Rapid Communications

Defective edge states and number-anomalous bulk-boundary correspondence in non-Hermitian topological systems

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Non-Hermitian topological systems exhibit properties very different from those of their Hermitian counterparts. An important puzzling issue for non-Hermitian topological systems is the existence of defective edge states beyond the usual bulk-boundary correspondence (BBC). In this Rapid Communication, to understand the existence of these defective edge states, the number-anomalous bulk-boundary correspondence (NA-BBC) theory, which distinguishes the non-Bloch BBC, is developed. With the one-dimensional non-Hermitian Su-Schrieffer-Heeger model taken as an example, the underlying physics of the defective edge states is explored. The defective edge states are a consequence of non-Hermitian coalescence from the anomalous edge Hamiltonian. In addition, with the help of a theorem, the number anomaly of the edge states in non-Hermitian topological systems becomes a mathematical problem in quantitative calculations when identifying the normal/non-normal non-Hermitian condition for the edge Hamiltonian and verifying the deviation of the BBC ratio from 1. In the future, NA-BBC theory can be generalized to higher-dimensional non-Hermitian topological systems (for example, the two-dimensional Chern insulator).

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Introduction. Topological systems, including topological insulators and topological superconductors, are new types of exotic quantum phases of matter that have become the forefront of condensed-matter physics for many years [1–5]. An important topological property for different types of topological systems is the bulk-boundary correspondence (BBC), i.e., bulk topological invariants that characterize topological systems as corresponding to unique gapless boundary (edge) states. Recently, non-Hermitian (NH) topological systems have been intensively studied in both theory [6–43] and experiments [44–52] . The topological properties of NH systems are very different from those of their Hermitian counterparts, including the fractional topological invariant and defective edge states [11,22], the breakdown of the usual BBC [16,19–21,35–37,39,42] and the NH skin effect [19,26,29,40,42]. Recently, within the framework of Altland-Zirnbauer theory, the classification of NH systems with topological bands was characterized by different symmetry-protected topological invariants [18,31,32].

Puzzle about defective edge states. An important *puzzling* issue for NH topological systems is the existence of defective edge states (DESs) beyond the usual BBC. A DES is edge state on the ends of an one-dimensional (1D) finite NH topological system with non-Hermitian coalescence, which was first discovered numerically by Lee in Ref. [11]. In Ref. [11], Lee noted that the *fractional winding number* guarantees the existence of DESs and a new type of BBC. However, due to the NH skin effect, the fractional winding number corresponding to DESs fails. In Ref. [19], Yao and Wang noted that the *non-Bloch topological invariant* rather than the fractional winding number characterizes the topological properties of the NH topological system. The energy spectra of the NH topological system under the open boundary condition (OBC) may differ from those under the periodic boundary condition (PBC). The new BBC is called a non-Bloch BBC or *NB-BBC*. With the help of the NB-BBC, people can obtain the correct topological phase diagram and know whether the edge states exist [19]. Instead of a single DES, according to the NB-BBC, two edge states are predicted, located either on each side separately or together on one (left or right) side only.

Thus, the situation becomes confusing: According to numerical calculations, DESs exist in NH topological systems. To accurately predict the existence of a single DES, one should obtain the topological phase diagram to know whether the edge states exist and check the number of edge states to know whether the edge states become defective. However, neither the fractional winding number nor the non-Bloch topological invariant accurately predict the existence of a single DES. The following question, thus, arises: *How can we understand the existence of the DESs*?

The key point of the answers is that the number of edge states becomes anomalous, which distinguishes the NB-BBC with a phase-boundary anomaly. We call the anomalous BBC with the number anomaly of the edge states the *NA-BBC*. In this Rapid Communication, we try to solve this puzzle after answering the following questions:

(1) What is the *underlying* physics for DESs and the NA-BBC?

(2) How can we *accurately* characterize the DESs and NA-BBC?

(3) Are the DESs *stable* in the thermodynamic limit?

(4) Do *universal* features exist for the NA-BBC?

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Quantitative description of number-anomalous bulk-boundary correspondence. First, we consider the boundary physics for an arbitrary 1D finite NH topological system of which the Hamiltonian is $\hat{H}_{\rm NH}$ ($\hat{H}_{\rm NH} \neq \hat{H}_{\rm NH}^{\dagger}$). From the classification of NH topological systems [31,32], in the topological phase, the topological invariant $\mathcal{Z}$ may guarantee the number of edge states $\mathcal{C}_{\rm finite}$ in a finite 1D system. The biorthogonal set for the quantum states of the boundary/edge modes is defined by $|\psi_k\rangle$ and $|\Psi_k\rangle$, i.e., $\hat{H}_{\rm NH}|\psi_k\rangle = E_k|\psi_k\rangle$, $\hat{H}_{\rm NH}^{\dagger}|\Psi_k\rangle = (E_k)^*|\Psi_k\rangle$ ($k = 1, \ldots, 2\mathcal{Z}$ is the state index) [53,54].

For the Hermitian case, the topological invariant $\mathcal{Z}$ may guarantee the number of edge states $\mathcal{C}_{\rm finite}$ on left/right end of which the eigenstates can be obtained by considering the semi-infinite topological systems. As a result, the usual BBC for finite Hermitian topological systems is denoted by $\mathcal{C}_{\rm finite} = 2\mathcal{Z}$. For NH topological systems, the situation changes. The NA-BBC may occur. The number of edge states cannot be $2\mathcal{Z}$ and becomes fractional, i.e.,

$$\mathcal{C}_{\rm finite} \neq 2\mathcal{Z}. \tag{1}$$

A special case of the NA-BBC is $\mathcal{C}_{\rm finite} = \mathcal{Z}$, which indicates the existence of a *singular* DES. We emphasize that the singular DES may have a distribution on both edges.

In this Rapid Communication, we focus on a 1D finite NH topological system with $\mathcal{Z} = 1$. To quantitatively characterize the edge physics for a 1D finite NH topological system, we introduce an *effective edge Hamiltonian*,

$$\hat{\mathcal{H}}_{\rm edge} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}, \tag{2}$$

where $h_{IJ} = \langle b^I|\hat{H}_{\rm NH}|b^J\rangle$, $I, J = 1, 2$. $\binom{|b^1\rangle}{|b^2\rangle}$ are the bases under a biorthogonal set of the edge states. With the help of $\hat{\mathcal{H}}_{\rm edge}$, the boundary phase diagram of the NH topological systems can be obtained in a straightforward manner.

To develop a clear quantitative description of the DESs and the corresponding NA-BBC, we introduce three new concepts:

Definition 1. Normal/non-normal NH condition: A NH Hamiltonian $\hat{H}$ $[\hat{H} \neq (\hat{H})^{\dagger}]$ *can be written as* $\hat{H} = \hat{H}_h + \hat{H}_a$, *where* $\hat{H}_h = \frac{1}{2}(\hat{H} + \hat{H}^{\dagger})$ *and* $\hat{H}_a = \frac{1}{2}(\hat{H} - \hat{H}^{\dagger})$ *are the Hermitian part and anti-Hermitian part, respectively. A Hamiltonian* $\hat{H}$ *is a normal NH if* $[\hat{H}_h, \hat{H}_a] = 0$; *a Hamiltonian* $\hat{H}$ *is a non-normal NH if* $[\hat{H}_h, \hat{H}_a] \neq 0$ [55].

Definition 2. State similarity of edge states: The state similarity for two edge states $|\psi_+\rangle$ *and* $|\psi_-\rangle$ *is* $|\langle\psi_+|\psi_-\rangle|$. *Here,* $|\psi_\pm\rangle$ *is the satisfied self-normalization condition, i.e.,* $|\langle\psi_\pm|\psi_\pm\rangle| \equiv 1$.

Definition 3. BBC ratio: The BBC ratio is defined by $\gamma_{\rm BBC} = 1 - |\langle\psi_+|\psi_-\rangle|/2$, *and* $\mathcal{C}_{\rm finite} = 2\gamma_{\rm BBC}$ *is the total number of edge states for a finite NH system.* $|\psi_\pm\rangle$ *are the two edge states.*

For a 1D finite NH topological system with $\mathcal{Z} = 1$, $\gamma_{\rm BBC}$ is a quantity that characterizes the number anomaly of the edge states. When $\hat{\mathcal{H}}_{\rm edge}$ obeys the normal NH condition, the eigenstates ($|\psi_+\rangle$ and $|\psi_-\rangle$) are orthogonal, and the number of edge states is 2 with $\gamma_{\rm BBC} = 1$. The usual BBC exists. When $\hat{\mathcal{H}}_{\rm edge}$ obeys the non-normal NH condition, the eigenstates ($|\psi_+\rangle$ and $|\psi_-\rangle$) become similar, and the number of edge states $\mathcal{C}_{\rm finite}$ is from 1 to 2 with $\frac{1}{2} < \gamma_{\rm BBC} < 1$. According to $\mathcal{C}_{\rm finite} \neq 2$, NA-BBC occurs. If $\gamma_{\rm BBC}$ is equal to 1/2, then a singular DES exists, and there exists one edge state ($\mathcal{C}_{\rm finite} = 1$). As a result, the existence of DESs and NA-BBC in NH topological systems becomes a well-defined mathematical problem under quantitative calculations. The detailed proof of this statement is given in the Supplemental Material [56]. In the following parts, we take the 1D nonreciprocal Su-Schrieffer-Heeger (SSH) model as an example to explore the underlying physics of NA-BBC and DESs.

The defective edge states as boundary exceptional points in the non-reciprocal Su-Schrieffer-Heeger model. The Hamiltonian for a non-reciprocal SSH model with N pairs of lattice sites is given by

$$\begin{aligned}\hat{H}_{\text{NH-SSH}} = {}& (t_1-\gamma)\sum_{n=1}^{N}|n,B\rangle\langle n,A| + (t_1+\gamma)\sum_{n=1}^{N}|n,A\rangle\langle n,B| \\ & + t_2\sum_{n=1}^{N-1}|n+1,A\rangle\langle n,B| + t_2\sum_{n=1}^{N-1}|n,B\rangle\langle n+1,A|.\end{aligned} \tag{3}$$

t_1 and t_2 describe the intracell and intercell hopping strengths, respectively. γ describes the unequal intracell hoppings. t_1, t_2, and γ are all real. A and B denote the sublattice subspace. In this Rapid Communication, we set $t_2 = 1$. The Bloch Hamiltonian for a nonreciprocal SSH model under the PBC is $\hat{H}_{\text{NH-SSH}}(k) = (t_1 + t_2 \cos k)\sigma_x + (t_2 \sin k + i\gamma)\sigma_y$ with sublattice symmetry with $\sigma_z\hat{H}_{\text{NH-SSH}}(k)\sigma_z = -\hat{H}_{\text{NH-SSH}}(k)$. σ_i denotes the Pauli matrices acting on the (A or B) sublattice subspace.

First, we consider the Hermitian SSH model with $\gamma = 0$. Now, the topological invariant is a winding number $w = \frac{1}{2\pi}\int_{-\pi}^{\pi}\partial_k\phi(k)dk$, where $\phi(k) = \tan^{-1}(d_y/d_x)$ with $d_y = t_2 \sin k$ and $d_x = t_1 + t_2 \cos k$. Two phases exist for the global bulk phase diagram: a topological phase with $w = 1$ in the region of $|t_1| < |t_2|$ and a trivial phase with $w = 0$ in the region of $|t_1| > |t_2|$. In the topological phase, the basis of the edge states is given by [57], $\binom{|b^1\rangle}{|b^2\rangle}$ of which the wave function is $|b^1\rangle = \frac{1}{\mathcal{N}}\sum_{n=1}^{N}(-\frac{t_1}{t_2})^{n-1}|n\rangle \otimes \binom{1}{0}$ or $|b^2\rangle = \frac{1}{\mathcal{N}}\sum_{n=0}^{N-1}(-\frac{t_1}{t_2})^{n}|N-n\rangle \otimes \binom{0}{1}$, where $\mathcal{N} = \sqrt{[1-(\frac{t_1}{t_2})^{2N}]/[1-(\frac{t_1}{t_2})^2]}$ is the normalization factor and $\binom{1}{0}$ and $\binom{0}{1}$ denote the state vectors of two sublattices. The orthogonal condition for the two edge states is $\langle b^1|b^2\rangle = 0$. To characterize the dynamics of the two edge states of the finite NH SSH model, we derive the effective edge Hamiltonian $\hat{\mathcal{H}}_{\rm edge} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$, where $h_{IJ} = \langle b^I|\hat{H}_{\text{NH-SSH}}(\gamma = 0)|b^J\rangle$, $I, J = 1, 2$. For this case, we have $h_{11} = h_{22} = 0$, $h_{12} = h_{21} = \Delta_0 = \frac{(t_1^2-t_2^2)}{t_2}(-\frac{t_1}{t_2})^N$ and obtain an effective edge Hamiltonian for the Hermitian SSH model of

$$\hat{\mathcal{H}}_{\rm edge} = \Delta_0\tau^x, \tag{4}$$

where τ^i denotes the Pauli matrices acting on the subspace of two edge states.

When considering unequal intracell hops ($\gamma \neq 0$), DESs appear. Correspondingly, the (non-Bloch) BBC becomes anomalous.

Under the OBC, to address the NH skin effect [19,20], we perform a similar transformation, i.e.,

$$\begin{aligned}\bar{H}_{\text{NH-SSH}}(k) &= (\hat{\mathcal{S}}_{\text{NHP}})^{-1}\hat{H}_{\text{NH-SSH}}(k)\hat{\mathcal{S}}_{\text{NHP}}\\ &= (\bar{t}_1 + \bar{t}_2\cos k)\sigma_x + \bar{t}_2 \sin k\sigma_y,\end{aligned} \tag{5}$$

where $\hat{\mathcal{S}}_{\text{NHP}}$ is the operation for the similar transformation. Under $\hat{\mathcal{S}}_{\text{NHP}}$, we remove the imaginary wave vector, i.e., $|\psi(k)\rangle \rightarrow |\bar{\psi}(k)\rangle = (\hat{\mathcal{S}}_{\text{NHP}})^{-1}|\psi(k - iq_0)\rangle$ by carrying out a NH transformation $U_{\text{NHP}} = \begin{pmatrix}1 & 0\\ 0 & e^{-q_0}\end{pmatrix}$ and a site-dependent scaling transformation $|n\rangle \rightarrow |\bar{n}\rangle = e^{-q_0(n-1)}|n\rangle$ (n denotes the cell number) $e^{q_0} = \sqrt{\frac{t_1-\gamma}{t_1+\gamma}}$. Consequently, the effective hopping parameters become $\bar{t}_1 = \sqrt{(t_1-\gamma)(t_1+\gamma)}$, and $\bar{t}_2 = t_2$. Now, the topological transition occurs at $|\bar{t}_1| = |\bar{t}_2|$.

In the topological phase $|\bar{t}_1| < |\bar{t}_2|$, the edge states are protected by the non-Bloch topological invariant determined by $\bar{H}_{\text{NH-SSH}}$, $\bar{w}$ [19,20]. Because $\bar{w} = 1$, based on the biorthogonal set, the basis of the edge states is given by [11,38,43] $\binom{|b^1\rangle}{|b^2\rangle}$ of which the wave function is $|b^1\rangle = \frac{1}{\bar{\mathcal{N}}}\sum_{n=1}^{N}(-\frac{\bar{t}_1}{\bar{t}_2})^{n-1}e^{q_0(n-1)}|n\rangle \otimes \binom{1}{0}$ or $|b^2\rangle = \frac{1}{\bar{\mathcal{N}}}\sum_{n=0}^{N-1}(-\frac{\bar{t}_1}{\bar{t}_2})^{n}e^{-q_0 n}|N-n\rangle \otimes \binom{0}{1}$, where $\bar{\mathcal{N}} = \mathcal{N}(t_1 \rightarrow \bar{t}_1,\ t_2 \rightarrow \bar{t}_2) = \sqrt{[1-(\frac{\bar{t}_1}{\bar{t}_2})^{2N}]/[1-(\frac{\bar{t}_1}{\bar{t}_2})^{2}]}$ is the normalization factor. Because the system with $\bar{t}_1 = \sqrt{(t_1-\gamma)(t_1+\gamma)} = 0$ is a singularity that corresponds to the exceptional points (EPs) of all bulk states, in this Rapid Communication, we focus on the case of $\bar{t}_1 \neq 0$.

Under the replacement $t_1 \rightarrow \bar{t}_1,\ t_2 \rightarrow \bar{t}_2$, the effective edge Hamiltonian becomes

$$\hat{\mathcal{H}}_{\text{edge}}(t_1, t_2) = \Delta_0\tau^x \rightarrow \bar{\mathcal{H}}_{\text{edge}}(\bar{t}_1, \bar{t}_2) = \bar{\Delta}\tau^x, \tag{6}$$

where $\bar{\Delta} = \frac{(\bar{t}_1^2-\bar{t}_2^2)}{\bar{t}_2}(-\frac{\bar{t}_1}{\bar{t}_2})^N$. The energy levels are correct when $E_{\pm} = \pm\bar{\Delta}$. However, although the energy levels are correct, the detailed numerical calculations *cannot* support this effective edge Hamiltonian $\bar{\mathcal{H}}_{\text{edge}}$, for example, in the numerical results,

$$h_{12} = \langle b^1|\hat{H}_{\text{NH-SSH}}|b^2\rangle \neq h_{21} = \langle b^2|\hat{H}_{\text{NH-SSH}}|b^1\rangle. \tag{7}$$

What is wrong with $\bar{\mathcal{H}}_{\text{edge}}$? The mismatch comes from overlooking the NH polarization effect on the two edge states.

To derive the correct result with the help of a similar transformation, we must consider an *additional* NH boundary polarization operation on the edge modes, i.e., $\tau^x \rightarrow U_{\text{edge}}^{-1}\tau^x U_{\text{edge}} = \tau^x\cosh(q_0N) - i\tau^y\sinh(q_0N)$, where

$$U_{\text{edge}} = \begin{pmatrix}1 & 0\\ 0 & e^{-q_0N}\end{pmatrix}. \tag{8}$$

After considering U_{edge}, the effective Hamiltonian $\bar{\mathcal{H}}_{\text{edge}}$ becomes *anomalous*, i.e.,

$$\begin{aligned}\check{\mathcal{H}}_{\text{eff}} &= (U_{\text{edge}})^{-1}\bar{\mathcal{H}}_{\text{edge}}(U_{\text{edge}})\\ &= \begin{pmatrix}0 & \bar{\Delta}e^{-q_0N}\\ \bar{\Delta}e^{q_0N} & 0\end{pmatrix} = \bar{\Delta}^{+}\tau^{+} + \bar{\Delta}^{-}\tau^{-},\end{aligned} \tag{9}$$

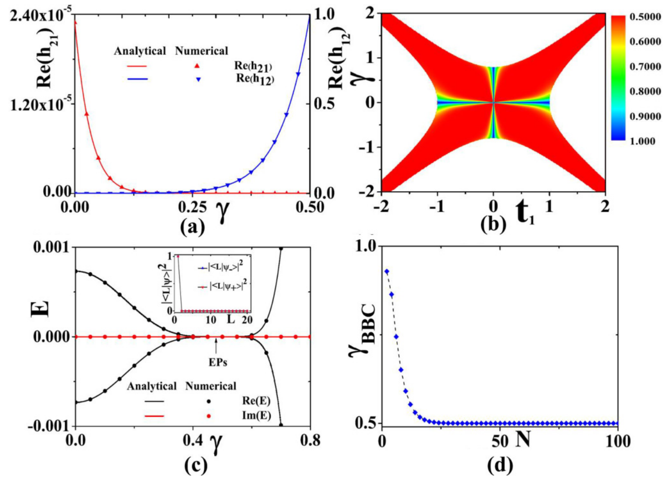

FIG. 1. (a) The numerical results and the analytical results for two matrix elements h_{12} and h_{21} of the effective edge Hamiltonian $\hat{\mathcal{H}}_{\text{edge}}$ for the cases of $t_1 = 0.5$ and $N = 15$. (b) The BBC ratio γ_{BBC}. The red region corresponds to boundary EPs. In the red region, the BBC ratio γ_{BBC} is approximately 0.5. The number of unit cells is $N = 10$. (c) The numerical results and the analytical results for the energy level of the edge states for the cases of $t_1 = 0.5$ and $N = 10$. The inset is the wave function of the edge states near the exceptional point with $\gamma = 0.49$. Here, L denotes the lattice site. (d) The BBC ratio via the lattice number N for the cases of $\gamma = 0.7$ and $t_1 = 0.2$. The result indicates that in the thermodynamic limit, the BBC ratio is 1/2. The BBC, thus, becomes number anomalous.

where $\bar{\Delta}^{+} = \bar{\Delta}\exp(-q_0N)$, $\bar{\Delta}^{-} = \bar{\Delta}\exp(q_0N)$. Under U_{edge}, the energy levels do not change with $E_{\pm} = \pm\bar{\Delta}$. In Fig. 1(a), we numerically calculate the two matrix elements for the case of $N = 15$ and eventually find a consistency between the numerical results and the analytical results. This edge Hamiltonian $\check{\mathcal{H}}_{\text{eff}}$ obeys the non-normal NH condition.

An interesting result is the existence of boundary EPs [58–60]. The condition for boundary EPs is $|\bar{\Delta}| = 0$. The solution for this equation is $N \rightarrow \infty$ or $|t_1| = |\gamma|$. As shown in Fig. 1(b), the red region corresponds to boundary EPs. Thus, the two edge states merge into one at boundary EPs $|t_1| = |\gamma|$ or $N \rightarrow \infty$ of which the wave function is $|\psi_{+}\rangle = |\psi_{-}\rangle = |b^1\rangle$ or $|b^2\rangle$. In the thermodynamic limit, the edge state for $E = 0$ becomes defective and is localized on either the left or the right edge.

Let us discuss the NA-BBC in this condition. The figure of γ_{BBC} in Fig. 1(b) is plotted for the NH topological system with a finite-size $N = 10$. For the cases of $t_1 = 0$ and $\gamma = 0$, the edge Hamiltonian $\check{\mathcal{H}}_{\text{eff}}$ obeys the normal NH condition, and the corresponding γ_{BBC} becomes 1. When $t_1 \neq 0$ and $\gamma \neq 0$, in the thermodynamic limit $N \rightarrow \infty$, the edge Hamiltonian becomes

$$\check{\mathcal{H}}_{\text{eff}} \rightarrow \bar{\Delta}\tau^{\pm} \quad (\tau^{\pm} = \tau^x \pm i\tau^y), \tag{10}$$

which obeys the non-normal NH condition. Now, we have the state similarity $|\langle\psi_{+}|\psi_{-}\rangle| = \tanh(2q_0N)$ equal to 1, and the BBC ratio γ_{BBC} becomes $\frac{1}{2}$. In particular, as shown in Fig. 1(d), the BBC for the NH topological system in the thermodynamic limit ($t_1 \neq 0$ and $\gamma \neq 0$) is anomalous.

Stability of defective edge states in the thermodynamic limit. To examine the stability of the DESs, we add an additional small uniform imaginary staggered potential $i\varepsilon\sigma_z$ on $\hat{H}_{\text{NH-SSH}}$. Using a similar approach, the anomalous edge Hamiltonian is obtained as

$$\breve{\mathcal{H}}_{\text{eff}} = \bar{\Delta}^+\tau^+ + \bar{\Delta}^-\tau^- + i\varepsilon\tau^z, \tag{11}$$

where $\bar{\Delta}^+ = \bar{\Delta}\exp(-q_0N)$, $\bar{\Delta}^- = \bar{\Delta}\exp(q_0N)$, $e^{q_0} = \sqrt{\frac{t_1-\gamma}{t_1+\gamma}}$, $\bar{\Delta} = \frac{(t_1^2-t_2^2-\gamma^2)}{t_2}(-\frac{t_1^2-\gamma^2}{t_2^2})^{N/2}$. A spontaneous (generalized) $\mathcal{PT}$-symmetry-breaking transition occurs at the boundary EPs $|\bar{\Delta}| = |\varepsilon|$. For a finite system, the DES for boundary EPs is described by $|\psi_+\rangle = |\psi_-\rangle \to (|b^1\rangle - ie^{q_0N}|b^2\rangle)$. This is DES without sublattice symmetry that is no more localized either on the left edge or on the right edge.

For a system with finite size, the anomalous edge Hamiltonian $\breve{\mathcal{H}}_{\text{eff}}$ always obeys the non-normal NH condition. However, in the thermodynamic limit, although $\bar{\Delta} \ll \varepsilon$, the situation becomes complex: For the case of $\bar{\Delta}^+ \ll \varepsilon$, $\bar{\Delta}^- \ll \varepsilon$ that corresponds to region $|t_1 \pm \gamma| < 1$, the edge Hamiltonian obeys the normal NH condition $\hat{\mathcal{H}}_{\text{edge}} \to i\varepsilon\tau^z$; for the case of $\bar{\Delta}^+ \gg \varepsilon$, $\bar{\Delta}^- \ll \varepsilon$ ($\bar{\Delta}^+ \ll \varepsilon$, $\bar{\Delta}^- \gg \varepsilon$), the edge Hamiltonian obeys the non-normal NH condition. In the region of $\bar{\Delta}^+ \ll \varepsilon$, $\bar{\Delta}^- \ll \varepsilon$, in the thermodynamic limit, arbitrary imaginary staggered potential breaking sublattice symmetry overcomes the effect from NH boundary polarization and moves the edge Hamiltonian away from boundary EPs. As a result, in the thermodynamic limit, in the region of $|t_1 \pm \gamma| < 1$, for a general NH SSH model without the protection of sublattice symmetry, the DES is *unstable*.

NA-BBC theorem for a number anomaly of the edge states in a general non-Hermitian topological system. To develop a general formula, we consider an arbitrary 1D NH topological system with a topological invariant $\mathcal{Z} = 1$. To explore the universal features for NA-BBC from DESs, we prove the following NA-BBC theorem.

NA-BBC theorem. The usual BBC is satisfied if the edge Hamiltonian $\hat{\mathcal{H}}_{\text{edge}}$ $[\hat{\mathcal{H}}_{\text{edge}} \neq (\hat{\mathcal{H}}_{\text{edge}})^\dagger]$ *for a NH topological system obeys the normal NH condition, i.e.,* $[\hat{\mathcal{H}}_{\text{edge},h}, \hat{\mathcal{H}}_{\text{edge},a}] = 0$. *Now,* γ_{BBC} *is* 1. *In contrast, the BBC becomes anomalous if* $\hat{\mathcal{H}}_{\text{edge}}$ *obeys the non-normal NH condition, i.e.,* $[\hat{\mathcal{H}}_{\text{edge},h}, \hat{\mathcal{H}}_{\text{edge},a}] \neq 0$. *Now,* γ_{BBC} *is smaller than* 1. *Here,* $\hat{\mathcal{H}}_{\text{edge},h} = \frac{1}{2}[\hat{\mathcal{H}}_{\text{edge}} + (\hat{\mathcal{H}}_{\text{edge}})^\dagger]$ and $\hat{\mathcal{H}}_{\text{edge},a} = \frac{1}{2}[\hat{\mathcal{H}}_{\text{edge}} - (\hat{\mathcal{H}}_{\text{edge}})^\dagger]$. In the Supplemental Material, we show the detailed proof of the NA-BBC theorem [56].

A special case of NA-BBC is $\gamma_{\text{BBC}} = 1/2$, which corresponds to boundary EPs with a singular DES. Now, the edge Hamiltonian $\hat{H}_{\text{edge}}$ $[\hat{H}_{\text{edge}} \neq (\hat{H}_{\text{edge}})^\dagger]$ obeys the following conditions: $\{\hat{H}_{\text{edge},h}, \hat{H}_{\text{edge},a}\} = 0$ and $\hat{H}_{\text{edge},h} = \hat{T}(i\hat{H}_{\text{edge},a})(\hat{T})^{-1}$. Here, $\hat{T}$ is a unitary Hermitian matrix.

Conclusions and discussion. Here, we draw a brief conclusion. In this Rapid Communication, the puzzle of "*how to understand the existence of DESs*" is solved. The NA-BBC theory for DESs, which distinguishes the NB-BBC from the NH skin effect, is developed. A rigorous theorem—the NA-BBC theorem—is proved. With this theorem, the NA-BBC of the number anomaly of the edge states is identified by verifying the (normal/non-normal) NH condition of the effective edge Hamiltonian $\hat{\mathcal{H}}_{\text{edge}}$. A special case of NA-BBC is $\gamma_{\text{BBC}} = 1/2$ with boundary EPs, which corresponds to a singular DES. For the NH SSH model in the region of $|t_1 \pm \gamma| < 1$ with the protection of sublattice symmetry, unequal intracell hoppings cause NH boundary polarization that drives the edge states to the boundary EPs in the thermodynamic limit. In addition, we emphasize that the quantitative theory for NA-BBC of the number anomaly of the edge states can be generalized to various types of NH topological systems, for example, the 1D NH SSH model with the next-next-nearest-neighbor hoppings, the 1D topological insulator with $\mathcal{Z} \neq 1$, 1D NH topological superconductors, and two-dimensional NH topological insulators. These issues will be presented in the future.

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[1] M. Z. Hasan and C. L. Kane, Rev. Mod. Phys. **82**, 3045 (2010).
[2] X.-L. Qi and S.-C. Zhang, Rev. Mod. Phys. **83**, 1057 (2011).
[3] J. Alicea, Rep. Prog. Phys. **75**, 076501 (2012).
[4] C.-K. Chiu, J. C. Y. Teo, A. P. Schnyder, and S. Ryu, Rev. Mod. Phys. **88**, 035005 (2016).
[5] A. Bansil, H. Lin, and T. Das, Rev. Mod. Phys. **88**, 021004 (2016).
[6] M. S. Rudner and L. S. Levitov, Phys. Rev. Lett. **102**, 065703 (2009).
[7] K. Esaki, M. Sato, K. Hasebe, and M. Kohmoto, Phys. Rev. B **84**, 205128 (2011).
[8] Y. C. Hu and T. L. Hughes, Phys. Rev. B **84**, 153101 (2011).
[9] S.-D. Liang and G.-Y. Huang, Phys. Rev. A **87**, 012118 (2013).
[10] B. Zhu, R. Lü, and S. Chen, Phys. Rev. A **89**, 062102 (2014).
[11] T. E. Lee, Phys. Rev. Lett. **116**, 133903 (2016).
[12] P. San-Jose, J. Cayao, E. Prada, and R. Aguado, Sci. Rep. **6**, 21427 (2016).
[13] D. Leykam, K. Y. Bliokh, C. Huang, Y. D. Chong, and F. Nori, Phys. Rev. Lett. **118**, 040401 (2017).
[14] H. Shen, B. Zhen, and L. Fu, Phys. Rev. Lett. **120**, 146402 (2018).
[15] S. Lieu, Phys. Rev. B **97**, 045106 (2018).
[16] Y. Xiong, J. Phys. Commun. **2**, 035043 (2018).
[17] K. Kawabata, Y. Ashida, H. Katsura, and M. Ueda, Phys. Rev. B **98**, 085116 (2018).
[18] Z. Gong, Y. Ashida, K. Kawabata, K. Takasan, S. Higashikawa, and M. Ueda, Phys. Rev. X **8**, 031079 (2018).
[19] S. Yao and Z. Wang, Phys. Rev. Lett. **121**, 086803 (2018).
[20] S. Yao, F. Song, and Z. Wang, Phys. Rev. Lett. **121**, 136802 (2018).
[21] F. K. Kunst, E. Edvardsson, J. C. Budich, and E. J. Bergholtz, Phys. Rev. Lett. **121**, 026808 (2018).
[22] C. Yin, H. Jiang, L. Li, R. Lü, and S. Chen, Phys. Rev. A **97**, 052115 (2018).
[23] K. Kawabata, K. Shiozaki, and M. Ueda, Phys. Rev. B **98**, 165148 (2018).
[24] V. M. M. Alvarez, J. E. B. Vargas, M. Berdakin, and L. E. F. F. Torres, Eur. Phys. J.: Spec. Top. **227**, 1295 (2018).

[25] H. Jiang, C. Yang, and S. Chen, Phys. Rev. A **98**, 052116 (2018).
[26] A. Ghatak and T. Das, J. Phys.: Condens. Matter **31**, 263001 (2019).
[27] J. Avila, F. Peñaranda, E. Prada, P. San-Jose, and R. Aguado, Commun. Phys. **2**, 133 (2019).
[28] L. Jin and Z. Song, Phys. Rev. B **99**, 081103(R) (2019); S. Lin, L. Jin, and Z. Song, *ibid.* **99**, 165148 (2019); K. L. Zhang, H. C. Wu, L. Jin, and Z. Song, *ibid.* **100**, 045141 (2019).
[29] C. H. Lee and R. Thomale, Phys. Rev. B **99**, 201103(R) (2019).
[30] T. Liu, Y.-R. Zhang, Q. Ai, Z. Gong, K. Kawabata, M. Ueda, and F. Nori, Phys. Rev. Lett. **122**, 076801 (2019).
[31] K. Kawabata, K. Shiozaki, M. Ueda, and M. Sato, Phys. Rev. X **9**, 041015 (2019).
[32] H. Zhou and J. Y. Lee, Phys. Rev. B **99**, 235112 (2019).
[33] C. H. Liu, H. Jiang, and S. Chen, Phys. Rev. B **99**, 125103 (2019).
[34] E. Edvardsson, F. K. Kunst, and E. J. Bergholtz, Phys. Rev. B **99**, 081302(R) (2019).
[35] L. Herviou, J. H. Bardarson, and N. Regnault, Phys. Rev. A **99**, 052118 (2019).
[36] K. Yokomizo and S. Murakami, Phys. Rev. Lett. **123**, 066404 (2019).
[37] F. K. Kunst and V. Dwivedi, Phys. Rev. B **99**, 245116 (2019).
[38] R. Chen, C.-Z. Chen, B. Zhou, and D.-H. Xu, Phys. Rev. B **99**, 155431 (2019).
[39] T. S. Deng and W. Yi, Phys. Rev. B **100**, 035102 (2019).
[40] F. Song, S. Yao, and Z. Wang, Phys. Rev. Lett. **123**, 170401 (2019).
[41] X.-W. Luo and C. Zhang, Phys. Rev. Lett. **123**, 073601 (2019).
[42] S. Longhi, Phys. Rev. Res. **1**, 023013 (2019).
[43] H. Jiang, R. Lü, and S. Chen, arXiv:1906.04700.
[44] J. M. Zeuner, M. C. Rechtsman, Y. Plotnik, Y. Lumer, S. Nolte, M. S. Rudner, M. Segev, and A. Szameit, Phys. Rev. Lett. **115**, 040402 (2015).
[45] S. Weimann, M. Kremer, Y. Plotnik, Y. Lumer, S. Nolte, K. G. Makris, M. Segev, M. C. Rechtsman, and A. Szameit, Nature Mater. **16**, 433 (2017).
[46] L. Xiao, X. Zhan, Z. H. Bian, K. K. Wang, X. Zhang, X. P. Wang, J. Li, K. Mochizuki, D. Kim, N. Kawakami, W. Yi, H. Obuse, B. C. Sanders, and P. Xue, Nat. Phys. **13**, 1117 (2017).
[47] M. A. Bandres, S. Wittek, G. Harari, M. Parto, J. Ren, M. Segev, D. N. Christodoulides, and M. Khajavikhan, Science **359**, 4005 (2018).
[48] H. Zhou, C. Peng, Y. Yoon, C. W. Hsu, K. A. Nelson, L. Fu, J. D. Joannopoulos, M. Soljačić, and B. Zhen, Science **359**, 1009 (2018).
[49] A. Cerjan, S. Huang, M. Wang, K. P. Chen, Y. Chong, and M. C. Rechtsman, Nat. Photonics **13**, 623 (2019).
[50] K. Wang, X. Qiu, L. Xiao, X. Zhan, Z. Bian, B. C. Sanders, W. Yi, and P. Xue, Nat. Commun. **10**, 2293 (2019).
[51] L. Xiao, T. Deng, K. Wang, G. Zhu, Z. Wang, W. Yi, and P. Xue, arXiv:1907.12566.
[52] T. Helbig, T. Hofmann, S. Imhof, M. Abdelghany, T. Kiessling, L. W. Molenkamp, C. H. Lee, A. Szameit, M. Greiter, and R. Thomale, arXiv:1907.11562.
[53] A. Mostafazadeh, J. Math. Phys **43**, 205 (2002).
[54] W. D. Heiss, J. Phys. A: Math. Theor. **45**, 444016 (2012).
[55] R. A. Horn and C. R. Johnson, *Matrix Analysis* (Cambridge University Press, Cambridge, UK, 1985).
[56] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevB.101.121116 for the detailed proof on the number-anomalous bulk-boundary correspondence theorem for non-Hermitian topological systems.
[57] J. Sirker, M. Maiti, N. P. Konstantinidis, and N. Sedlmayr, J. Stat. Mech.: Theory Exp. (2014) P10032.
[58] C. M. Bender and S. Boettcher, Phys. Rev. Lett. **80**, 5243 (1998).
[59] C. M. Bender, D. C. Brody, and H. F. Jones, Phys. Rev. Lett. **89**, 270401 (2002).
[60] C. M. Bender, Rep. Prog. Phys. **70**, 947 (2007).