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Subdiffusion in the Anderson model on the random regular graph

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We study the finite-time dynamics of an initially localized wave packet in the Anderson model on the random regular graph (RRG) and show the presence of a subdiffusion phase coexisting both with ergodic and putative nonergodic phases. The full probability distribution $\Pi(x,t)$ of a particle to be at some distance x from the initial state at time t is shown to spread subdiffusively over a range of disorder strengths. The comparison of this result with the dynamics of the Anderson model on $\mathbb{Z}^d$ lattices, $d > 2$, which is subdiffusive only at the critical point implies that the limit $d \to \infty$ is highly singular in terms of the dynamics. A detailed analysis of the propagation of $\Pi(x,t)$ in space-time (x,t) domain identifies four different regimes determined by the position of a wave front $X_{\text{front}}(t)$, which moves subdiffusively to the most distant sites $X_{\text{front}}(t) \sim t^{\beta}$ with an exponent $\beta < 1$. Importantly, the Anderson model on the RRG can be considered as proxy of the many-body localization transition (MBL) on the Fock space of a generic interacting system. In the final discussion, we outline possible implications of our findings for MBL.

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Introduction. The common belief that generic, isolated quantum systems thermalize as a result of their own dynamics has been challenged by a recent line of works showing that strong enough disorder can prevent them reaching thermal equilibrium [1,2]. This phenomenon, referred to as many-body localization (MBL) [1–7], generalizes the concept of Anderson localization [8] to the case of interacting particles, and has an important bearing on our understanding of quantum statistical mechanics.

Although MBL has been extensively studied [3,6,7], many of its aspects are still under intense debate. For example, only little is known on the nature of the MBL transition [9–13]. Recent numerical results show that the critical point of the transition may have been previously underestimated [14,15] and critical exponents extracted with exact numerics seems to violate general constraints (i.e., so-called Harris bounds) [3,16]. Even the nature of the ergodic phase is not completely settled. For instance, subdiffusive dynamics has been observed on finite time scales and system sizes [17–24], but its mechanism and asymptotic limit are far from being clear [14,25–28].

Numerically these difficulties originate from the exponentially increasing complexity of the problem with system sizes, which makes the resolution of these open issues an extremely hard task. One way to overcome this problem is to consider approximate calculation methods like matrix product states [14,29–32] in order to increase significantly system sizes. Another way is to find proxies of interesting observables in more tractable models, which can reproduce the salient intrinsic features of MBL systems [33–38]. In this work we take the latter route considering an Anderson model on a hierarchical

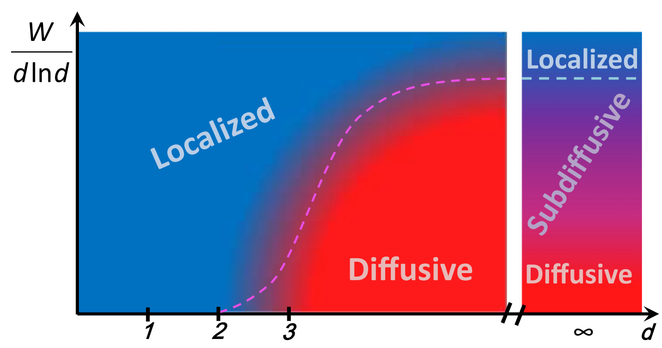

FIG. 1. Schematic phase diagram of the anomalous transport of the Anderson model in d dimension. In $d = 1, 2$ the system is fully localized at any finite disorder. For $d > 2$ the system has an Anderson transition at disorder strength $W = W_{\text{AT}}$; for $W < W_{\text{AT}}$ the transport is diffusive and subdiffusive only at the critical point. At small $d - 2 \ll 1$ the critical disorder $W_{\text{AT}} \sim (d-2)$ (linear behavior of dashed line), while at large d it is given by $W_{\text{AT}} \sim d \ln d$ (dashed line saturation). The limit $d \to \infty$ is given by the Anderson model on the RRG with the branching number K. The latter limit is characterized by three distinct phases: a diffusive, subdiffusive, and a localized one.

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treelike structure as a proxy for more realistic many-body systems. Indeed, in [1,39], the idea of mapping interacting disordered electrons to an effective Anderson model on a section of Fock space was used to give evidence of the stability of the MBL phase. Recently, this paradigm and the hierarchical structure of the Fock space of generic many-body systems have revived interest in the Anderson problem on locally treelike structures, such as the random regular graph (RRG).

Combining the hierarchical structure of the Fock space with the simplicity of regular graphs, i.e., the fixed branching number, RRG can be considered as a natural choice to approximate MBL systems [33–35] and hope to overcome some of the numerical difficulties that have been mentioned earlier. Apart from the fact that RRG gives a new emphasis on the field of Anderson localization, independently also its own physics is extremely rich [40–49]. For instance, it has been shown that there is a possibility of a nonergodic extended (NEE) phase composed of critical states and placed between the ergodic and the localized phase [33,41–44]. Nevertheless, it has been argued that this NEE phase might merely be a finite-size effect and would disappear in the thermodynamic limit [34,50–61]. However, this intricate question is far from being resolved.

Following the mapping of RRG to the Fock space of many-body systems, one expects the ergodic phase of wave functions on RRG to be qualitatively mapped to the validity range of the eigenstate thermalization hypothesis (ETH) [62–64] for many-body eigenstates. Furthermore, it has been recently suggested analytically and numerically confirmed that subleading corrections of ETH assumptions may lead to slow dynamics of local observables after quench instead of a diffusive one [17].

Motivated by the above-mentioned mapping, we study the spreading of an initially localized wave packet in the Anderson model on the RRG as a probe of different dynamical phases. In many-body systems, this can be considered as a proxy for the nonequilibrium dynamics of local operators after quench [33,65,66] and also as a direct measure for entanglement propagation [65,66]. We give evidence of existence of subdiffusive dynamical phase over an entire range of parameters both in a part of the phase diagram where most of the works [34,42–47,50–61] agree on ergodic nature of eigenstates according to standard wave-function analysis and in a putative nonergodic phase [33,41–44]. Moreover, it is important to point out that the dynamics of the Anderson model on $\mathbb{Z}^d$, $d > 2$, is believed to be diffusive within its ergodic phase and subdiffusive only at the critical point [67,68]. Thus the found subdiffusive phase in the limiting dimension $d \to \infty$ of the RRG provides a further example of the importance of dimensionality in the physics of localization, beside the well known example of fully localized systems in $d = 1, 2$; see Fig. 1.

Model and methods. The Anderson model on the RRG is defined as

$$\hat{H} = -\sum_{\substack{x,y \\ x\sim y}}^{L} |x\rangle\langle y| + \sum_{x}^{L} h_x |x\rangle\langle x|, \tag{1}$$

where x counts L site states $|x\rangle$ on the RRG. The first sum in $\hat{H}$ runs over sites (x, y) that are connected $(x \sim y)$ on the RRG with fixed branching number (the number of neighbors of each site is fixed to $K + 1 = 3$). $\{h_x\}$ independent random variables distributed uniformly between $[-W/2, W/2]$. This model is known to have an Anderson localization transition at $W_{\rm AT} \approx 18.1 \pm 0.1$ [34,44,48].

We are interested in studying the full propagation of a wave function initially localized in a neighborhood of a site state $|x_0\rangle$, and having energy concentrated in a window of size δE around the center of the band, $E = 0$.

A standard description for the dynamics employs the distribution function $\Pi(x, t)$ [69] which determines the probability to find the particle at time t in some state at distance x from the initial one

$$\Pi(x, t) = \overline{\frac{\sum_{y:d(y,x_0)=x} |\langle y|\hat{P}_{\Delta E} e^{-i\hat{H}t} \hat{P}_{\Delta E}|x_0\rangle|^2}{\sum_y |\langle y|\hat{P}_{\Delta E}|x_0\rangle|^2}}. \tag{2}$$

The sum in Eq. (2) runs over all states $|y\rangle$ located at distance $d(y, x_0) = x$ from the initial state $|x_0\rangle$. The distance $d(y, x_0)$ is defined as the shortest path's length that connects two sites on the RRG. Importantly, in the many-body setting this distance is related to the Hamming metric of the Fock space [39,65,66,70]. The computation of the Hamming distance between two Fock states involves only the measure of local observables, and has been measured experimentally in the MBL context, specially using it as a witness for entanglement propagation [65].

The overline in Eq. (2) indicates the average over disorder, graph ensemble, and initial states $|x_0\rangle$. $\hat{P}_{\Delta E} = \sum_{E\in\Delta E} |E\rangle\langle E|$ is the projector onto eigenstates of $\hat{H}$ with energy E from a small energy shell $E \in \Delta E = [-\delta E/2, \delta E/2]$ around the middle of the spectrum of $\hat{H}$. In particular, we consider δE to be a small fraction f $(f = 1/8)$ of the entire bandwidth E_{BW} $(\delta E = f E_{BW})$.

The usage of the projector is motivated by several reasons. First, $\hat{P}_{\Delta E}$ avoids the localized eigenstates at the edge of the spectrum [71]. Second, the initialization of the system in the microcanonical state with well-defined energy $E \in \Delta E$ in a small interval in the middle of the spectrum mimics ETH assumptions of many-body physics and under otherwise equal conditions prefers thermalization. Thus slow nondiffusive propagation of such projected wave packet should rule out the possibility of a fully ergodic phase (equivalent to random matrix theory [72]). Finally, the projector can be used as a dynamical indicator to distinguish a fully ergodic system from a nonergodic one [73,74]. In a fully ergodic phase, as a consequence of level repulsion, the return probability, $\Pi(0, t)$, takes a standard form [74] given by

$$\frac{\Pi(0, t)}{\Pi(0, 0)} = \left(\frac{\sin \delta E t}{\delta E t}\right)^2. \tag{3}$$

The projector $\hat{P}_{\Delta E}$ slightly spreads the initial delta-function-like state $|x_0\rangle$ to the wave packet $\hat{P}_{\Delta E}|x_0\rangle$ with a finite width. This initialization supports the semiclassical description of wave-packet propagation in the system. We ensure that our results do not change significantly with δE, provided it is not too big [75].

As a further measure of the spread of the wave packet, we study the first moment of $\Pi(x, t)$,

$$X(t) = \sum_x x \Pi(x, t). \tag{4}$$

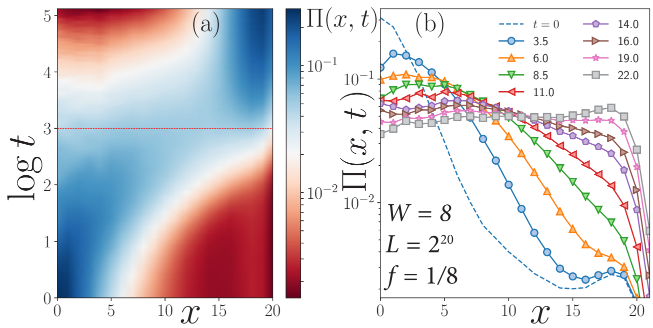

FIG. 2. Probability $\Pi(x,t)$ for the particle to be at distance x from the initial state at time t. (a) $\Pi(x,t)$ versus distance x and time t in a color plot. Blue (red) color corresponds to high (low) values of $\Pi(x,t)$ and shows the propagation of the initially localized wave packet with the initial size $X(0)\simeq 2$ through the uniform distribution over distance $\Pi(x,t_{\rm Th})\simeq$ const (see the dashed line $t_{\rm Th}\approx 22$) to the uniform distribution over sites $\Pi(x,\infty)\simeq \mathcal{N}(x)/L$. (b) Cross section of the color plot in panel (a) at several times below $t_{\rm Th}$, showing how the wave front propagates to the diameter of the graph. All plots are shown at the most representative disorder amplitude $W=8$ for fixed system size $L=2^{20}$.

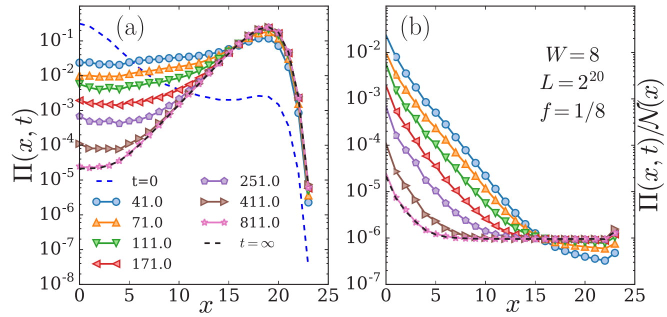

FIG. 3. Probability $\Pi(x,t)$ versus distance x at large times $t>t_{\rm Th}$. (a) The cross sections of Fig. 2(a) at several times above $t_{\rm Th}$, when the propagation front has already reached the diameter of the graph $X_{\rm front}>D=\ln L/\ln K$. $\Pi(x,t)$ relaxes from the uniform distribution in the distance $\Pi(x,t_{\rm Th})\simeq$ const to the uniform distribution over sites $\Pi(x,\infty)\simeq\mathcal{N}(x)/L$. Dashed line shows the initial distribution $\Pi(x,0)$ as a guide for eyes. (b) The distribution from panel (a) renormalized by the mean number of sites $\mathcal{N}(x)$ at some distance x from an initial site state $|x_0\rangle$. This figure gives evidence of the space-time factorization Eq. (8), once the front has already passed, $X_{\rm front}(t)>D$. The parameters are the same as in Fig. 2.

The wave-packet width at time $t=0$ induced by $\hat{P}_{\Delta E}$ can be simply estimated by $X(0)=\sum_x x\Pi(x,0)$.

In Ref. [73] we have shown that for small values of W $(0<W<0.16W_{\rm AT}\simeq 3)$ the return probability $\Pi(0,t)$ is consistent with the result of Eq. (3), confirming that the system is in a fully ergodic phase [76,77]. For larger disorder, $W\in[0.4W_{\rm AT},0.7W_{\rm AT}]\simeq[8,13]$, $\Pi(0,t)$ decays as a stretched exponential $\sim e^{-\Gamma t^{\beta(W)}}$, where the exponent is well approximated by

$$\beta(W)\simeq 1-W/W_{\rm AT},\quad 0.4W_{\rm AT}\lesssim W\lesssim 0.7W_{\rm AT} \tag{5}$$

and goes to zero at the Anderson transition. As a consequence, the drastic change in the time evolution of $\Pi(0,t)$ gives evidence of the existence of two *dynamically distinct phases*.

As a side remark, before coming to the results, we stress the difference between the wave-packet propagation on hierarchical structures and d-dimensional lattices like $\mathbb{Z}^d$. It is well known that the return probability for a classical unbiased random walk on a Bethe lattice with branching number K decays exponentially fast in time $\sim e^{-\Omega(K)t}$ [78] due to the exponential growth K^x of the number of sites with the distance x from an initial point $|x_0\rangle$. Instead, in $\mathbb{Z}^d$ lattices the typical behavior is diffusive $\sim t^{-d/2}$ as the number of sites at distance x grows algebraically $\mathcal{N}(x)\sim x^{d-1}$. Thus the diffusive propagation on hierarchical tree lattices is characterized by a linear growth of the width of the wave packet with time $X(t)\sim t$ [see Eq. (4)] unlike $X(t)\sim t^{1/2}$ in d-dimensional lattices. Noticing this difference, we call the propagation in RRG subdiffusive if $X(t)\sim t^{\beta}$ with $\beta<1$.

In this work, we show that, as time increases, $\Pi(x,t)$ relaxes forming a wave front $X_{\rm front}(t)$ that moves subdiffusively to the most distant sites, as shown in Fig. 2. More specifically the propagation of $\Pi(x,t)$ can be divided into four regions in space-time domain (x,t) depending on the position of the moving front $X_{\rm front}(t)$.

(i) At large distances (small times), $x>X_{\rm front}(t)$, the wave front has not yet reached x, and the distribution is nearly unperturbed

$$\Pi(x,t)\approx\Pi(x,0),\quad x>X_{\rm front}(t) \tag{6}$$

(see the red area at small times in Fig. 2 and the plateau at short times in the inset of Fig. 4).

(ii) At $x\simeq X_{\rm front}(t)$ in proximity of the front propagation, $\Pi(x\lesssim X_{\rm front}(t),t)$ renormalized by its maximal value $\Pi(X_{\rm front}(t),t)$ collapses to a universal function

$$\Pi(x,t)-\Pi(x,\infty)=\Pi(X_{\rm front}(t),t)f(X_{\rm front}(t)-x) \tag{7}$$

with the semiclassical (x,t) front propagation governed by the parameter $X_{\rm front}(t)-x$, as shown in Fig. 4(b).

In particular, the front moves subdiffusively, $X_{\rm front}(t)\sim t^{\beta(W)}$, where $\beta(W)$ is given by Eq. (5) [79].

(iii) At larger times (smaller distances within the wave packet), $x<X_{\rm front}(t)$, $\Pi(x,t)$ shows space-time factorization

$$\Pi(x,t)-\Pi(x,\infty)=g(x)[\Pi(0,t)-\Pi(0,\infty)], \tag{8}$$

with respect to the return probability $\Pi(0,t)$ and a certain function $g(x)$, as shown in Figs. 3 and 4(a). Thus, in this regime, the relaxation is dictated by the return probability which is connected to the front of propagation by the following relation:

$$\Pi(0,t)\sim\exp[-\lambda X_{\rm front}(t)],\quad \lambda>0. \tag{9}$$

(iv) Eventually at very long times $\Pi(x,t)$ saturates at the uniform distribution over sites $\Pi(x,\infty)=\mathcal{N}(x)/L$, where $\mathcal{N}(x)\sim K^x$ is the mean number of sites at some distance x from an initial site state $|x_0\rangle$ and L is the number of sites; see Fig. 3.

Stages (i) and (ii) are presented only for times $t<t_{\rm Th}$ corresponding to front propagation inside the graph $X_{\rm front}(t)<D$, where $D\simeq\ln L/\ln K$ is the diameter of the graph. At larger times only relaxation with the return probability (iii) and saturation (iv) stages are relevant.

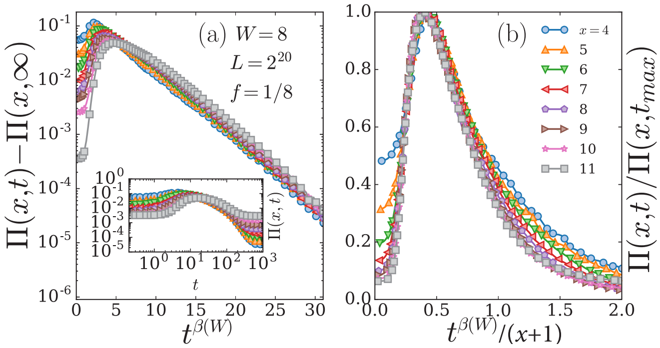

FIG. 4. Collapse of the probability $\Pi(x,t)$ versus t at different distances x. (a) Space-time factorization of $\Pi(x,t)$ Eq. (8) at large times $t > t_{\rm Th}(x)$ corresponding to the relaxation inside the wave packet $x < X_{\rm front}(t)$. This relaxation is proportional to the return probability $\Pi(0,t) \sim e^{-\Gamma t^{\beta(W)}}$. The time axis is properly rescaled with the power $\beta(W) = 1 - W/W_{\rm AT}$, Eq. (5), to emphasize stretched exponential decay of $\Pi(x,t)$. The inset shows $\Pi(x,t)$ on a log-log scale for all four stages of the evolution: initial distribution $\Pi(x,t) \approx \Pi(x,0)$ before wave front coming $x > X_{\rm front}(t)$, the maximum, the common tail after front passing, and the eventual saturation. (b) Collapse of $\Pi(x,t)$ Eq. (7) around the wave front confirming subdiffusive propagation $X_{\rm front}(t) \sim t^{\beta(W)}$. The parameters are the same as in Fig. 2.

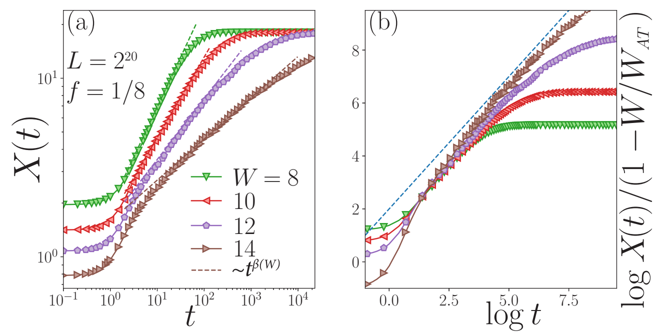

FIG. 5. Subdiffusive wave-packet spreading. (a) Wave-packet width $X(t) = \sum_x x\Pi(x,t)$ versus time t on a log-log scale for different disorder strengths supplemented by a guide for eyes $\sim t^{\beta(W)}$ (dashed lines), with $\beta(W) = 1 - W/W_{\rm AT}$, Eq. (5). (b) Collapse of wave-packet width $\ln X(t)/\beta(W)$ from panel (a) showing the unit slope versus t (dashed line) in increasing time interval with growing disorder amplitude W. The parameters are the same as in Fig. 2.

Results. We focus our attention on intermediate disorder values $8 \leqslant W \leqslant 14$, for which the return probability shows slow dynamics $\Pi(0,t) \sim e^{-\Gamma t^{\beta(W)}}$, with $\beta(W)$ given by Eq. (5) [80]. The propagation of $\Pi(x,t)$ versus distance x and time t at fixed system size $L = 2^{20} \approx 10^6$ is shown in Fig. 2(a). At small times, $\delta E\ t \sim O(1)$, the wave-packet width is small $X(0) \sim 2$, Eq. (4). As time evolves, the wave packet spreads in the form of wave-front $X_{\rm front}(t)$ [81] which transfers most of its weight to the most distant sites $[X(\infty) \simeq D]$, as shown in Fig. 2.

Although the dynamics on the RRG is typically not isotropic, the time scale $t_{\rm Th}$ at which the wave front reaches the diameter could be seen as a natural choice for the Thouless time analogous to the time that a charge needs to propagate through a diffusive conductor [82]. The wave-front propagation at times $t < t_{\rm Th}$, $X_{\rm front}(t) < D$, can be seen in Fig. 2(a) and is explicitly shown in Fig. 2 (b). Already for time $t_{\rm Th} \approx 22$, as emphasized in the color plot of Fig. 2(a) (red dashed line), the main core has lost most of its amplitude $\Pi(x \leqslant 5, t_{\rm Th})/\Pi(x \leqslant 5, 0) \sim 10^{-1}$ and $\Pi(x,t)$ becomes nearly uniform over the distance, $\Pi(x, t_{\rm Th}) \simeq$ const.

Figure 3(a) shows $\Pi(x,t)$ as a function of x at large times, $t > t_{\rm Th}$, when the front has already reached the diameter of the graph. In this regime, $\Pi(x,t)$ relaxes uniformly in distance x to the equiprobable configuration on the graph $\Pi(x,\infty) = \frac{\mathcal{N}(x)}{L}$ [dashed line in Fig. 3(a)]. Thus, at these times, $\Pi(x,t) - \Pi(x,\infty)$ is factorized in (x,t) according to Eq. (8), with $g(0) = 1$ due to the uniform relaxation seen as well for $x = 0$.

Detailed analysis shows that the factorization works beyond the limit, $t > t_{\rm Th}$, provided the wave front crossed the observation point $x < X_{\rm front}(t)$; see Fig. 4(a). Subtracting from $\Pi(x,t)$ its long time limit $\Pi(x,\infty)$ results in the collapse in Eq. (8) of the curves for any $x < X_{\rm front}(t)$. It is important to note that in Fig. 4 the time axis is rescaled as $t^{\beta(W)}$, with $\beta(W)$ given by Eq. (5) to emphasize the stretched-exponential time relaxation shown to be true for the return probability in Ref. [73], $\Pi(0,t) - \Pi(0,\infty) \simeq e^{-\Gamma t^{\beta(W)}}$. Moreover, raw $\Pi(x,t)$ is shown in the inset of Fig. 4(a) on a log-log scale to demonstrate nearly unperturbed short-time behavior of $\Pi(x,t)$ in Eq. (6).

In order to analyze the time dependence of the wave-front propagation $X_{\rm front}(t)$ in Fig. 4(b) we collapse the curves dividing $\Pi(x,t)$ by its maximum, $\Pi(X_{\rm front}(t), t)$, and rescale the time $t^{\beta(W)}$ in order to collapse the position of the maximum. This collapse allows us to extract the following subdiffusive wave-front evolution:

$$X_{\rm front}(t) \simeq \Gamma(W) t^{\beta(W)}, \quad \beta(W) < 1. \tag{10}$$

Moreover, the collapse of the curves, Fig. 4(b), implies the simple exponential dependence, Eq. (9), of the return probability versus $X_{\rm front}(t)$ with a certain decay rate λ, $f(z) = e^{-\lambda z}$, Eq. (7) [83]. The front-propagation collapse, Eq. (7), is shown to work also for different disorder strengths in the range of interest $8 \leqslant W \leqslant 13$ [80].

As a further consequence, the Thouless time, defined as the time when the wave front reaches the graph diameter, scales as $t_{\rm Th} \sim (\frac{\ln L}{K})^{1/\beta(W)}$. The similar scaling of the Thouless time calculated for MBL systems in the subdiffusive phase [84,85] supports the idea that wave-packet dynamics on RRG is a good proxy for MBL systems.

Finally, we analyze the first moment $X(t)$, Eq. (4), of the radial probability distribution $\Pi(x,t)$. Figure 5(a) shows the algebraic growth of $X(t)$ in time for several W

$$X(t) \sim t^{\beta(W)}, \tag{11}$$

with the same subdiffusive exponent $\beta(W)$, Eq. (5), as in the wave-front propagation $X_{\rm front}(t)$. Furthermore, the curves $X(t)$ can be reasonably well collapsed for the range of disorder strengths by considering the rescaled function $\frac{\ln X(t)}{\beta(W)}$ versus $\ln t$. This result is checked to be robust with respect to the finite-size effects and to the variation of the fraction f [80].

Conclusion and discussions. In this work, we provide evidence of the existence of a subdiffusive phase for a finite range of parameters by probing the dynamics of an initially localized particle on the RRG via the probability distribution $\Pi(x,t)$ to detect it at distance x at time t.

The relaxation of $\Pi(x,t)$ is characterized by the formation of a semiclassical wave front $X_{\text{front}}(t)$, moving subdiffusively to the most distant sites. Remarkably, as soon as the wave front passed the observation point x, the space-time factorization, $\Pi(x,t) = g(x)h(t)$, is found.

The Anderson model on RRG gives the first example of an entire subdiffusive phase as the systems on $\mathbb{Z}^d$ lattices are either localized for any finite disorder in $d = 1, 2$ or show subdiffusion only at the critical point, $d > 2$ [67,68].

It is important to note that the existence of a subdiffusive phase is not in contrast with the possibility that the eigenfunctions are ergodic in terms of the inverse participation ratio (IPR) scaling as the inverse of the volume. To emphasize further, the IPR scaling is a statement about the nature of the fluctuations of the eigenfunctions equivalent to the long time limit ($t \to \infty$) of certain dynamical observables. On the other hand, in our study we probe the time evolution of a wave packet and far away from the aforementioned limit. Thus our study excludes the scenario that the system is fully ergodic at $W \geqslant 8$, which is a stronger requirement than just IPR ergodicity discussed above [86].

We have to mention that some works [61,87] claim only diffusive propagation [$\beta(W) = 1$] for all $W < W_{\text{AT}}$; however, in the mathematically rigorous work [87], an absolutely continuous spectrum is assumed, which may not be so. We do not report any crossover to diffusivity for our available system sizes and time scales. Although we cannot completely rule out this possibility in the thermodynamic limits $L \to \infty$ and $t \to \infty$, the above finite-time subdiffusive dynamics is highly relevant for corresponding experiments in many-body systems.

In addition, the Anderson model on the RRG can be considered as a proxy for the dynamics of more realistic MBL systems. Our finding thus opens the possibility to have a subdiffusive dynamical phase in Fock space, that might imply slow relaxation of local observables. This possible implication of subdiffusive spatial dynamics in MBL systems [17–22,25] from slow Fock space dynamics may give rise to a different mechanism which does not invoke the existence of Griffiths effects [11,12,22]. Recent works (see, e.g., Ref. [88]) show that in MBL systems the subdiffusive phase is also consistent with a weakly ergodic phase confirming RRG as the commonly believed proxy.

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