


London penetration depth at zero temperature and near the superconducting transition

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A simple relation is established between the zero- T penetration depth $\lambda(0)$ and the slope of $\lambda^{-2}(T)$ near T_c , similar to Helfand-Werthamer's relation for $H_{c2}(0)$ and the slope of $H_{c2}(T)$ at T_c for the isotropic s -wave case with nonmagnetic scattering [E. Helfand and N. R. Werthamer, *Phys. Rev.* **147**, 288 (1966)]. When the scattering parameter $\rho = \hbar v/2\pi T_c \ell$ (v is the Fermi velocity and ℓ is the mean free path) varies from 1 to 10, the coefficient of proportionality between $\lambda^{-2}(0)$ and $T_c(d\lambda^{-2}/dT)_{T_c}$ changes from 0.43 to 0.38. Combining this relation with the Rutgers thermodynamic identity, one can express $\lambda(0)$ in terms of the slope $(dH_{c2}/dT)_{T_c}$ and the density of states.

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I. INTRODUCTION

In the seminal work by Helfand and Werthamer, a simple relation between zero-temperature upper critical field $H_{c2}(0)$ and the slope of $H_{c2}(T)$ at T_c was established for the isotropic s -wave superconductors with nonmagnetic scattering [1]:

$$H_{c2}(0) = \mu_h T_c \left(\frac{dH_{c2}}{dT} \right)_{T_c}. \quad (1)$$

With increasing transport scattering, the numerical factor μ_h varies from 0.73 in the clean limit to 0.69 in the dirty case, i.e., $\mu_h \approx 0.7$ with $\sim 4\%$ accuracy. This relation is commonly used to estimate $H_{c2}(0)$ when only high-temperature data for $H_{c2}(T)$ are available. It is shown below that a similar relation exists for the penetration depth $\lambda(T)$:

$$\lambda^{-2}(0) = \mu_\lambda T_c \left(\frac{d\lambda^{-2}}{dT} \right)_{T_c}, \quad (2)$$

with $\mu_\lambda \approx 0.4$ in a range of scattering parameters covering nearly all practical transport scattering rates.

Combining Eqs. (1) and (2) with Rutgers thermodynamic identity [2], which connects the slopes of H_{c2} and of λ^{-2} at T_c , one obtains a useful relation between difficult to measure $\lambda(0)$ and usually available slope $(dH_{c2}/dT)_{T_c}$.

II. $\lambda^{-2}(0)$ IN TERMS OF $T_c(d\lambda/dT)_{T_c}$

The penetration depth in isotropic BCS superconductors is given by

$$\lambda^{-2} = \frac{16\pi^2 e^2 T N(0) \Delta^2 v^2}{3c^2} \sum_{\omega>0} \frac{1}{\beta^2 \beta'}. \quad (3)$$

Here, $\hbar\omega = \pi T(2n+1)$ defines the Matsubara frequencies, $N(0)$ is the one-spin density of states at the Fermi level,

v is the Fermi velocity, $\Delta(T)$ is the gap parameter, $\beta^2 = \hbar^2 \omega^2 + \Delta^2$, $\beta' = \beta + \hbar/2\tau$, and τ is the transport scattering time. One can find this result in the book by Abrikosov, Gor'kov, and Dzyaloshinskii [3]. It can also be derived using the Eilenberger quasiclassical version of the BCS theory; see, e.g., Refs. [4,5].

At zero temperature, one replaces the sum with an integral according to $2\pi T \sum_{\omega} \rightarrow \int_0^{\infty} d(\hbar\omega)$ to obtain [6,7]

$$\lambda^{-2}(0) = \frac{4\pi^2 e^2 N(0) v^2}{3c^2 \eta} \left(1 + \frac{4 \tan^{-1} \frac{\eta-1}{\sqrt{1-\eta^2}}}{\pi \sqrt{1-\eta^2}} \right), \quad (4)$$

where the scattering parameter

$$\eta = \frac{\hbar}{2\tau \Delta_0} = \frac{\pi}{2} \frac{\xi_0}{\ell}, \quad (5)$$

$\xi_0 = \hbar v/\pi \Delta_0$ is the BCS zero- T coherence length, and ℓ is the transport mean free path. Equation (4) works for any $\eta > 0$. For $\eta > 1$, it can be written in explicitly real form by replacing $\tan^{-1} \rightarrow -\tanh^{-1}$ and $\sqrt{1-\eta^2} \rightarrow \sqrt{\eta^2-1}$.

Near T_c , $\Delta^2 = 8\pi^2 T_c(T_c - T)/7\zeta(3)$, $\beta \approx \hbar\omega = \pi T_c(2n+1)$, $\beta' \approx \pi T_c(2n+1+\rho)$ with

$$\rho = \frac{\hbar v}{2\pi T_c \ell} = e^{-\gamma} \eta, \quad (6)$$

where $\gamma \approx 0.577$ is the Euler constant. The sum in Eq. (3) is expressed in terms of digamma functions ψ . Doing the algebra, one obtains the slope at T_c :

$$\frac{d\lambda^{-2}}{dT} = \frac{64\pi^2 e^2 N(0) v^2}{21\zeta(3) c^2 T_c} \frac{1}{\rho^2} \left[\psi\left(\frac{1+\rho}{2}\right) - \psi\left(\frac{1}{2}\right) - \frac{\pi^2 \rho}{4} \right]. \quad (7)$$

Following Ref. [1], one defines the quantity

$$\mu_\lambda^{-1} = -\frac{T_c}{\lambda^{-2}(0)} \left(\frac{d\lambda^{-2}}{dT} \right)_{T_c} = -\left(\frac{d\rho_s}{dt} \right)_{t=1}, \quad (8)$$

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where $t = T/T_c$, and $\rho_s(t) = \lambda^2(0)/\lambda^2(t)$ is commonly called the superfluid density. We then obtain from Eqs. (7) and (4)

$$\mu_\lambda^{-1} = \frac{16e^\gamma}{7\pi\zeta(3)\rho^2} \left[\psi\left(\frac{1+\rho}{2}\right) - \psi\left(\frac{1}{2}\right) - \frac{\pi^2\rho}{4} \right] / \left(1 + \frac{4 \tan^{-1} \frac{\eta-1}{\sqrt{1-\eta^2}}}{\pi\sqrt{1-\eta^2}} \right), \quad \eta = \rho e^\gamma. \quad (9)$$

The top panel of Fig. 1 shows μ_λ versus ρ . In the clean limit, $\mu_\lambda = 0.5$, as it should. With increasing ρ , μ_λ decreases by about 30%, but most of this change happens close to the clean limit where $0 < \rho < 1$. The relative change of μ_λ is about 9% when ρ varies in the realistic range from 1 to 10. Note also that for scattering parameters $\rho \sim 100$ the dirty limit is reached with $\mu_\lambda = 7\zeta(3)/4\pi e^\gamma \approx 0.376$.

Curiously enough, the empirical two-fluid model with $\lambda^{-2} = \lambda^{-2}(0)(1-t^4)$ yields $\mu_\lambda = 0.25$. For the sake of completeness, we mention the strong pair-breaking (the gapless situation) for which both λ^{-2} and H_{c2} have the same T dependence [8]: $\lambda^{-2} = \lambda^{-2}(0)(1-t^2)$ and $H_{c2} = H_{c2}(0)(1-t^2)$. This gives $\mu_\lambda = \mu_h = 0.5$.

III. RELATION BETWEEN $\lambda^{-2}(0)$ AND $(dH_{c2}/dT)_{T_c}$

The slopes of H_{c2} at T_c are given in Ref. [1] for isotropic type-II materials with arbitrary transport scattering:

$$-\left. \frac{dh}{dt} \right|_{t=1} = 3\rho^2 \left[\psi\left(\frac{1}{2}\right) - \psi\left(\frac{\rho+1}{2}\right) + \frac{\rho}{2} \psi'\left(\frac{1}{2}\right) \right]^{-1}, \quad (10)$$

where t is the reduced temperature and

$$h = H_{c2} \frac{\hbar^2 v^2}{2\pi T_c^2 \phi_0}. \quad (11)$$

Comparing this with the slope of λ^{-2} near T_c , Eq. (7), one sees that the ρ dependences of these two quantities are inverse so that their product is ρ -independent:

$$\left(\frac{d\lambda^{-2}}{dT} \frac{dH_{c2}}{dT} \right)_{T_c} = \frac{128\pi^4}{7\zeta(3)\phi_0} N(0). \quad (12)$$

Remarkably, the scattering parameter does not enter this relation at all. The only material parameter on the right-hand side is the density of states $N(0)$. The physical reason for this result can be traced to the Rutgers thermodynamic relation [9], in which the product of slopes in Eq. (12) is proportional to the transport-scattering-independent specific-heat jump at T_c (Anderson's theorem) [10].

In particular, this relation can be checked in the clean limit where at T_c

$$\frac{d\lambda^{-2}}{dT} = -\frac{16\pi e^2 N(0) v^2}{3c^2 T_c}, \quad \frac{dH_{c2}}{dT} = -\frac{24\pi \phi_0 T_c}{7\zeta(3) \hbar^2 v^2}. \quad (13)$$

In the dirty limit, we have

$$\frac{d\lambda^{-2}}{dT} = -\frac{16\pi^3 e^2 N(0) v^2}{21\zeta(3) c^2 T_c \rho}, \quad \frac{dH_{c2}}{dT} = -\frac{24\phi_0 T_c \rho}{\pi v^2 \hbar^2}. \quad (14)$$

The slopes of λ^{-2} at T_c follow from Eq. (7); one can find slopes of H_{c2} in Ref. [1].

Both clean and dirty limits are not quite realistic. According to Fig. 1, within the range $1 < \rho < 10$ one has approximate relations:

$$\left(\frac{d\lambda^{-2}}{dT} \right)_{T_c} \approx -\frac{\lambda^{-2}(0)}{0.4T_c}, \quad (15)$$

while

$$\left(\frac{dH_{c2}}{dT} \right)_{T_c} \approx -\frac{H_{c2}(0)}{0.7T_c}. \quad (16)$$

Using Eq. (12) along with (15) and (16), one obtains

$$H_{c2}(0) \lambda^{-2}(0) \approx 415 \frac{T_c^2 N(0)}{\phi_0}. \quad (17)$$

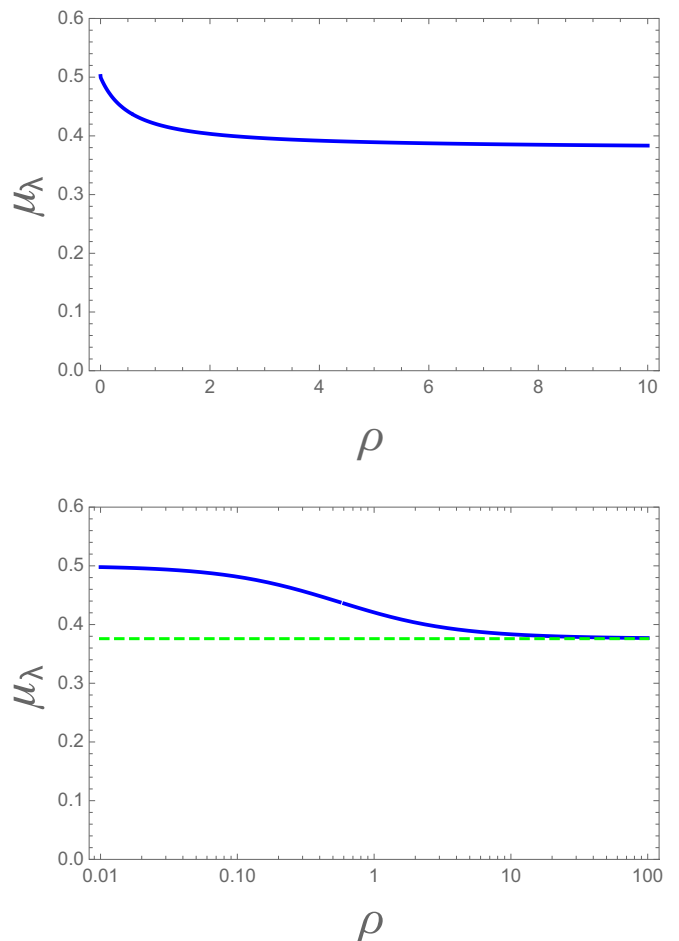


FIG. 1. Top panel: μ_λ vs ρ . Lower panel: the same on a semilog scale; the dashed line is the dirty limit value.

This, however, holds if there is no low-temperature paramagnetic limiting of H_{c2} . Utilizing only Eq. (15), one gets

$$-\lambda^{-2}(0) \left(\frac{dH_{c2}}{dT} \right)_{T_c} \approx 593 \frac{T_c N(0)}{\phi_0}, \quad (18)$$

a potentially useful relation, since the paramagnetism is not involved here and one can express $\lambda^{-2}(0)$ in terms of the slope $(dH_{c2}/dT)_{T_c}$, T_c , and the density of states $N(0)$. Noting that $N(0) = 3\gamma/2\pi^2$ (γ is the coefficient in the linear low- T dependence of the specific heat) and the slope $H'_{c2}(T_c)$ are usually accessible, one can estimate a difficult to measure $\lambda(0)$. Note: in Eq. (18), temperatures are given in energy units (erg); with temperature in Kelvins, one has

$$\begin{aligned} -\lambda^{-2}(0) \left(\frac{dH_{c2}}{dT} \right)_{T_c} \left(\frac{\text{G}}{\text{cm}^2 \text{K}} \right) \\ \approx 593 \frac{k_B^2 T_c N(0)}{\phi_0} \approx 4.5 \times 10^8 \left(\frac{1}{\text{cm}^2 \text{G}} \right) T_c(\text{K}) \gamma \left(\frac{\text{erg}}{\text{cm}^3 \text{K}^2} \right). \end{aligned} \quad (19)$$

This is perhaps the most useful result of our paper since it relates a difficult to measure $\lambda(0)$ to easily accessible slope $(dH_{c2}/dT)_{T_c}$.

IV. DISCUSSION

The above arguments hold for isotropic s -wave materials with nonmagnetic scattering. In the presence of pair-breaking or for other than s -wave order parameter and general Fermi surfaces, the Anderson theorem does not work, and Eq. (12) is not expected to be valid. Equations (2) may still hold, however with the factor μ_λ changing significantly with scattering, unlike the situation considered here. On the other hand, if the order parameter is constant at the Fermi surface of any shape (including multiband structures), there is no obvious reason for our results to be inapplicable.

To show how the obtained results can be applied for real materials, we estimate $\lambda(0)$ of V_3Si and Nb_3Sn using data of Orlando *et al.* [11]. For a sample of V_3Si with $T_c = 16.4$ K, $dH_{c2}/dT = -1.84 \times 10^4$ Oe/K, and $\gamma = 2.2 \times 10^4$ erg/cm³ K², Eq. (19) yields $\lambda(0) \approx 106$ nm. Near T_c , $\lambda = \lambda_{\text{GL}}/\sqrt{1-t}$ and Ref. [11] provides $\lambda_{\text{GL}} = 62$ nm. In the clean limit, we have $\lambda(0) = \lambda_{\text{GL}}\sqrt{2} \approx 88$ nm, which is reasonably close to 106 nm given uncertain scattering parameters for these data [as shown below, the dirty limit assumption would give $\lambda(0) = 1.63\lambda_{\text{GL}} \approx 102$ nm].

For Nb_3Sn with $T_c = 17.9$ K, $dH_{c2}/dT = -1.83 \times 10^4$ Oe/K, and $\gamma = 1.1 \times 10^4$ erg/cm³ K², one obtains $\lambda(0) \approx 144$ nm, which corresponds to $\lambda_{\text{GL}} = \lambda(0)/\sqrt{2} = 102$ nm, whereas Ref. [11] cites $\lambda_{\text{GL}} = 64$ nm (the dirty limit assumption would have given $\lambda_{\text{GL}} \approx 88$ nm).

Another example is $\text{Rh}_9\text{In}_4\text{S}_4$ with $T_c = 2.25$ K, $dH_{c2}/dT = -1.69 \times 10^4$ G/K, $\gamma = 34 \times 10^4$ erg/mol K² = 0.21×10^4 erg/cm³ K², and $\lambda_{\text{GL}} = 575$ nm [12]. The ratio of zero- T BCS coherence length to the mean free path for the sample studied was $\xi_0/\ell \sim 20$ – 200 , i.e., it is the dirty limit. From Eq. (3) with $\eta \gg 1$ one obtains the known expression

for the dirty limit:

$$\lambda^{-2} = \frac{8\pi^2 e^2 N(0) D}{c^2 \hbar} \Delta \tanh \frac{\Delta}{2T} \quad (20)$$

($D = v\ell/3$ is the diffusivity). It is now readily shown that

$$\frac{\lambda^2(0)}{\lambda_{\text{GL}}^2} = \frac{4\pi^2 T_c}{7\zeta(3)\Delta_0}. \quad (21)$$

Since $\Delta_0/T_c \approx 1.76$, we estimate $\lambda(0) \approx 1.63\lambda_{\text{GL}} \approx 939$ nm.

On the other hand, Eq. (19) yields $\lambda(0) \approx 892$ nm. A reasonable agreement between our model and the data of this case might be due to the fact that the strong scattering washes away anisotropies of the order parameter, thus making the material “more BCS-like.” Besides, the strong scattering excludes the possibility of other than s -wave symmetry since even the transport scattering for non- s -wave symmetry is pair-breaking, and superconductivity disappears well before the dirty limit is reached.

The multiband MgB_2 is an example in which our model should not work. Still, taking $T_c \approx 39$ K, $dH_{c2}/dT = -0.41 \times 10^4$ G/K as given in Ref. [13] along with $\gamma = 7.2 \times 10^2$ erg/cm³ K² as provided by Ref. [14], with the help of Eq. (19) we estimate $\lambda(0) \approx 176$ nm. Reference [14] cites 185 nm obtained from thermodynamic data, and a close value of 180 nm reported from analysis of microwave response [15]. One may say that the proximity of these numbers does not mean much. On the other hand, it shows that formulas based on the penetration depth property represented by Eq. (2) are quite robust and can be used for rough estimates in a variety of situations, similar to what is commonly done with the H_{c2} property of Eq. (1).

Still, since the above derivation of $\lambda(0)$, Eq. (4), has been done for s -wave superconductors with only transport scattering, one should not expect consequences of this equation to hold in the presence of pair-breaking, be it due to the spin-flip or to other than s -wave order parameter. To check this, we turn to well-studied CeCoIn_5 , a clean superconductor with $T_c = 2.3$ K and $\gamma = 3 \times 10^4$ erg/cm³ K² [16]. The penetration depth in this material turned out [17] to satisfy with high accuracy the relation $\lambda = \lambda(0)/\sqrt{1-t^2}$ established by Abrikosov-Gor'kov for a strong pair-breaking [8]. The fit to the data gives $\lambda(0) = 358$ nm and $dH_{c2}/dT = -11.5 \times 10^4$ G/K [17]. We thus have all information needed to calculate $\lambda(0)$ with the help of Eq. (19), which gives $\lambda(0) = 608$ nm. The discrepancy is larger yet if we compare with $\lambda(0) = 196$ nm, as found in microwave measurements [18]. Clearly, our model fails in this case indicating the pair-breaking as a possible culprit.

Also, our model fails being applied to KFe_2As_2 with $T_c = 3.5$ K, $\gamma = 1.53 \times 10^4$ erg/cm³ K², and $dH_{c2,c}/dT = -0.6$ T/K for the field along the c direction [19–21]. Equation (19) of the model yields $\lambda(0) = 158$ nm while the literature values are close to 200 nm [19,22–24]. One of the reasons for disagreement is a possible d -wave order parameter and a relatively strong anisotropy of superconducting properties.

In conclusion, we consider isotropic s -wave superconductors without pair-breaking scattering, but with arbitrary

potential scattering. We establish new relations of the London penetration depth at $T = 0$ with its variation at T_c and with the slope of the upper critical field H_{c2} . This gives a relatively simple way to estimate difficult to measure $\lambda(0)$ from measured H_{c2} near T_c and the specific heat [needed to estimate the density of states $N(0)$ at the Fermi level]. On the other hand, if the obtained estimate turns out to be unreasonable or very different from independent measurement, this may signal the unconventional superconductivity or pair-breaking in the studied material. We therefore believe that our work

provides a useful practical tool for researchers dealing with experimental superconductivity.

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