# Nieh-Yan anomaly: Torsional Landau levels, central charge, and anomalous thermal Hall effect

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The Nieh-Yan anomaly is the anomalous breakdown of the chiral U(1) symmetry caused by the interaction between torsion and fermions. We study this anomaly from the point of view of torsional Landau levels. It was found that the torsional Landau levels are gapless, while their contributions to the chiral anomaly are canceled, except those from the lowest torsional Landau levels. Hence, the dimension is effectively reduced from $(3+1)$-dimensional to $(1+1)$-dimensional. We further show that the coefficient of the Nieh-Yan anomaly is the free-energy density in $(1+1)$ dimensions. Especially, at finite temperature, the thermal Nieh-Yan anomaly is proportional to the central charge. The anomalous thermal Hall conductance in Weyl semimetals is then shown to be proportional to the central charge, which is the experimental fingerprint of the thermal Nieh-Yan anomaly.

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## I. INTRODUCTION

Although torsion naturally arises as the curvature tensor of the translational gauge fields [1], in gravity, its observable effects are relatively small and are usually neglected. By contrast, torsion is attracting more and more attention in condensed-matter physics. Torsion can emerge from dislocations [2–5], the temperature gradient [6–9], background rotation, and the order parameter of a Fermi superfluid or topological superconductors [10–12]. Especially, in Dirac and Weyl semimetals, due to their gapless spectrum and strong spin-orbit coupling [13,14], torsion has led to rich physical phenomena, for example, chiral zero modes trapped in dislocations [15,16], the chiral torsional magnetic effect [17], and other viscoelastic responses [18–22].

Topological phases of matter are closely related to quantum anomalies [23,24]. Similarly, both the chiral anomaly and mixed axial gravitational anomaly are important to understand Dirac and Weyl semimetals [25–39]. Torsion can lead to chiral current nonconservation as well, which is known as the Nieh-Yan anomaly [40,41]. However, compared to other anomalies, the Nieh-Yan anomaly depends on the cutoff and thus the specific ultraviolet physics, which is still controversial [42,43]. The Nieh-Yan anomaly lies in the intersection of condensed-matter physics and high-energy physics, topology, and geometry. Meanwhile, Dirac and Weyl semimetals have provided an ideal platform to study these phenomena on a tabletop system.

Recently, it was suggested in Ref. [44] that there might be an extra thermal term in the Nieh-Yan anomaly, i.e.,

$$\frac{1}{\sqrt{|g|}}\partial_\mu\sqrt{|g|}\,j^{5\mu}=\left(\frac{\Lambda^2}{4\pi^2}-\frac{T^2}{12}\right)\frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{|g|}}\partial_\mu e^a_\nu\partial_\rho e^b_\sigma\eta_{ab}, \quad (1)$$

where $g_{\mu\nu}$ is the metric and $g=\det g_{\mu\nu}$. $\Lambda$ is the cutoff, $T$ is the temperature, $e^a_\mu$ is the vierbeins, and for simplicity, the spin connection is set to zero. From the view of Poincaré gauge theory, this anomaly has a form similar to the Adler-Bell-Jackiw anomaly [45,46]. But what is the physical meaning and physical mechanism behind this anomaly? Can we understand this anomaly equation from some kind of "Landau level" [47]? Compared to the zero-temperature Nieh-Yan term, the coefficient of the $T^2$ term is dimensionless, so it is also tempting to ask if they are related to any kind of topological invariant and why the coefficient is $\frac{T^2}{12}$.

In this paper, we derive the energy spectrum for Weyl fermions under torsional magnetic fields. The energy spectrum turns out to be sharply different from its magnetic counterparts. Namely, all the Landau levels collapse together at the zero-momentum point. Interestingly, only the lowest torsional Landau levels matter as far as the axial currents are concerned because the effect from the higher torsional Landau levels is canceled exactly. Thus, the $(3+1)$-dimensional system is effectively reduced to pairs of $(1+1)$-dimensional Dirac fermions. Especially, the axial current is proportional to the free-energy density of the effective $(1+1)$-dimensional Dirac fermions. If the torsional electric fields are further applied, the effective velocity of the lowest torsional Landau levels is changed. Since the free-energy density of $(1+1)$-dimensional conformal fields is proportional to central charge as well as the inverse of velocity [48,49], the torsional electric fields change the axial current and thus lead to chiral anomaly. Hence, we have found that the coefficient of the thermal Nieh-Yan anomaly in Eq. (1) is

$$\frac{T^2}{12}=\left(\frac{\pi c}{6}T^2\right)\left(\frac{1}{2\pi}\right), \quad (2)$$

where $\frac{\pi c}{6}T^2$ is the free-energy density of some $(1+1)$-dimensional conformal fields (the lowest torsional Landau

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levels), $\frac{1}{2\pi}$ is from the level degeneracy, and $c = 1$ is the central charge. The $\Lambda^2$ term in Eq. (1) is from the vacuum energy, for example, the band depth in Weyl semimetals, so it depends on concrete materials. By contrast, the thermal term is proportional to the central charge, which is universal. Finally, we show that the anomalous thermal Hall effect in Weyl semimetals is induced by the thermal Nieh-Yan anomaly, which serves as the experimental signature of the thermal Nieh-Yan anomaly.

The rest of this paper is organized as follow. In Sec. II, the energy spectrum for Weyl fermions under torsional magnetic fields is derived. In Sec. III, both the chiral torsional effect and the Nieh-Yan anomaly are calculated from the lowest torsional Landau levels. In Sec. IV, an effective model is constructed from the lowest torsional Landau levels, which relates the thermal Nieh-Yan anomaly to the central charge. In Sec. V, we derive the anomalous thermal Hall effect in Weyl semimetals from the thermal Nieh-Yan anomaly. We summarize the main results of this paper in Sec. VI.

## II. TORSIONAL MAGNETIC FIELDS AND TORSIONAL LANDAU LEVELS

Let us consider the following action for Weyl fermions in curved space-time,

$$S = \int d^d x e \frac{1}{2}\left[\bar{\psi} e^\mu_a \gamma^a (i\partial_\mu)\psi - \bar{\psi}(i\overleftarrow{\partial}_\mu) e^\mu_a \gamma^a \psi\right], \tag{3}$$

where the spin connection is set to zero and $a,\ \mu = 0,\ 1,\ 2,\ 3$ denote the locally flat coordinates (coordinate vectors $e^\mu_a \partial_\mu$) and curved coordinates, respectively. $\gamma^a$ is the gamma matrix, i.e., $\gamma^0 = \sigma^0 \otimes \tau^1$ and $\gamma^i = \sigma^i \otimes (-i\tau^2)$, where both $\sigma$ and $\tau$ stand for the Pauli matrices. The vierbein $e^\mu_a$ and its inverse $e^a_\nu$ satisfy $e^\mu_a e^b_\mu = \delta^b_a$. In addition, these vierbeins can be realized in Weyl semimetals, for example, by dislocation [2–5], temperature gradient, [6,8] and global rotation.

Now, for simplicity, let us consider a specific configuration of the vierbeins, namely, $e^a_\mu = \delta^a_\mu + w^a_\mu$ and $w^a_\mu = \frac{1}{2}\delta^a_3 \tilde{T}^3_B(0,\ -y,\ x,\ 0)$, $\tilde{T}^3_B > 0$, which means that the torsional magnetic fields are applied along the $z$ direction. The corresponding Hamiltonian is

$$H_s = s\left[p_z\sigma^3 + \left(\hat{p}_x + \frac{1}{2}\tilde{T}^3_B y p_z\right)\sigma^1 + \left(\hat{p}_y - \frac{1}{2}\tilde{T}^3_B x p_z\right)\sigma^2\right], \tag{4}$$

where $p_z$ is a good quantum number and $s = \pm 1$ denotes the chirality. Compared to the magnetic case, this Hamiltonian looks like Weyl fermions under magnetic fields with charge $p_z$. The dispersion relation of this Hamiltonian can be straightforwardly derived, i.e.,

$$\mathcal{E}_s = \begin{cases} s|p_z| & n = 0, \\ \pm\sqrt{p_z^2 + 2\left|n\tilde{T}^3_B p_z\right|} & |n| \geqslant 1, \end{cases} \tag{5}$$

where the level degeneracy is $\frac{1}{2\pi}|p_z|\tilde{T}^3_B$ and the spectrum is shown in Fig. 1. The energy spectrum with $|n| \geqslant 1$ is the same for Weyl fermions with different chiralities. So only the lowest torsional Landau levels can distinguish fermions with different chiralities. However, compared to the magnetic Landau levels, there are two main differences. First, all of the Landau levels collapse together at $p_z = 0$, which is because the torsional magnetic charge $p_z$ vanishes at this point. Second, the lowest torsional Landau level is of the form $s|p_z|$ rather than $sp_z$ in the magnetic case, which is because $p_z$ reverses its sign at $p_z = 0$.

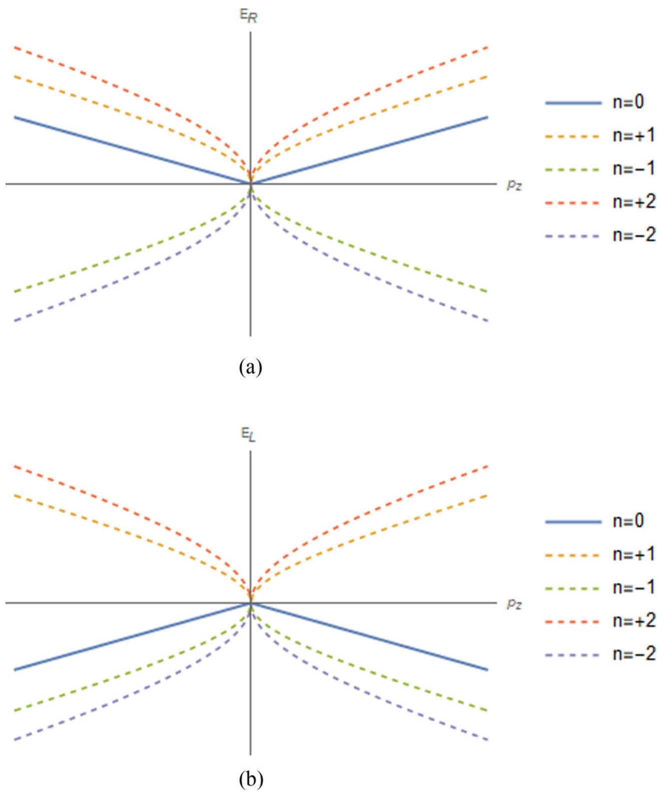


FIG. 1. Torsional Landau levels. Top: the energy spectrum for right-handed Weyl fermions under torsional magnetic fields. Bottom: the energy spectrum for left-handed Weyl fermions. The energy for the right-handed lowest torsional Landau level is positive, while it is negative for the left-handed one.

## III. NIEH-YAN TERM FROM THE LOWEST TORSIONAL LANDAU LEVELS

After deriving the torsional Landau levels, it is natural to ask if we can extract the Nieh-Yan anomaly from these levels and what the physical meaning of this anomaly is. In this section, we shall show that regardless of the gapless nature in the energy spectrum, the higher torsional Landau levels ($|n| \geqslant 1$) do not contribute to the anomaly equation. Thus, the Nieh-Yan anomaly arises from the lowest torsional Landau levels, and the system is effectively reduced to $(1+1)$-dimensional Dirac fermions. In addition, the prefactor of the Nieh-Yan term is the free-energy density of the effectively $(1+1)$-dimensional Dirac fermions.

Now we further turn on the torsional electric fields, i.e., $\tilde{T}^3_E = \partial_0 e^3_z - \partial_z e^3_0$. For simplicity, we set $e^3_0 = 0$ and $e^3_z = 1 + \Phi$, $\Phi \ll 1$. Due to the "minimal coupling," i.e., $\sigma^a \to e^\mu_a \sigma^a$, only $\sigma^3$ in the Hamiltonian are modified, and the dispersion relation now becomes

$$\mathcal{E}_s = \begin{cases} s(1-\Phi)|p_z| & n=0, \\ \pm\sqrt{|(1-\Phi)p_z|^2 + 2|n\tilde{T}_B^3 p_z|} & n^2 \geqslant 1, \end{cases} \tag{6}$$

where for $e^a_\mu = \delta^a_\mu + w^\mu_a$, $e^\mu_a \simeq \delta^\mu_a - \delta^\nu_a\delta^\mu_b w^b_\nu$ and thus $e^z_3 \simeq 1-\Phi$. This shows that the torsional electric fields affect the lowest Landau levels by modifying their slope (or velocity), i.e., $|p_z| \to (1-\Phi)|p_z|$.

The axial charge density $j^{5\mu}|_{\mu=0}$ can be written as

$$j^{50} = \sum_n \sum_s s\int_{-\infty}^{+\infty} \frac{dp_z}{2\pi} n_F(\mathcal{E}^n_s)\left(\frac{|p_z|\tilde{T}^3_B}{2\pi}\right), \tag{7}$$

where $n = 0, \pm 1, \dots$ is used to label the torsional Landau levels, $n_F(\mathcal{E}^n_s) = \frac{1}{\exp(\beta\mathcal{E}^n_s)+1}$ is the Fermi-Dirac distribution function, and $\frac{|p_z|\tilde{T}^3_B}{2\pi}$ is the level degeneracy. Because for $|n| \geqslant 1$, $\mathcal{E}^n_R = \mathcal{E}^n_L$, $\sum_{|n|>1}\sum_s s n_F(\mathcal{E}^n_s)|p_z| = 0$, so the contribution from the higher torsional Landau levels ($|n \geqslant 1$) to $j^{50}$ are canceled exactly.

Now, the axial charge density becomes

$$\begin{aligned}\left(\sum_s s j^0_s\right)/[\tilde{T}^3_B/(2\pi)] &= \int_{-\infty}^{+\infty}\frac{dp_z}{2\pi}|p_z|\left\{\frac{1}{\exp[\beta(1-\Phi)|p_z|]+1} - \frac{1}{\exp[-\beta(1-\Phi)|p_z|]+1}\right\} \\ &= \frac{1}{(1-\Phi)^2}2\int_{-\infty}^{+\infty}\frac{d\epsilon}{2\pi}\epsilon\frac{1}{\exp(\beta\epsilon)+1},\end{aligned} \tag{8}$$

where $\epsilon \equiv (1-\Phi)|p_z|$. If we regard $(1-\Phi)$ as the effective velocity, then terms in the second line are almost the energy density of the $(1+1)$-dimensional Dirac fermions. By "almost," we mean that the coefficient of $|p_z|$ in the integrand is not $(1-\Phi)$. In the last line, $2\int_{-\infty}^{+\infty}\frac{d\epsilon}{2\pi}\epsilon\frac{1}{\exp(\beta\epsilon)+1}$ is the energy density of two different kinds of fermions. This suggests the close relation between the Nieh-Yan anomaly and the energy density. It is also known that in two-dimensional conformal field theory, the free energy density is proportional to the central charge [48,49], so this equation also hints at the close relation between the Nieh-Yan anomaly and the central charge. This connection will be explored in the next section.

However, Eq. (8) is actually divergent, so we need to perform the integration carefully. To be more concrete, we rewrite Eq. (8) as

$$\frac{2}{(1-\Phi)^2}\left\{\int_{-\infty}^{+\infty}\frac{d\epsilon}{2\pi}\epsilon\left[\frac{1}{\exp(\beta\epsilon)+1} - \theta(-\epsilon)\right] + \int_{-\infty}^{+\infty}\frac{d\epsilon}{2\pi}\epsilon\theta(-\epsilon)\right\}. \tag{9}$$

The first term in the brackets is now convergent, i.e.,

$$\int_{-\infty}^{+\infty}\frac{d\epsilon}{2\pi}\frac{\epsilon}{2\pi}\left[\frac{1}{\exp(\beta\epsilon)+1} - \theta(-\epsilon)\right] = \frac{T^2}{4!}.$$

But the second term can be regularized by introducing a hard energy cutoff, i.e., $-\Lambda < \epsilon < \Lambda$. Then, the integral becomes

$$\int_{-\Lambda}^{\Lambda}\frac{d\epsilon}{2\pi}\epsilon\theta(-\epsilon) = -\frac{1}{4\pi^2}\Lambda^2,$$

where the positive $\epsilon$ part in the integral is eliminated by $\theta(-\epsilon)$ and the negative part is regularized by $-\Lambda$. Hence, $\Lambda$ measures the depth of the vacuum, and it stands for the vacuum energy. In reality, the depth of the vacuum is not universal. For example, it might depend on the concrete materials. By contrast, the term proportional to $T^2$ can be universal. As we shall show in the next section, it is proportional to the central charge.

By summing everything together,

$$j^{50} = \left(\frac{1}{12}T^2 - \frac{1}{4\pi^2}\Lambda^2\right)\tilde{T}^3_B + \left(\frac{1}{6}T^2 - \frac{1}{2\pi^2}\Lambda^2\right)\Phi\tilde{T}^3_B. \tag{10}$$

The first term in the parentheses is the chiral torsional effect obtained in Ref. [21]. In addition to the thermal energy, the vacuum energy can affect the chiral torsional effect as well, which leads to the $\Lambda^2$ term.

We can also study the chiral anomaly from Eq. (10). For example, by recasting $\partial_t j^{50}$ in a covariant form,

$$\frac{1}{\sqrt{|g|}}\partial_\mu\sqrt{|g|}j^{5\mu} = \left(\frac{\Lambda^2}{4\pi^2} - \frac{T^2}{12}\right)\frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{|g|}}\partial_\mu e^a_\nu\partial_\rho e^b_\sigma\eta_{ab}, \tag{11}$$

which is the Nieh-Yan term with a thermal contribution. The term proportional to $T^2$ can be understood as follows. Under torsional electric fields, the velocity of the lowest torsional Landau levels is changed. For example, in Fig. 2, the velocities

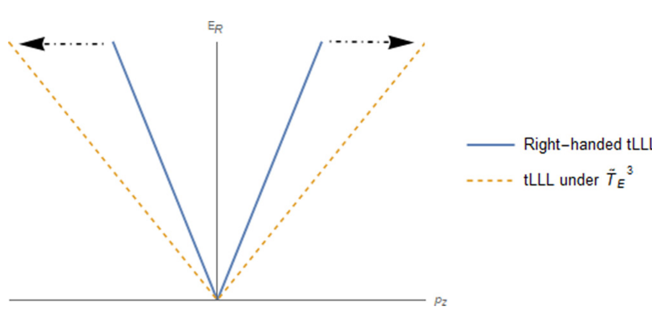


FIG. 2. An illustration of the Nieh-Yan anomaly under torsional electric fields $\tilde{T}_E = \partial_t\Phi$ (tLLL stands for the torsional lowest Landau levels). The torsional electric fields change the slope of the lowest Landau level, for example, from the thick line (velocity $v = 1$) to the dashed line [velocity $v = (1-\Phi)$]. Because the free-energy density of two-dimensional conformal field theory is $\frac{\pi c T^2}{6}\frac{1}{v}$ [48,49], the change in chiral current density is $\Delta j^{50} = (\frac{1}{v^2} - 1)(\frac{\pi c T^2}{6})(\frac{\tilde{T}^3_B}{2\pi})$, where $\frac{\tilde{T}^3_B}{2\pi}$ is from the level degeneracy. Thus, $\frac{\Delta j^{50}}{\Delta t} = \frac{c}{6}T^2\tilde{T}^3_E\tilde{T}^3_B$.

(or slopes) of the thick solid line and dashed line are $v = 1$ and $v = (1-\Phi)$, respectively, where $\Phi = \tilde{T}_E^3 \Delta t$. It is known that for two-dimensional conformal fields, the free-energy density at velocity $v$ is $\mathcal{F}(v) = \frac{\pi c}{6} T^2 \frac{1}{v}$ [48,49], where $c$ is the central charge and $c = 1$ for the $(1+1)$-dimensional Dirac fermions here. In Eq. (8), the chiral density is $[\frac{1}{v} \times \mathcal{F}(v)] \frac{\tilde{T}_B^3}{2\pi}$, where $\frac{1}{v}$ is because the torsional Landau level degeneracy is not affected by the torsional electric fields. Thus, the change in chiral density is $\Delta j^{50} = [\frac{1}{(1-\Phi)^2} - 1](\frac{\pi c}{6} T^2)(\frac{\tilde{T}_B^3}{2\pi})$ or $\Delta j^{50} \simeq \frac{c}{6} T^2 \tilde{T}_E^3 \tilde{T}_B^3 \Delta t$, with $c = 1$.

This implies that the prefactor of the Nieh-Yan term is actually the free-energy density. Especially, the coefficient of the first term is from

$$\frac{T^2}{12} = \left(\frac{c\pi T^2}{6}\right)\left(\frac{1}{2\pi}\right), \tag{12}$$

where $c = 1$, $\frac{1}{2\pi}$ is from the level degeneracy and $\frac{c\pi T^2}{6}$ is the free-energy density. This strongly suggests the close relation between the central charge and the Nieh-Yan anomaly.

## IV. THE (1 + 1)-DIMENSIONAL EFFECTIVE MODEL AND CENTRAL CHARGE

In the last section, we derived both the Nieh-Yan anomaly and the chiral torsional effect from the lowest torsional Landau levels, which seem to relate to the central charge. In this section, we shall study this connection by projecting the $(3+1)$-dimensional system onto its lowest torsional Landau levels, which is effectively the $(1+1)$-dimensional Dirac theory.

Since only the lowest Landau levels contribute to the anomaly equation, we can project our $(3+1)$-dimensional system onto the lowest torsional Landau levels. Then, the effective Lagrangian is

$$\mathcal{L} = \psi^\dagger (i\partial_t + |p_z|\sigma^3)\psi. \tag{13}$$

The corresponding chiral current defined in two dimensions is

$$j_c^\mu = \bar{\psi}\Gamma^\mu\Gamma^5\psi, \tag{14}$$

where $\Gamma^\mu$ and $\Gamma^5$ are the gamma matrices defined on the $(1+1)$-dimensional space-time. We can further recast the Lagrangian above as

$$\mathcal{L} = \bar{\psi}'(i\partial_\mu\Gamma^\mu)\psi', \tag{15}$$

where $\psi'(p_z) = \binom{\psi_R}{\psi_L}$ for $p_z > 0$ and $\psi'(p_z) = \binom{\psi_L}{\psi_R}$ for $p_z < 0$.

Hence, in terms of $\psi'$, Eq. (14) can be written as $j_c^\mu(p_z) = \mathrm{sgn}(p_z)\bar{\psi}'\Gamma^\mu\Gamma^5\psi'$. Notice that the level degeneracy of the torsional Landau level is $\frac{1}{2\pi}\tilde{T}_B^3|p_z|$, so the actual chiral current in the $(3+1)$-dimensional space-time is $j^{5\mu} = (\frac{\tilde{T}_B^3}{2\pi})(\bar{\psi}'\Gamma^\mu\Gamma^5 p_z\psi')$. By using the identity $\Gamma^5\Gamma^\mu = \epsilon^{\mu\nu}\Gamma_\nu$, $j^{5\mu} = (\epsilon^{\mu\nu}\bar{\psi}'\Gamma_\nu p_z\psi')(\frac{\tilde{T}_B^3}{2\pi})$, where, up to equations of motion, $\bar{\psi}'\Gamma_\nu p_\mu\psi'$ is the canonical energy-momentum tensor from the Noether theorem. Hence, we shall calculate $T_{\mu\nu}$ to obtain $j^{5\mu}$, i.e.,

$$j^{5\mu} = \epsilon^{\mu\nu}T_{\nu 3}\left(\frac{\tilde{T}_B^3}{2\pi}\right), \tag{16}$$

where $\mu,\ \nu = 0,\ 3$. Since we are most interested in the finite-temperature chiral current, we compactify the temporal direction to a circle of radius $\beta = T^{-1}$. By using the Schwarzian derivative, one can obtain [50]

$$T_{33} = \frac{c\pi T^2}{6}, \quad T_{00} = \frac{c\pi T^2}{6}, \tag{17}$$

where $c$ is the central charge. Roughly speaking, $c$ counts the degrees of freedom. To see this, let us compactify the spatial direction (to a circle of radius $T^{-1}$) instead, and the absolute value of the energy density is $|T\sum_n (2\pi nT)| = \frac{\pi T^2}{6}$, which means that each independent mode will contribute a factor $\frac{\pi T^2}{6}$ to the energy density, and thus, $c$ counts the number of different modes. By inserting Eq. (17) back into Eq. (16),

$$j^{50} = \left(\frac{cT^2}{12}\right)\tilde{T}_B^3,$$

which is exactly the chiral torsional effect we obtained in Eq. (10). This can be recast in a covariant form as

$$j^{5\mu} = -\frac{cT^2}{12}\frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{|g|}}\eta_{ab}\delta_\nu^a\partial_\rho e_\sigma^b, \tag{18}$$

and thus, the chiral anomaly is

$$\frac{1}{\sqrt{|g|}}\partial_\mu\sqrt{|g|}j^{5\mu} = -\left(\frac{cT^2}{12}\right)\frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{|g|}}\partial_\mu e_\nu^a\partial_\rho e_\sigma^b\eta_{ab}, \tag{19}$$

which is similar to the results obtained in the last section. Since the expectation value of the energy-momentum tensor is known to be the free-energy density [50], we have shown explicitly that the prefactor of the Nieh-Yan anomaly and the chiral torsional effect is the free-energy density. Interestingly, we have shown that both the chiral torsional effect and the thermal Nieh-Yan anomaly are proportional to the central charge.

Compared to our results in Eq. (10), terms proportional to cutoff do not appear. This is because the vacuum energy is from the normal ordering of the creation and annihilation operators, which is secretly thrown away in conformal field theory due to the constraints of translational symmetry and rotational symmetry. However, in realistic materials, both the translational symmetry and the rotational symmetry can be broken by the ultraviolet physics. This means that the $\Lambda^2$ term might exist in the Nieh-Yan anomaly, but it is not universal and depends on the concrete systems.

## V. ANOMALOUS THERMAL HALL EFFECT IN WEYL SEMIMETALS

In this section, we shall apply the thermal Nieh-Yan anomaly to Weyl semimetals. The anomalous thermal Hall effect naturally arises as the experimental signature of the thermal Nieh-Yan anomaly. The anomalous thermal Hall conductance is then shown to be proportional to the central charge.

The anomaly equation in Eq. (19) implies that $\langle j^{5\mu}\rangle = -\frac{cT^2}{12}\frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{|g|}}e_\nu^a\partial_\rho e_\sigma^b\eta_{ab}$. Hence, the effective action in Weyl semimetals can be written as

$$S_{\rm eff} = -\int d^4x\sqrt{|g|}b_\mu\langle j^{5\mu}\rangle + \cdots$$
$$= \frac{cT^2}{12}\int d^4x\epsilon^{\mu\nu\rho\sigma}b_\mu e^a_\nu\partial_\rho e^b_\sigma\eta_{ab} + \cdots, \tag{20}$$

where $b^\mu$ is the separation between Weyl nodes in the energy-momentum space and we have kept only terms with linear dependence on $b$. By performing variation of the vierbeins, the energy-momentum response current is given as

$$T^\mu_a = -\frac{cT^2}{6}\eta_{ab}\frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{|g|}}b_\nu\partial_\rho e^b_\sigma. \tag{21}$$

Especially, for $a = 0$, $T^\mu_0$ is the energy current, which is given as

$$\boldsymbol{j}_T = \frac{cT}{6}\boldsymbol{b}\times(\nabla T). \tag{22}$$

Here $(\boldsymbol{j}_T)^i = T^i_0$ is the thermal current, and we have used $\partial_0 e^0_i - \partial_i e^0_0 = (T^{-1}\nabla T)_i$ [7,8]. This is the anomalous thermal Hall effect in Weyl semimetals, and it is shown to be proportional to the central charge, which matches exactly that calculated from the Kubo formula by using the lattice model in the low-temperature limit [51]. Because the central charge closely relates to the conformal anomaly in two-dimensional space-time, the anomalous thermal Hall effect in Weyl semimetals is thus expected to be protected by topology. In addition, compared to the anomalous quantum Hall effect in Weyl semimetals, i.e., $\boldsymbol{j} = \frac{1}{2\pi^2}\boldsymbol{b}\times\boldsymbol{E}$, the ratio of the anomalous thermal Hall conductivity and the anomalous Hall conductivity is $\frac{c\pi^2 T}{3}$, which matches the Wiedemann-Franz law exactly.

Equation (21) also tells us how the temperature affects the momentum transport in dislocations, i.e., $\boldsymbol{j}_{p_m} = \frac{cT^2}{6}b_0\nabla\times\boldsymbol{e}^m$, where $b_0$ is the chiral chemical potential, $(\boldsymbol{j}_{p_m})^i = T^i_m$ is the $p_m$ momentum current, and $(\boldsymbol{e}^m)_i = -e^{a=m}_{\mu=i}$. Thus, the total momentum transferred along a dislocation is

$$J_{p_m} = \int_M (d\boldsymbol{S})\cdot\boldsymbol{j}_{p_m} = -\frac{cT^2 b_0}{6}b^m_{\rm bur},$$

where $M$ is a surface area containing dislocations and $\boldsymbol{b}_{\rm bur}$ is the Burger vector, i.e., $\oint dx^\mu\wedge dx^\nu\partial_\mu e^a_\nu = b^m_{\rm bur}$.

## VI. CONCLUSION

In summary, we have calculated the torsional Landau levels, from which the Nieh-Yan anomaly is derived. It was shown that the coefficient of the Nieh-Yan anomaly is the free-energy density of the lowest torsional Landau levels. By projecting the system onto the lowest torsional Landau levels consisting of the $(1+1)$-dimensional Dirac fermions, we related the Nieh-Yan anomaly to the central charge and thus the conformal anomaly. The anomalous thermal Hall effect in Weyl semimetals arises as the direct consequence of the thermal Nieh-Yan anomaly, which is shown to be proportional to the central charge and can be regarded as the experimental signature of the thermal Nieh-Yan anomaly. We have clarified the physical mechanism behind the Nieh-Yan anomaly and revealed the topological nature of the thermal Nieh-Yan anomaly.

For time-reversal symmetry-protected topological insulators, we assume that the negative-mass insulators are the topologically nontrivial ones. The effective action can be obtained by performing a chiral transformation to reverse the sign of the mass. Similarly, the corresponding torsional effective action for these topological insulators can be derived from our anomaly equation here, i.e.,

$$S_{\rm TI} = \frac{\pi cT^2}{24}\int d^4x\eta_{ab}\epsilon^{\mu\nu\rho\sigma}\partial_\mu e^a_\nu\partial_\rho e^b_\sigma, \tag{23}$$

which suggests the existence of the thermal counterpart of the magnetoelectric effect in topological insulators.

In addition, the descent relation between the chiral anomaly and the parity anomaly suggests that there is a corresponding thermal parity anomaly in $(2+1)$ dimensions originating from the thermal Nieh-Yan anomaly, i.e.,

$$S_{\rm TH} = \frac{\pi cT^2}{12}\int \eta_{ab}\epsilon^{\mu\nu\rho}e^a_\mu\partial_\nu e^b_\rho, \tag{24}$$

which is proportional to the central charge. This can be used to describe the thermal Hall effect and maybe the topological Hall viscosity.

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[1] F. W. Hehl, P. von der Heyde, G. D. Kerlick, and J. M. Nester, Rev. Mod. Phys. **48**, 393 (1976).
[2] M. Katanaev and I. Volovich, Ann. Phys. (N.Y.) **216**, 1 (1992).
[3] T. L. Hughes, R. G. Leigh, and E. Fradkin, Phys. Rev. Lett. **107**, 075502 (2011).
[4] T. L. Hughes, R. G. Leigh, and O. Parrikar, Phys. Rev. D **88**, 025040 (2013).
[5] O. Parrikar, T. L. Hughes, and R. G. Leigh, Phys. Rev. D **90**, 105004 (2014).
[6] J. M. Luttinger, Phys. Rev. **135**, A1505 (1964).
[7] A. Shitade, Prog. Theor. Exp. Phys. **2014**, 123I01 (2014)
[8] G. Tatara, Phys. Rev. Lett. **114**, 196601 (2015).
[9] B. Bradlyn and N. Read, Phys. Rev. B **91**, 125303 (2015).
[10] G. E. Volovik, *The Universe in a Helium Droplet* (Oxford University Press, Oxford, 2003).
[11] O. Golan and A. Stern, Phys. Rev. B **98**, 064503 (2018).
[12] J. Nissinen, arXiv:1909.05846.

[13] N. P. Armitage, E. J. Mele, and A. Vishwanath, Rev. Mod. Phys. **90**, 015001 (2018).
[14] A. Burkov, Annu. Rev. Condens. Matter Phys. **9**, 359 (2018).
[15] M. N. Chernodub and M. A. Zubkov, Phys. Rev. B **95**, 115410 (2017).
[16] Z.-M. Huang, L. Li, J. Zhou, and H.-H. Zhang, Phys. Rev. B **99**, 155152 (2019).
[17] H. Sumiyoshi and S. Fujimoto, Phys. Rev. Lett. **116**, 166601 (2016).
[18] L. Sun and S. Wan, Europhys. Lett. **108**, 37007 (2014).
[19] Y. You, G. Y. Cho, and T. L. Hughes, Phys. Rev. B **94**, 085102 (2016).
[20] G. Palumbo and J. K. Pachos, Ann. Phys. (N.Y.) **372**, 175 (2016).
[21] Z. V. Khaidukov and M. A. Zubkov, JETP Lett. **108**, 670 (2018).
[22] Y. Ferreiros, Y. Kedem, E. J. Bergholtz, and J. H. Bardarson, Phys. Rev. Lett. **122**, 056601 (2019).
[23] S. Ryu, J. E. Moore, and A. W. W. Ludwig, Phys. Rev. B **85**, 045104 (2012).
[24] E. Witten, Rev. Mod. Phys. **88**, 035001 (2016).
[25] A. Vilenkin, Phys. Rev. D **20**, 1807 (1979).
[26] A. Vilenkin, Phys. Rev. D **22**, 3080 (1980).
[27] K. Fukushima, D. E. Kharzeev, and H. J. Warringa, Phys. Rev. D **78**, 074033 (2008).
[28] K. Landsteiner, E. Megías, and F. Pena-Benitez, Phys. Rev. Lett. **107**, 021601 (2011).
[29] A. A. Zyuzin and A. A. Burkov, Phys. Rev. B **86**, 115133 (2012).
[30] M. A. Stephanov and Y. Yin, Phys. Rev. Lett. **109**, 162001 (2012).
[31] J. Zhou, H. Jiang, Q. Niu, and J. Shi, Chin. Phys. Lett. **30**, 027101 (2013).
[32] C.-Z. Li, L.-X. Wang, H. Liu, J. Wang, Z.-M. Liao, and D.-P. Yu, Nat. Commun. **6**, 10137 (2015).
[33] B. Z. Spivak and A. V. Andreev, Phys. Rev. B **93**, 085107 (2016).
[34] H. Li, H. He, H.-Z. Lu, H. Zhang, H. Liu, R. Ma, Z. Fan, S.-Q. Shen, and J. Wang, Nat. Commun. **7**, 10301 (2016).
[35] Q. Li, D. E. Kharzeev, C. Zhang, Y. Huang, I. Pletikosic, A. V. Fedorov, R. D. Zhong, J. A. Schneeloch, G. D. Gu, and T. Valla, Nat. Phys. **12**, 550 (2016).
[36] Z. Jia, C. Li, X. Li, J. Shi, Z. Liao, D. Yu, and X. Wu, Nature Comm. **7**, 13013 (2016).
[37] E. V. Gorbar, V. A. Miransky, I. A. Shovkovy, and P. O. Sukhachov, Phys. Rev. B **96**, 085130 (2017).
[38] Z.-M. Huang, J. Zhou, and S.-Q. Shen, Phys. Rev. B **96**, 085201 (2017).
[39] M. Stone and J. Y. Kim, Phys. Rev. D **98**, 025012 (2018).
[40] H. Nieh and M. Yan, Ann. Phys. (N.Y.) **138**, 237 (1982).
[41] H. Nieh and M. Yan, J. Math. Phys. **23**, 373 (1982).
[42] O. Chandía and J. Zanelli, Phys. Rev. D **63**, 048502 (2001).
[43] D. Kreimer and E. W. Mielke, Phys. Rev. D **63**, 048501 (2001).
[44] J. Nissinen, G. E. Volovik, arXiv:1909.08936.
[45] S. L. Adler, Phys. Rev. **177**, 2426 (1969).
[46] J. S. Bell and R. Jackiw, Nuovo Cimento A **60**, 47 (1969).
[47] H. B. Nielsen and M. Ninomiya, Phys. Lett. B **130**, 389 (1983).
[48] I. Affleck, Phys. Rev. Lett. **56**, 746 (1986).
[49] H. W. J. Blöte, J. L. Cardy, and M. P. Nightingale, Phys. Rev. Lett. **56**, 742 (1986).
[50] P. Di Francesco, P. Mathieu, and D. Sénéchal, *Conformal Field Theory*, Graduate Texts in Contemporary Physics (Springer-Verlag, New York, 2012).
[51] E. V. Gorbar, V. A. Miransky, I. A. Shovkovy, and P. O. Sukhachov, Phys. Rev. B **96**, 155138 (2017).