (1) as

$$\begin{pmatrix} u_\uparrow(\boldsymbol{r}) \\ u_\downarrow(\boldsymbol{r}) \\ v_\uparrow(\boldsymbol{r}) \\ v_\downarrow(\boldsymbol{r}) \end{pmatrix} = \begin{pmatrix} f(x) \\ f(x) \\ \varphi(x) \\ -\varphi(x) \end{pmatrix}. \tag{3}$$

We can get rid of $U(x) - \mu$ terms by writing $[f(x), \varphi(x)] \to \exp[i\omega(x)][f(x), \varphi(x)]$ and properly choosing $\omega(x)$. Thus we come to the differential equations

$$(E + iv\partial_x)f(x) + \Delta(x)\varphi(x) = 0,$$
$$(E - iv\partial_x)\varphi(x) + \Delta^*(x)f(x) = 0. \tag{4}$$

Let us divide the first equation of (4) by $\Delta(x)$ and act on the resulting equation with $(E - iv\partial_x)$. Then let us divide the second equation (4) by $\Delta^*(x)$ and act on the resulting equation with $(E + iv\partial_x)$. Thus we come to

$$E^2\psi(x) = \left(-v^2\partial_x^2 + \hat{W}^2 + iv\hat{\sigma}_z\frac{\partial \hat{W}}{\partial x}\right)\psi(x). \tag{5}$$

Here,

$$\psi(x) = \begin{pmatrix} f(x) \\ \varphi(x) \end{pmatrix}, \quad \hat{W}(x) = \begin{pmatrix} 0 & \Delta(x) \\ \Delta^*(x) & 0 \end{pmatrix}. \tag{6}$$

The equation in the case of the backward motion will be the same as Eq. (5).

The right-hand side of Eq. (5) is identical to the Hamiltonian of Witten's supersymmetric quantum mechanics, see Ref. [18], but for $\hat{W}(x)$ being the matrix, not the scalar function. It should be emphasized, that semiconducting structures are often described by a Hamiltonian having supersymmetric structure, see Ref. [26]. The different possibilities for Andreev levels resulting from Eq. (5) are discussed in the next section.

III. DIFFERENT SOLUTIONS FOR ANDREEV LEVELS

The most typical case considered for the Josephson junction formed by two superconductors separated by a layer of topological insulator corresponds to the function $\Delta(x)$ changing abruptly from $|\Delta|e^{i\chi/2}$ to $|\Delta|e^{-i\chi/2}$ at $x = 0$. In this situation the integration of Eq. (5) near $x = 0$ provides the jump of the derivatives of f, φ near $x = 0$. Matching these with the solutions decaying into the depth one can obtain [13]

$$E^2 = \Delta^2 \cos^2 \frac{\chi}{2}. \tag{7}$$

However, even if the deposited superconductors are the same, the effective $\Delta(x)$ in Eq. (5) involves the tunneling amplitudes (see Appendix in Ref. [27]) and these can be different. So the general situation for a short junctions corresponds to $\Delta(x)$ on the left equal to $\Delta_1 e^{i\chi/2}$ and on the right to $\Delta_2 e^{-i\chi/2}$. The parameters $\Delta_{1,2}$ are taken to be real and positive. Then in the same manner one can obtain

$$E^2 = \frac{\Delta_1^2 \Delta_2^2 \sin^2 \chi}{\Delta_1^2 + \Delta_2^2 - 2\Delta_1 \Delta_2 \cos \chi} \tag{8}$$

with the following important stipulation, identical to the Ref. [28]. The solution is valid only for $\cos \chi < \Delta_1/\Delta_2$ (it is assumed for definiteness that $\Delta_1 < \Delta_2$). For $\cos \chi > \Delta_1/\Delta_2$ there are no Andreev bound states. This statement holds irrespective of the transmission coefficient of the given conducting channel. The expression (8), naturally, corresponds to results of Ref. [28] and to Ref. [29], where the asymmetric case with the presence of Zeeman field was considered.

The Appendix deals with short conventional asymmetric ($\Delta_1 \neq \Delta_2$) junction. Emphasis is laid on the analytic properties of the Josephson current. It is demonstrated that the usual formula for the Josephson current in terms of the Andreev levels energy can give the result essentially different from the correct one.

The absence of the Andreev bound states for $\cos \chi > \Delta_1/\Delta_2$ limits the validity of widely used reasoning about the adiabatic evolution of forward and backward states in the context of the 4π periodic Josephson current. This reasoning relies on the protection of the forward and backward states due to the time reversal symmetry with the resulting possibility of nonequilibrium occupancy of these states.

Let us now explore another possibility for the Andreev bound states under the assumption that the tunneling is soft, in a sense that the resulting $\Delta(x)$ distribution is given by (the superconductors are close)

$$\Delta(x) = \frac{\Delta_L}{1 + e^{x/a}} + \frac{\Delta_R}{1 + e^{-x/a}}. \tag{9}$$

Here a is the length parameter, characterizing the tunneling Hamiltonian. So this model assumes that the superconducting electrodes are close, but the tunneling has some distance scale a. Taking the function $\Delta(x)$ deep in the left and deep in the right to be $\Delta_L = |\Delta| e^{i\chi/2}$, $\Delta_R = |\Delta| e^{-i\chi/2}$, we obtain

$$\Delta(x) = |\Delta| \cos \frac{\chi}{2} - i|\Delta| \sin \frac{\chi}{2} \tanh \frac{x}{a}. \tag{10}$$

We see from Eq. (10), that the main feature of this model is the change of the superconducting phase on the scale a. If a is small, we recover the previous model of the abrupt change of χ, if a is not that small, we have the gradual change of χ with the distance. Hence we come to the equation

$$E^2 \psi = \left[-v^2 \partial_x^2 + |\Delta|^2 \cos^2 \frac{\chi}{2} + |\Delta|^2 \sin^2 \frac{\chi}{2} \tanh^2 \frac{x}{a} + \frac{v}{a \cosh^2(x/a)} |\Delta| \sin \frac{\chi}{2} \hat{\sigma}_x \right] \psi. \tag{11}$$

Making the unitary transformation to diagonalize $\hat{\sigma}_x$

$$\frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{12}$$

we come to the typical structure of the supersymmetric quantum mechanics

$$E^2 \psi = \left[-v^2 \partial_x^2 + |\Delta|^2 \cos^2 \frac{\chi}{2} + W^2(x) + v W'(x) \hat{\sigma}_z \right] \psi. \tag{13}$$

Here,

$$W(x) = |\Delta| \sin \frac{\chi}{2} \tanh \frac{x}{a}. \tag{14}$$

Using results of Ref. [18] we get the following expressions for the eigenvalues:

$$E_n^2 = \Delta^2 - \left(\Delta \left| \sin \frac{\chi}{2} \right| - n \frac{v}{a} \right)^2. \tag{15}$$

The values of n are $0, 1, \dots$ and up to the largest integer less or equal $\Delta |\sin(\chi/2)| a/v$. Excluding $n = 0$, the levels are doubly degenerate.

To be more specific, let us describe the structure of the Andreev levels for $0 < \chi < 2\pi$. There is a level $E = |\Delta| \cos(\chi/2)$ corresponding to the forward motion, the level with the opposite sign $E = -|\Delta| \cos(\chi/2)$, corresponding to the backward motion. If allowed by the condition

$$n \frac{v}{a} < \Delta |\sin(\chi/2)|, \tag{16}$$

there are also levels with $n > 0$, given by

$$E_n = \pm \sqrt{\Delta^2 - \left(\Delta \left| \sin \frac{\chi}{2} \right| - n \frac{v}{a} \right)^2}. \tag{17}$$

Each of the levels (17) is doubly degenerate—one state corresponds to the forward motion and the other to the backward motion. Note, that in the case of Andreev levels with several n, the corresponding states are realized in different ranges of χ. It contrasts with the case of states (8) and the similar case of the short junction with several conducting channels, characterized by different transmission coefficients. All the Andreev states in the latter situation are realized for $\cos \chi < \Delta_1/\Delta_2$.

One can assume, that substituting the values (15) into the usual expression for the Josephson current in terms of the Andreev bound states

$$I_J^{(A)} = -2e \sum_n E_n{}'(\chi) \tanh \frac{E_n(\chi)}{2T}, \tag{18}$$

one will have the answer [let us remind, that the summation in Eq. (18) goes over positive Andreev levels]. However, as explained in the next section this assumption is inaccurate. The value of the current essentially differs from that given by Eq. (18).

The similar discrepancy can also appear in the case of the conventional SNS ballistic contact with equal gaps on two sides. The Green's function in Matsubara technique can be calculated as discussed in Ref. [30], where the expression (16), (17) for the Josephson current is presented. Similarly to the Appendix the sum over Matsubara frequencies can be transformed as the sum over the poles plus the integral along the branch cut. Naturally, the poles coincide with the usual expression for Andreev levels, i.e., for a long junction they have a standard form

$$E_n = \frac{v}{2d}[\pi(2n+1) \pm \chi]. \tag{19}$$

Here v is the Fermi velocity, d is the length of the normal layer and n is an integer number. The integral over the branch cut can be transformed to the integral from Δ to ∞. It can be checked that for $\Delta \sim v/d$ the branch cut contribution is comparable to the contributions from the poles. The similar effect takes place in the case under consideration, when we have the effective gap. The next section deals with the construction of the Green's function in Matsubara technique for the effective gap distribution given by Eq. (10).

IV. JOSEPHSON CURRENT IN THE CASE OF THE SUPERSYMMETRIC SPECTRUM OF ANDREEV STATES

Equation (4) exactly coincides with the Eq. (5) of Ref. [30] after the replacement $E \to i\omega_n$. Reference [30] is devoted to the construction of the Green's functions and calculation of the Josephson current in multiple-barrier structures. The forward and the backward motion in that reference, naturally, correspond to the expansion around $\exp(\pm ik_F x)$, where k_F is the Fermi momentum. The situation considered in the current context corresponds to the previous one with the stipulation of the perfect transmission due to Klein tunneling for the normal incidence mode. The Green's functions can be constructed in the same way as before.

So we are dealing with the system of equations

$$\begin{aligned}(i\omega_n + iv\partial_x)f(x) + \Delta(x)\varphi(x) = 0,\\ (i\omega_n - iv\partial_x)\varphi(x) + \Delta^*(x)f(x) = 0,\end{aligned} \tag{20}$$

where $\omega_n = (2n+1)\pi T$ is the Matsubara frequency. Taking the sum and the difference of the equations (20) for the distribution of $\Delta(x)$ given by Eq. (10), we come to the following equation on $y(x) = f(x) + \varphi(x)$:

$$\cosh^2 x\, y''(x) + [\nu(\nu+1) - \mu^2\cosh^2 x]y(x) = 0. \tag{21}$$

The coordinate x was rescaled by the factor a, so that the new coordinate x is dimensionless. Also

$$\mu^2 = \frac{a^2}{v^2}\left(|\Delta|^2 + \omega_n^2\right), \quad \nu = -\frac{a}{v}|\Delta|\sin\frac{\chi}{2}. \tag{22}$$

The decaying solutions of Eq. (21) with Re $\mu > 0$ are Ferrers function P_ν^μ (see online information in Ref. [31]):

$$\begin{aligned}&\mathrm{P}_\nu^{-\mu}(\tanh x), \quad \text{when } x \to +\infty\\ &\mathrm{P}_\nu^{-\mu}(-\tanh x), \quad \text{when } x \to -\infty.\end{aligned} \tag{23}$$

So we know solutions of Eq. (20) with the proper behavior. In constructing the Green's function $G_{\omega_n}(x, x')$ (depending on Matsubara frequencies and two coordinates x, x'), we take the linear combinations of products of solutions of Eq. (20) on functions, depending on argument x'. Then imposing the condition of needed behavior at $x \to x'$, similarly to Ref. [30], we obtain the Green's function. Calculating the Josephson current at $x = 0$, one has in the case of one conducting

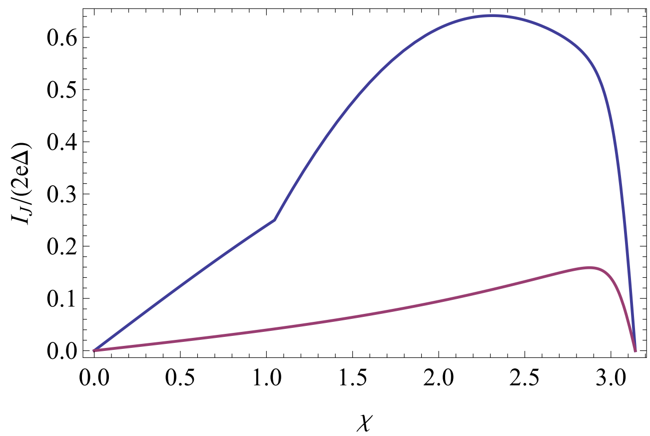

FIG. 2. The phase dependence of the Josephson current described by Eqs. (25) (lower curve) and (18) (upper curve) at $\pi T/|\Delta| = 0.1$ and $a|\Delta|/v = 1$.

channel:

$$I_J = -eT\nu\cot\frac{\chi}{2}\sum_{\omega_n>0}\left[\mathrm{P}_\nu^{-\mu}(0)\left(\mathrm{P}_{-\nu}^{-\mu}\right)'(0) - \mathrm{P}_{-\nu}^{-\mu}(0)\left(\mathrm{P}_\nu^{-\mu}\right)'(0)\right] \times\left[\frac{1}{\left(\mathrm{P}_{-\nu}^{-\mu}\right)'(0)\left(\mathrm{P}_\nu^{-\mu}\right)'(0)} + \frac{1}{\mu^2-\nu^2}\frac{1}{\mathrm{P}_\nu^{-\mu}(0)\mathrm{P}_{-\nu}^{-\mu}(0)}\right]. \tag{24}$$

Taking from the Ref. [31] the values of Ferrers functions and their derivatives at $x = 0$, we come to

$$I_J = -eT\pi\nu\cot\frac{\chi}{2}\sum_{\omega_n>0}2^{1-2\mu}\Gamma(\mu-\nu)\Gamma(\mu+\nu) \times\left[\frac{\mu-\nu}{\Gamma^2(1+\frac{\mu}{2}-\frac{\nu}{2})\Gamma^2(\frac{1}{2}+\frac{\mu}{2}+\frac{\nu}{2})} - \frac{\mu+\nu}{\Gamma^2(1+\frac{\mu}{2}+\frac{\nu}{2})\Gamma^2(\frac{1}{2}+\frac{\mu}{2}-\frac{\nu}{2})}\right]. \tag{25}$$

This expression gives the final result for the Josephson current in the case of one conducting channel and perfect transmission (zero scattering potential). Naturally, in the case of short junction $a \to 0$, we recover from Eq. (25) the standard expression

$$I_J = e|\Delta|\sin\frac{\chi}{2}\tanh\left(\frac{|\Delta|\cos(\chi/2)}{2T}\right). \tag{26}$$

The gamma function does not have zeros in the complex plane. So the poles of the expression (25) at ω_n taking the imaginary values are defined by the poles of the gamma function at nonpositive integers. Thus we recover the Andreev levels (15). In the same manner the Andreev levels can be recovered from the expression for the equilibrium current in conventional Josephson junction as discussed in Ref. [32].

At the same time the expression for the Josephson current provided by Eq. (25) essentially differs from that given by Eq. (18) [with the Andreev spectrum (15)] as demonstrated in the Figs. 2 and 3. There is no cause to wonder having in mind that $\Delta(x)$ in Eq. (4) is the effective function, not the genuine superconducting gap. However, it is also demonstrated in

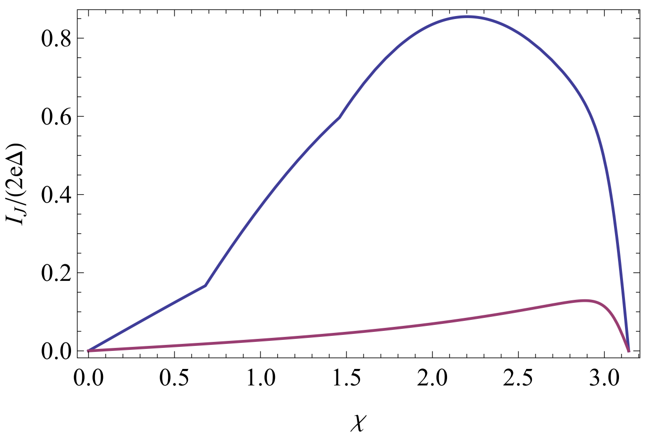

FIG. 3. The phase dependence of the Josephson current described by Eqs. (25) (lower curve) and (18) (upper curve) at $\pi T/|\Delta| = 0.1$ and $a|\Delta|/v = 2$.

the Appendix that even for the gap corresponding to the real superconducting gap the current defined by the usual expression (18) can give an essentially inaccurate answer.

V. CONCLUSION

To summarize, it is shown that the analog of the supersymmetric quantum mechanics of Witten is possible in the case of the Josephson junctions, see Eqs. (13) and (15). This realization relies on the distribution of effective gap $\Delta(x)$ given by Eq. (10). Such a distribution (plausible in the junction formed by a topological insulator) corresponds to the smoothly changing phase in the junction layout shown schematically in the Fig. 1.

More careful treatment of the Josephson current through this junction allows us to come to the explicit result (25) [with notations introduced in Eq. (22)]. The Josephson current provided by Eq. (25) strongly deviates from the usual expression for the Josephson current (18). Nevertheless, the pole structure is not modified, and corresponding Andreev levels can be studied by means of the circuit quantum electrodynamics.

The Appendix demonstrates, that even in a simple realization the current provided by Eq. (18) can give an inaccurate result. It is compared with the derived general analytic expression (A10) for the Josephson current through a small junction.

ACKNOWLEDGMENTS

Useful discussions with M. Yu. Kupriyanov, A. G. Semenov, and A. D. Zaikin are gratefully acknowledged. This work was supported by the Presidium of the Russian Academy of Sciences, Program of Fundamental Research No. 13.

APPENDIX: JOSEPHSON CURRENT IN A SMALL ASYMMETRIC JUNCTION

Let us consider a Josephson current through a point contact, the dimensions of which are smaller than the superconducting coherence length. The Josephson current through it is given by Eq. (19) of Ref. [32]

$$I_J = 4eT\sum_{\omega>0}\sum_n \frac{T_n f_1 f_2 \sin\chi}{2 + T_n(g_1 g_2 + f_1 f_2\cos\chi - 1)}. \quad \text{(A1)}$$

Here the summation goes over Matsubara frequencies and n conducting channels with transmission coefficients T_n. The Matsubara Green's functions are given by

$$g_{1,2} = \frac{\omega}{\sqrt{\omega^2 + \Delta_{1,2}^2}}, \quad f_{1,2} = \frac{\Delta_{1,2}}{\sqrt{\omega^2 + \Delta_{1,2}^2}}. \quad \text{(A2)}$$

Here $\Delta_{1,2}$ are real and positive. The expression for the Josephson current (A1) results from the calculation of the quasiclassical Green's function with account of Zaitsev's boundary conditions [15]. Eq. (A1) is essentially equivalent to Eq. (35) from the Ref. [15].

In the case of symmetric $\Delta_1 = \Delta_2$ Josephson junction the square roots in the denominator of Eq. (A1) disappear and the contribution to the sum comes from the poles in the complex plane (Andreev levels)

$$I_J = \frac{e\Delta\sin\chi}{2}\sum_n \frac{T_n}{\sqrt{1 - T_n\sin^2(\chi/2)}} \times \tanh\frac{\Delta\sqrt{1 - T_n\sin^2(\chi/2)}}{2T}. \quad \text{(A3)}$$

Introducing the Andreev levels

$$E_n(\chi) = \Delta\sqrt{1 - T_n\sin^2(\chi/2)}, \quad \text{(A4)}$$

we can rewrite the expression (A3) in the form (18).

The analytic expression for the Josephson current becomes more complicated in the asymmetric case $\Delta_1 \neq \Delta_2$. It amounts to finding the following sum for the given transmission channel

$$\sum_{k=0}^{\infty} \frac{1}{\sqrt{(2k+1)^2 + a^2}\sqrt{(2k+1)^2 + b^2} + \gamma_n[(2k+1)^2 + ab\cos\chi]}. \quad \text{(A5)}$$

Here $\gamma_n = T_n/(2 - T_n)$ lies in the range $0 \leqslant \gamma_n \leqslant 1$ and we choose $a \leqslant b$ for definiteness. Using the methods of the theory of functions of a complex variable, one can rewrite this sum as

$$\sum_{k=0}^{\infty} \frac{1}{\sqrt{(2k+1)^2 + a^2}\sqrt{(2k+1)^2 + b^2} + \gamma_n[(2k+1)^2 + ab\cos\chi]} = \frac{1}{2}\int_a^b \frac{dx\tanh(\pi x/2)\sqrt{(x^2 - a^2)(b^2 - x^2)}}{(x^2 - a^2)(b^2 - x^2) + \gamma_n^2(x^2 - ab\cos\chi)^2}$$
$$+ \theta\left(\frac{a}{b} - \cos\chi\right)\frac{\pi\gamma_n\tanh(\pi x_0/2)}{4(1 - \gamma_n^2)x_0}\left[\frac{a^2 + b^2 - 2ab\cos\chi}{\sqrt{(a^2 - b^2)^2 + 4ab\gamma_n^2(a - b\cos\chi)(b - a\cos\chi)}} - 1\right]. \quad \text{(A6)}$$

Here $\theta(x)$ is the Heaviside step function and

$$x_0^2 = \frac{1}{2(1-\gamma_n^2)}\Big[a^2+b^2-2ab\gamma_n^2\cos\chi - \sqrt{(a^2-b^2)^2+4ab\gamma_n^2(a-b\cos\chi)(b-a\cos\chi)}\Big]. \quad \text{(A7)}$$

We take for x_0 the positive square root of this expression. The right-hand side of Eq. (A6) consists of the integral along the cut and the pole (Andreev level) contribution. The pole $\pm ix_0$ exists only for $\cos\chi < a/b$ irrespective of γ_n, i.e., irrespective of the transmission coefficient.

Using Eq. (A6) we can rewrite the expression for the Josephson current as

$$I_J = -2e\theta\left(\frac{\Delta_1}{\Delta_2}-\cos\chi\right)\sum_n E_n'(\chi)\tanh\frac{E_n(\chi)}{2T} + \frac{2e}{\pi}\Delta_1\Delta_2\sin\chi\sum_n\gamma_n\int_{\Delta_1}^{\Delta_2}\frac{dx\tanh(x/2T)\sqrt{(x^2-\Delta_1^2)(\Delta_2^2-x^2)}}{(x^2-\Delta_1^2)(\Delta_2^2-x^2)+\gamma_n^2(x^2-\Delta_1\Delta_2\cos\chi)^2}. \quad \text{(A8)}$$

We take for definiteness $\Delta_1 \leqslant \Delta_2$. The square of the Andreev level energy in the nth conducting channel is given by

$$E_n^2(\chi) = \frac{1}{2(1-\gamma_n^2)}\Big[\Delta_1^2+\Delta_2^2-2\Delta_1\Delta_2\gamma_n^2\cos\chi - \sqrt{(\Delta_1^2-\Delta_2^2)^2+4\Delta_1\Delta_2\gamma_n^2(\Delta_1-\Delta_2\cos\chi)(\Delta_2-\Delta_1\cos\chi)}\Big] \quad \text{(A9)}$$

and this expression coincides with the Eq. (23) from the Ref. [28]. Taking the zero-temperature ($T \ll \Delta_1$) integrals [33] one comes to the expression for the equilibrium current of a small junction

$$I_J = -2e\theta\left(\frac{\Delta_1}{\Delta_2}-\cos\chi\right)\sum_n E_n'(\chi) + \frac{2e}{\pi}\Delta_1\sin\chi\sum_n\frac{\gamma_n}{1-\gamma_n^2}\Bigg\{K\left(1-\frac{\Delta_1^2}{\Delta_2^2}\right) - \frac{1}{\tilde\epsilon_n^2-E_n^2}\Bigg[(\tilde\epsilon_n^2-\Delta_1^2)\Pi\left(\frac{\Delta_2^2-\Delta_1^2}{\Delta_2^2-\tilde\epsilon_n^2},1-\frac{\Delta_1^2}{\Delta_2^2}\right)+(\Delta_1^2-E_n^2)\Pi\left(\frac{\Delta_2^2-\Delta_1^2}{\Delta_2^2-E_n^2},1-\frac{\Delta_1^2}{\Delta_2^2}\right)\Bigg]\Bigg\}. \quad \text{(A10)}$$

Here

$$\tilde\epsilon_n^2(\chi) = \frac{1}{2(1-\gamma_n^2)}\Big[\Delta_1^2+\Delta_2^2-2\Delta_1\Delta_2\gamma_n^2\cos\chi + \sqrt{(\Delta_1^2-\Delta_2^2)^2+4\Delta_1\Delta_2\gamma_n^2(\Delta_1-\Delta_2\cos\chi)(\Delta_2-\Delta_1\cos\chi)}\Big] \quad \text{(A11)}$$

differs from the square of the Andreev level (A9) by a sign in front of the square root. The complete elliptic integrals are defined as

$$K(x) = \int_0^{\pi/2}\frac{d\theta}{\sqrt{1-x\sin^2\theta}}, \qquad \Pi(x,y) = \int_0^{\pi/2}\frac{d\theta}{(1-x\sin^2\theta)\sqrt{1-y\sin^2\theta}}. \quad \text{(A12)}$$

The expression (A10) reproduces all the known results for the small junction. Naturally, the case of the symmetric junction is trivial, since the current reduces to the result (A3), see Ref. [34]. This expression reproduces the so-called KO-1 and KO-2 results for the diffusive and clean limits respectively (see Ref. [32]). To obtain KO-1 result it is necessary to additionally average over the Dorokhov distribution of the transmission coefficients $P(T_n) \propto 1/(T_n\sqrt{1-T_n})$.

The asymmetric case is less trivial. In the case of the strong asymmetry ($\Delta_2 \gg \Delta_1$) Eq. (A10) results within the next to logarithmic accuracy

$$I_J = \frac{2e}{\pi}\Delta_1\sin\chi\sum_n\gamma_n\left[\ln\left(\frac{4\Delta_2}{\Delta_1}\right)-\frac{\arctan\sqrt{\gamma_n^{-2}-1}}{\sqrt{\gamma_n^{-2}-1}}\right] - \sum_n\frac{2e\Delta_1\gamma_n^2\sin\chi}{\sqrt{1-\gamma_n^2\cos^2\chi}}\left[\cos\chi\,\theta(-\cos\chi)+\frac{|\cos\chi|}{\pi}\times\arccos(\gamma_n|\cos\chi|)\right]. \quad \text{(A13)}$$

For the single perfectly transparent conducting channel $\gamma_n = 1$ this expression leads to

$$I_J = \frac{2e}{\pi}\Delta_1\sin\chi\left[\ln\left(\frac{4\Delta_2}{\Delta_1}\right)-1\right]-\frac{2e}{\pi}\Delta_1\chi\cos\chi. \quad \text{(A14)}$$

Equation (A14) is written for $-\pi < \chi < \pi$, for other phases the Josephson current is continued 2π periodically. Equation

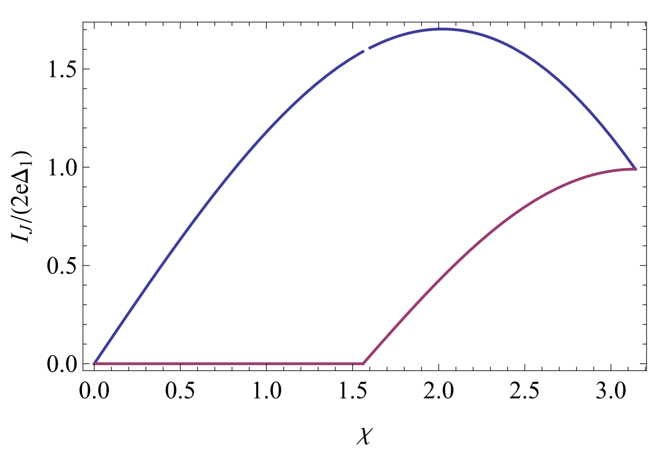

FIG. 4. The zero-temperature Josephson current (upper curve) of a perfectly transmitting conducting channel with $\Delta_2/\Delta_1 = 100$ and the Andreev level contribution to this current given by Eq. (18) (lower curve).

(A14) corresponds to the results of Ref. [19] and improves the next to logarithmic accuracy.

In order to find the Josephson current in the diffusive limit, it is again necessary to average Eq. (A13) with the Dorokhov distribution. Using the value of the integral

$$\int_0^1 \frac{\gamma \arccos(\gamma|\cos\chi|)d\gamma}{\sqrt{1-\gamma^2}\sqrt{1-\gamma^2\cos^2\chi}} = \frac{\pi}{2|\cos\chi|}\ln(1+|\cos\chi|), \tag{A15}$$

one comes to

$$I_J = \frac{\pi\Delta_1}{2eR_N}\sin\chi\ln\left(\frac{2\Delta_2}{\Delta_1(1+\cos\chi)}\right). \tag{A16}$$

Here R_N is the normal state resistance of the junction. Eq. (A16) corresponds to Eq. (12) of Ref. [32].

The message of the Appendix is illustrated in the Fig. 4, drawn for $\Delta_2/\Delta_1 = 100$ and the transmission coefficient equal to 1. It can be seen, that the Andreev level contribution [given by Eq. (18)] essentially differs from the total result.

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