# Electromagnetic response of superconductors in the presence of multiple collective modes

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Collective-mode fluctuations play an essential role in ensuring the electromagnetic response of superfluids is gauge invariant. The contribution of these fluctuations, however, is known to drop out from the Meissner response of uniform superfluids. The same phenomenon is not so established in the context of nonuniform superfluids. To clarify this issue, we revisit how collective modes appear in the Meissner effect. We find that their contribution vanishes both in uniform and nonuniform systems, unless an external length scale is present—as in Fulde-Ferrell or finite-sized superfluids. As examples, we consider $s$-wave and chiral $p+ip$ superconductors. To facilitate this analysis, we formulate a path-integral matrix methodology for computing the response of fermionic fluids in the presence of multiple collective modes. Closed-form expressions are provided, incorporating effects from phase and amplitude of the superconducting order parameter and electronic density fluctuations. All microscopic symmetries and invariances are manifestly satisfied in this approach, and it can be straightforwardly extended to other scenarios.

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## I. INTRODUCTION

Collective modes in superfluids and superconductors play a pivotal role in understanding gauge invariance in a many-particle context [1–3]. These modes comprise amplitude and phase fluctuations of the order parameter [4,5], and in the context of neutral superfluids the presence of the phase mode is evinced as a longitudinal sound oscillation [6–8]. Observation of the amplitude mode in a condensed-matter context, while possible, is rather challenging [9]. Some particular cases where this mode was indeed observed include systems with emergent Lorentz invariance [10–12] and superconductors coupled to either charge-density waves [13,14] or optical modes [15,16]. Collective modes in general provide nontrivial examples of the rich physics associated with broken symmetries and nontrivial ordering [17,18].

In contrast, the Meissner effect is conventionally understood as a "transverse" response [19,20], where "longitudinal" collective modes are not thought to participate. This issue was addressed, and partly clarified, in Ref. [21]. There it was shown that in nonuniform superfluids the longitudinal collective modes can possibly appear in what are termed—in the context of uniform systems—transverse response functions. In addition, Ref. [22] provided an explicit calculation of the electromagnetic (EM) response of the Fulde-Ferrell (FF) superfluid, which consists of finite-momentum Cooper pairs, and showed that the amplitude mode gives a significant contribution to the superfluid density. These issues motivate the current work, where we investigate the superfluid response for systems with nonuniform pairing, such as $p$-wave superfluids [23,24] and superconductors [25,26], and we provide a more general understanding on the type of superconductor where collective modes can contribute to the Meissner response.

We define a uniform superfluid or superconductor to be one where the order parameter two-point function is both translation and rotation invariant. A nonuniform system is one that is not uniform, and as such it violates either one or both of the conditions above. In the case of uniform $s$-wave superconductors, gauge invariance and the uniformity of the gap establishes that there is no collective-mode contribution to the Meissner effect [27]. When isotropy is broken, however, this argument needs to be revisited [21].

Phase fluctuations of the order parameter must be included to derive a gauge-invariant EM response [5]. On top of this, one can also consider amplitude fluctuations of the order parameter, and these have been shown [5] to be necessary to satisfy a thermodynamic sum rule, namely the compressibility sum rule [28,29]. Of particular interest is the response in $p$-wave superfluids [23–26] and also in systems with other pairing symmetries [30]. A complete calculation of the EM response for a chiral $p+ip$ system, in the presence of Coulomb and amplitude and phase fluctuations of the order parameter, has not been, to the best of our knowledge, presented in the literature, and the question of the Meissner response for such a system was unaddressed in Ref. [21]. In this paper we show that collective modes do not contribute to the Meissner effect in either uniform $s$-wave or nonuniform $p$-wave superconductors. More generally, our results show that collective modes do not contribute to the Meissner effect, independent of the pairing symmetry, in any superconductor that does not display an external wave-vector scale (e.g., finite-momentum pairing).

In order to derive this result, we develop a method for computing the gauge-invariant EM response of an electronic system with multiple collective modes present. Our analysis is based on an extension of the path-integral formulation of Ref. [5] and matrix-linear-response approaches of

 

Refs. [4,28,31]. One of our central results is to demonstrate how these collective modes can be incorporated in comprehensive and illuminating EM response tensors using singular-value decompositions. For pedagogical purposes we consider several examples of application, including the Coulomb screening in a normal metal and phase fluctuations in the EM response of a superfluid. We demonstrate the power of our formulation by obtaining the manifestly gauge-invariant EM response tensor for superconductors with amplitude, phase, and Coulomb fluctuations present. More generally, our results are applicable to a variety of scenarios beyond the scope of this work. They are relevant in any situation where energy scales compete, leading to intertwined ordering [17], or where symmetries provide multidimensional order parameters. The study of the contribution of distinct collective modes to the EM response tensor provides a direct method to access signatures of broken symmetries and nontrivial ordering.

The paper is organized as follows: in Sec. II we outline general formulas for the electromagnetic susceptibility tensors; a careful derivation of these formulas along with their covariance properties is provided in Appendices A and B. Following this, Sec. III provides a set of applications of these formulas, including: Coulomb screening, phase fluctuations in a superfluid, the gapping of phase modes in a superconductor by Coulomb screening, and finally the mixing of phase modes with amplitude-Higgs modes in a charged superconductor. This section contains our algebraic approach to screening by use of singular-value decompositions. Finally, Sec. IV addresses our discussions regarding the Meissner effect and we conclude in Sec. V. Appendices C–E provide further details on several relevant calculations.

## II. ELECTROMAGNETIC RESPONSE TENSOR

The starting point of our analysis is a fermionic system subject to a set of collective fluctuating degrees of freedom. The latter are described by a set of generalized coordinates, denoted by $\mathbf{\Delta}$, which should be thought of as a vector of Hubbard-Stratonovich decoupling fields. In the presence of an external EM probe $A$, we consider the dynamics of the EM response at the mean-field level, which is defined by the following conditions for each component $\Delta_a$ of $\mathbf{\Delta}$:

$$\left.\frac{\delta S_{\text{eff}}[\mathbf{\Delta}, A]}{\delta \mathbf{\Delta}_a(x)}\right|_{\mathbf{\Delta}=\mathbf{\Delta}_{\text{mf}}[A]} = 0. \tag{2.1}$$

Here $S_{\text{eff}}$ is the effective action for the fluctuating degrees of freedom, in the presence of the external EM probe, obtained after integration over the fermionic degrees of freedom [32]. The solutions to the mean-field equations, $\mathbf{\Delta}_{\text{mf}}[A]$, are no longer arbitrary fluctuating degrees of freedom to be functionally integrated over, but rather they are functions determined by the external EM probe [4,5,33]. As a result, the mean-field EM response tensor reads

$$K^{\mu\nu}_{\text{mf}}(x, y) = \left.\frac{\delta^2 S_{\text{eff}}[\mathbf{\Delta}_{\text{mf}}[A], A]}{\delta A_\mu(x)\delta A_\nu(y)}\right|_{A=0}. \tag{2.2}$$

Note that $K^{\mu\nu}(x, y) = K^{\nu\mu}(y, x)$. In this paper imaginary time will be used and thus $A^\mu = (A_0, \mathbf{A}) = (iA_t, \mathbf{A})$.

To evaluate these derivatives it is necessary to use a functional chain rule and differentiate all terms with dependence on the vector potential. This manipulation, together with an application of the mean-field equations in Eq. (2.1), is presented in Appendix A; the result is a matrix form for the mean-field-level EM response, namely,

$$K^{\mu\nu}_{\text{mf}}(x, y) = Q^{\mu\nu}(x, y) - \int_{z,z'} \{R^{\mu a}(x, z) \times [S^{-1}(z, z')]^{ab} R^{b\nu}(z', y)\}, \tag{2.3}$$

where

$$Q^{\mu\nu}(x, y) = \left.\frac{\delta^2 S_{\text{eff}}[\mathbf{\Delta}, A]}{\delta A_\mu(x)\delta A_\nu(y)}\right|_{A=0,\mathbf{\Delta}=\mathbf{\Delta}_{\text{mf}}[0]}, \tag{2.4}$$

$$R^{\mu a}(x, y) = R^{a\mu}(y, x) = \left.\frac{\delta^2 S_{\text{eff}}[\mathbf{\Delta}, A]}{\delta A_\mu(x)\delta \mathbf{\Delta}_a(y)}\right|_{A=0,\mathbf{\Delta}=\mathbf{\Delta}_{\text{mf}}[0]}, \tag{2.5}$$

and

$$S^{ab}(x, y) = \left.\frac{\delta^2 S_{\text{eff}}[\mathbf{\Delta}, A]}{\delta \mathbf{\Delta}_a(x)\delta \mathbf{\Delta}_b(y)}\right|_{A=0,\mathbf{\Delta}=\mathbf{\Delta}_{\text{mf}}[0]}. \tag{2.6}$$

Here the derivatives with respect to the gauge field $A$ act only on the explicit vector-potential dependence. In the second contribution of Eq. (2.3), we emphasize that the matrix $S^{ab}$ must be computed first, as in Eq. (2.6), and then inverted before being inserted into Eq. (2.3). In other words, Eq. (2.3) does not involve the inverse of each matrix element of Eq. (2.6), but rather the elements of the inverse of the matrix itself.

This expression contains several insightful properties. First, it manifestly decouples into two contributions which correspond, respectively, to the bubble and collective-mode linear responses. Second, as shown in Appendix B, this expression is reparametrization covariant, i.e., it does not change form under a basis transformation of $\mathbf{\Delta}$. This means that all fluctuations are considered symmetrically, in an unbiased manner. In the context of superconductivity, for example, Eq. (2.3) can be equally used for considering fluctuations in the real and imaginary parts of the superconducting pairing strength [31], or for fluctuations in the radial and phase degrees of freedom, as we shall do later in the paper. Third, by writing this expression in real space it affords greater generality and can thus be used, for example, in the presence of either impurities or defects occurring in collective-mode order parameters. For a translation-invariant system, the momentum-space representation is more tractable and reads

$$K^{\mu\nu}_{\text{mf}}(q) = Q^{\mu\nu}(q) - R^{\mu a}(q)[S^{-1}(q)]^{ab} R^{b\nu}(q), \tag{2.7}$$

where, for example,

$$Q^{\mu\nu}(x, y) = Q^{\mu\nu}(x - y) = \int_q e^{-iq\cdot(x-y)} Q^{\mu\nu}(q). \tag{2.8}$$

We use the short-hand notation $\int_q = TL^d \sum_{i\Omega_m} \int \frac{d\mathbf{q}}{(2\pi)^d}$, where $L$ is a length scale, $d$ is the number of spatial dimensions, $T$ is the temperature, and $\Omega_m$ is a bosonic Matsubara frequency. Natural units $c = \hbar = k_B = 1$ are used throughout the paper.

## III. GENERAL APPLICATIONS

In this section we present several applications of Eq. (2.7). For the benefit of the reader, in the following subsections we take a pedagogical approach and start with a rather detailed calculation of the application of Eq. (2.3) in two familiar scenarios: Sec. III A electrostatic screening and Sec. III B gauge-invariant response in superfluids due to phase fluctuations. With the mathematical procedures well established, we will then move on at a progressively faster pace: in Sec. III C we study the next simplest possible scenario—a superconductor with phase fluctuations—and here we introduce the concept of folding the effects of competing fluctuations using singular-value decompositions. The dénouement of this section is Sec. III D, where we put all this methodology together to compute the EM response tensor in the nontrivial case of concomitantly fluctuating Coulomb and superconducting phase and amplitude degrees of freedom. To clarify our terminology, a superconductor is a charged system with Coulomb interactions present and a superfluid is a neutral system.

### A. Screening due to electrostatic interactions

Consider an interacting electronic system in $D = d + 1$ space-time dimensions with an action given by

$$\begin{aligned} S[A] = &- \int d^D x d^D x' \psi_\sigma^\dagger(x) \mathcal{G}_0^{-1}[A](x, x') \psi_\sigma(x') \\ &+ \frac{e^2}{2} \int d^D x d^D x' \delta n(x) V(x - x') \delta n(x') \\ &+ ie \int d^D x A_t(x) n_0, \end{aligned} \tag{3.1}$$

where $\delta n(x) = \psi_\sigma^\dagger(x)\psi_\sigma(x) - n_0$, with $n_0$ the constant background density, $\sigma = \downarrow, \uparrow$ is a spin index (summed if repeated), and the (bare) inverse Green's function is

$$\mathcal{G}_0^{-1}[A](x, x') = -[\partial_\tau - ieA_t(x) + h(\hat{\mathbf{p}} - e\mathbf{A})]\delta(x - x'). \tag{3.2}$$

The single-particle Hamiltonian, denoted by $h(\mathbf{p})$, is kept general at this stage. For concreteness, we assume instantaneous interactions: $V(x - x') = V(\mathbf{x} - \mathbf{x}')\delta(\tau - \tau')$. Throughout the paper we shall interchangeably refer to electronic density fluctuations as Coulomb fluctuations. The generating functional for electromagnetic response is then

$$\mathcal{Z}[A] = \int \mathcal{D}[\psi^\dagger, \psi] e^{-S[A]}. \tag{3.3}$$

We are interested in how nonuniform charge distributions affect the EM response of this system. Thus it is natural to consider decoupling the electrostatic interaction terms via a Hubbard-Stratonovich decomposition as

$$\mathcal{Z}[A] \sim \int \mathcal{D}\varphi e^{-S_{\text{eff}}[\varphi, A]}. \tag{3.4}$$

Defining $\beta = 1/T$ and $\mathcal{G}_0^{-1}[\varphi, A] = \mathcal{G}_0^{-1}[A_t + \varphi, \mathbf{A}]$, the effective action is

$$\begin{aligned} S_{\text{eff}}[\varphi, A] = &\int d^D x d^{D-1} x' \frac{\varphi(\mathbf{x}, \tau)\varphi(\mathbf{x}', \tau)}{2V(\mathbf{x} - \mathbf{x}')} \\ &+ ie \int d^D x [A_t(x) + \varphi(x)] n_0 \\ &- \operatorname{Tr} \ln\left( - \beta \mathcal{G}_0^{-1}[\varphi, A]\right). \end{aligned} \tag{3.5}$$

The capitalized trace denotes a trace over all space-time/momentum-frequency and internal (uncapitalized trace) degrees of freedom:

$$\operatorname{Tr} \ln\left( - \beta \mathcal{G}_0^{-1}[\varphi, A]\right) = \int d^D x \operatorname{tr}\langle x | \ln\left( - \beta \mathcal{G}_0^{-1}[\varphi, A]\right) | x \rangle. \tag{3.6}$$

In this language we obtain the building blocks for Eq. (2.3) (which are tantamount to undressed polarization tensors). In fact, due to translation invariance, we can focus on the expressions in momentum space used in Eq. (2.7). For instance [34],

$$\begin{aligned} Q^{\mu\nu}(q) &\equiv \left. \frac{\delta^2 S_{\text{eff}}[\varphi, A]}{\delta A_\mu(-q) \delta A_\nu(q)} \right|_{A,\varphi=0} \\ &= - \left. \frac{\delta^2 \operatorname{Tr} \ln\left(-\beta \mathcal{G}_0^{-1}[\varphi, A]\right)}{\delta A_\mu(-q) \delta A_\nu(q)} \right|_{A,\varphi=0}. \end{aligned} \tag{3.7}$$

Similarly, noticing that $A_0 = iA_t$ and that all terms involving $\varphi$ appear in the Green's function as $iA_t + i\varphi$, one finds

$$R^{\mu\varphi}(q) \equiv \left. \frac{\delta^2 S_{\text{eff}}[\varphi, A]}{\delta A_\mu(-q) \delta \varphi(q)} \right|_{A,\varphi=0} = iQ^{\mu 0}(q), \tag{3.8}$$

$$\begin{aligned} S^{\varphi\varphi}(q) &\equiv \left. \frac{\delta^2 S_{\text{eff}}[\boldsymbol{\Delta}, A]}{\delta \varphi(-q) \delta \varphi(q)} \right|_{A,\varphi=0} \\ &= V^{-1}(q) - Q^{00}(q). \end{aligned} \tag{3.9}$$

Conveniently, all building blocks can be expressed in terms of the undressed polarization tensor $Q^{\mu\nu}(q)$. An in-depth analysis of these expressions is provided in Appendix C.

Applying Eq. (2.7) now becomes a simple matter (we drop the $q$-dependence label for simplicity):

$$\begin{aligned} K_{\text{mf}}^{\mu\nu} &= Q^{\mu\nu} - \left(iQ^{\mu 0}\right)(V^{-1} - Q^{00})^{-1}(iQ^{0\nu}) \\ &= Q^{\mu\nu} + \frac{Q^{\mu 0} V Q^{0\nu}}{1 - V Q^{00}} \equiv \widetilde{Q}^{\mu\nu}. \end{aligned} \tag{3.10}$$

The last definition will be used throughout later sections of the paper. The above result reproduces the screening effect of Coulomb fluctuations. In particular, the RPA charge-charge susceptibility [29] is obtained:

$$K_{\text{mf}}^{00} = \frac{Q^{00}}{1 - V Q^{00}}. \tag{3.11}$$

### B. EM response for superfluids (with no amplitude fluctuations)

Another simple application of Eq. (2.7) concerns the gauge-invariant EM response tensor for superfluids with phase fluctuations of the order parameter. In superfluids where the mean-field order parameter takes on a finite vacuum expectation value the global U(1) symmetry is spontaneously broken. To restore gauge invariance, the phase fluctuations of the order parameter must be included. In this section we consider a superfluid where the amplitude of the order parameter is rigidly pinned down to its mean-field value, but allow the phase to depend on the external EM probe.

It is straightforward to analyze this scenario with our present approach. Consider a set of nonrelativistic spin-$\frac{1}{2}$ particles, with free Hamiltonian $h(\mathbf{p}) = \mathbf{p}^2/(2m) - \mu$, interacting instantaneously with each other via an attractive, translation invariant but possibly anisotropic potential $g(\mathbf{x}-\mathbf{x}')$. In the presence of an external probe field $A$, the action reads

$$S[A] = -\int d^Dxd^Dx'\psi_\sigma^\dagger(x)\mathcal{G}_0^{-1}[A](x,x')\psi_\sigma(x') - \int d^Dxd^Dx'\psi_\uparrow^\dagger(x)\psi_\downarrow^\dagger(x')g(x-x')\psi_\downarrow(x')\psi_\uparrow(x) + ie\int d^DxA_tn_0. \tag{3.12}$$

Here $g(x-x') = g(\mathbf{x}-\mathbf{x}')\delta(\tau-\tau')$.

Preparing again for the mean-field treatment of the problem, we now perform a Hubbard-Stratonovich decomposition in the Cooper channel to arrive at the generating functional

$$\mathcal{Z}[A] \sim \int \mathcal{D}[\Delta,\Delta^*]\mathcal{D}[\psi^\dagger,\psi]e^{-S_{\text{bos}}}e^{-S_{\text{el}}}, \tag{3.13}$$

where the bosonic contribution to the action is

$$S_{\text{bos}} = ie\int d^DxA_tn_0 + \int d^Dxd^{D-1}x'\frac{|\Delta(\mathbf{x},\mathbf{x}',\tau)|^2}{g(\mathbf{x}-\mathbf{x}')} \tag{3.14}$$

and the electronic contribution is

$$S_{\text{el}} = -\int d^Dxd^Dx'\psi_\sigma^\dagger(x)\mathcal{G}_0^{-1}[A](x,x')\psi_\sigma(x') - \int d^Dxd^{D-1}x' \times [\psi_\uparrow^\dagger(\mathbf{x},\tau)\Delta(\mathbf{x},\mathbf{x}',\tau)\psi_\downarrow^\dagger(\mathbf{x}',\tau) + \text{H.c.}]. \tag{3.15}$$

Before integrating out the fermions, remember that the symmetry of the interaction potential $g(\mathbf{x}-\mathbf{x}')$ is decisive in determining the symmetry structure of the pairing field. Due to the homogeneity of the problem (in the absence of strong driving external EM fields), it is advantageous to use relative and center-of-mass coordinates to describe the pairing field:

$$\Delta(\mathbf{x},\mathbf{x}',\tau) \to \Delta\left(\mathbf{x}-\mathbf{x}', \frac{\mathbf{x}+\mathbf{x}'}{2}, \tau\right). \tag{3.16}$$

We ignore spin-orbit coupling. In this case, spherical anisotropy in the pairing potential can be captured in a gradient expansion of $\Delta$,

$$\Delta\left(\mathbf{x}-\mathbf{x}', \frac{\mathbf{x}+\mathbf{x}'}{2}, \tau\right) = \left|\Delta_s\left(\frac{\mathbf{x}+\mathbf{x}'}{2},\tau\right)\right|e^{i\Phi_s\left(\frac{\mathbf{x}+\mathbf{x}'}{2},\tau\right)}\delta(\mathbf{x}-\mathbf{x}') + \left|\Delta_p\left(\frac{\mathbf{x}+\mathbf{x}'}{2},\tau\right)\right|e^{i\Phi_p\left(\frac{\mathbf{x}+\mathbf{x}'}{2},\tau\right)}(\partial_x + i\partial_y)\delta(\mathbf{x}-\mathbf{x}') + \cdots, \tag{3.17}$$

where we favor an amplitude-phase coordinate choice. In general, the pairing potential will select only one term in Eq. (3.17); the structure we chose for the interaction, in fact, favors opposite-spin pairing by construction. Nevertheless, we can remain fairly general and write

$$S_{\text{bos}} = ie\int d^DxA_tn_0 + \int d^Dx\frac{|\Delta(\mathbf{x},\tau)|^2}{\widetilde{g}}, \tag{3.18}$$

where $\widetilde{g}$ is a renormalized value for $g$, and

$$S_{\text{el}} = -\int d^Dxd^Dx'\psi_\sigma^\dagger(x)\mathcal{G}_0^{-1}[A](x,x')\psi_\sigma(x') - \int d^Dx[\Delta(\mathbf{x},\tau)\psi_\uparrow^\dagger(\mathbf{x},\tau)\hat{D}\psi_\downarrow^\dagger(\mathbf{x},\tau) + \text{H.c.}], \tag{3.19}$$

where $\Delta(\mathbf{x},\tau) = \rho(x)e^{i\theta(x)}$ for a general amplitude and phase and $\hat{D}$ corresponds to a differential operator that depends on the symmetry channel. In Appendix D we consider an explicit application of this to a spinless $p$-wave problem.

We are now ready to integrate out the fermions; introducing the Nambu spinor $\Psi = (\psi_\uparrow, \psi_\downarrow, \psi_\uparrow^\dagger, \psi_\downarrow^\dagger)^T$, the electronic part of the action becomes

$$S_{\text{el}} = -\frac{1}{2}\int d^Dxd^Dx'\Psi^\dagger(x)\mathcal{G}^{-1}[A](x,x')\Psi(x'), \tag{3.20}$$

where the (inverse) Nambu-space Green's function is

$$\mathcal{G}^{-1}[A](x,x') = -\begin{pmatrix} \left[\partial_\tau - ie\widetilde{A}_t + \left(\frac{[\hat{\mathbf{p}}-e\widetilde{\mathbf{A}}]^2}{2m} - \mu\right)\right] & -\rho(x)i\sigma_y\hat{D} \\ \rho(x)i\sigma_y\hat{D}^\dagger & \left[\partial_\tau + ie\widetilde{A}_t - \left(\frac{[\hat{\mathbf{p}}+e\widetilde{\mathbf{A}}]^2}{2m} - \mu\right)\right] \end{pmatrix}\delta(x-x'), \tag{3.21}$$

$\sigma_y$ acts on the spin degrees of freedom. As customarily performed [25,35,36], we have rotated away the superconducting phase, which is conveniently absorbed by the gauge fields as $\widetilde{A}_\mu = A_\mu - \frac{1}{2e}\partial_\mu\theta$. The generating functional thus becomes

$$\mathcal{Z}[A] \sim \int \mathcal{D}[\Delta,\Delta^*]e^{-S_{\text{eff}}[\Delta,\Delta^*,A]}, \tag{3.22}$$

where the effective action is (dropping the $A$-dependence label)

$$S_{\text{eff}}[\Delta,\Delta^*,A] = S_{\text{bos}} - \tfrac{1}{2}\text{Tr}\ln(-\beta\mathcal{G}^{-1}), \tag{3.23}$$

with $S_{\text{bos}}$ as in Eq. (3.18) and one should keep in mind the factor of $\frac{1}{2}$ due to Nambu doubling.

At this point we consider the mean-field response. In this section, we will neglect fluctuations of the superconducting amplitude, setting $\rho(x) \to \rho_0$. It is then possible to use the relationship between $\widetilde{A}_\mu$ and $A_\mu$ to write

$$\begin{aligned}\frac{\delta S_{\text{eff}}[\theta, A]}{\delta\theta(x)} &= \int dy \frac{\delta S_{\text{eff}}[\theta, A]}{\delta\partial_\alpha\theta(y)}\frac{\delta\partial_\alpha\theta(y)}{\delta\theta(x)} \\ &= -\partial_\alpha \frac{\delta S_{\text{eff}}[\theta, A]}{\delta\partial_\alpha\theta(x)} \\ &= \frac{1}{2e}\partial_\alpha \frac{\delta S_{\text{eff}}[\theta, A]}{\delta A_\alpha(x)}. \end{aligned} \tag{3.24}$$

The factor of $2e$ can be safely absorbed as it will drop out from the correlation functions; we will omit it from now on. This allows us to once again write all the momentum-space tensors in terms of the undressed polarization tensors $Q^{\mu\nu}$, namely,

$$R^{\mu\theta}(q) = iQ^{\mu\beta}(q)q_\beta, \tag{3.25}$$

$$R^{\theta\nu}(q) = -iq_\alpha Q^{\alpha\nu}(q), \tag{3.26}$$

$$S^{\theta\theta}(q) = q_\lambda Q^{\lambda\sigma}(q)q_\sigma. \tag{3.27}$$

At the mean-field level $\theta$ is a constant and drops out from the Green's functions. Notice that the Green's functions appearing in $Q^{\mu\nu}(q)$ in this case correspond to Eq. (3.21) with $\widetilde{A}_\mu = 0$ and $\rho(x) \to \rho_0$. Implementing Eq. (2.7), the EM response is then

$$\begin{aligned} K^{\mu\nu}_{\text{mf}} &= Q^{\mu\nu} - \big(iQ^{\mu\beta}q_\beta\big)\big(q_\lambda Q^{\lambda\sigma}q_\sigma\big)^{-1}\big(-iq_\alpha Q^{\alpha\nu}\big) \\ &= Q^{\mu\nu} - \frac{Q^{\mu\beta}q_\beta q_\alpha Q^{\alpha\nu}}{q_\lambda Q^{\lambda\sigma}q_\sigma} \equiv \Pi^{\mu\nu}. \end{aligned} \tag{3.28}$$

This is the general form of the EM response tensor for a neutral superfluid, independent of the pairing symmetry. The gapless fluctuating phase degree of freedom is crucial to ensure gauge invariance, which the form above manifestly obeys: $q_\mu K^{\mu\nu}_{\text{mf}}(q) = K^{\mu\nu}_{\text{mf}}(q)q_\nu = 0$. Setting $q_\lambda Q^{\lambda\sigma}q_\sigma = 0$ recovers the well-known result of Anderson and Bogoliubov [37,38]: The EM response has a pole corresponding to a long-wavelength sound mode (with speed $c_s = v_F/\sqrt{3}$ at $T = 0$) induced by phase fluctuations of the order parameter. An equivalent result has been found in previous literature [35,39], where gauge invariance is recovered by including, and integrating over, the fluctuations of the phase of the order parameter. In our approach no such functional integration is necessary; gauge invariance is guaranteed by the implicit dependence of the order parameter on the external vector potential in Eq. (2.1).

### C. EM response for superconductors (with no amplitude fluctuations)

With the previous results established, for our first nontrivial application of Eq. (2.7) we consider a charged superconductor with both phase and Coulomb fluctuations present. This problem was also considered in Ref. [25], in the context of the EM response of a chiral $p + ip$ superconductor, via sequential functional integration of the Coulomb and phase degrees of freedom. It is natural to ask what the form of the EM response would be if this procedure were performed in the opposite order, and this will be addressed in what follows. In our case, the results from the previous sections allow the response to be written as

$$\begin{aligned} K^{\mu\nu}_{\text{mf}} &= Q^{\mu\nu} - \begin{pmatrix} iQ^{\mu 0} \\ iQ^{\mu\beta}q_\beta \end{pmatrix}^T \\ &\quad\times \begin{pmatrix} V^{-1} - Q^{00} & -Q^{0\beta}q_\beta \\ q_\alpha Q^{\alpha 0} & q_\lambda Q^{\lambda\sigma}q_\sigma \end{pmatrix}^{-1} \begin{pmatrix} iQ^{0\nu} \\ -iq_\alpha Q^{\alpha\nu} \end{pmatrix} \\ &= Q^{\mu\nu} - \frac{1}{(V^{-1} - Q^{00})q_\lambda \widetilde{Q}^{\lambda\sigma}q_\sigma} \begin{pmatrix} Q^{\mu 0} \\ Q^{\mu\beta}q_\beta \end{pmatrix}^T \\ &\quad\times \begin{pmatrix} -q_\lambda Q^{\lambda\sigma}q_\sigma & Q^{0\beta}q_\beta \\ q_\alpha Q^{\alpha 0} & V^{-1} - Q^{00} \end{pmatrix} \begin{pmatrix} Q^{0\nu} \\ q_\alpha Q^{\alpha\nu} \end{pmatrix}. \end{aligned} \tag{3.29}$$

The Coulomb-screened EM response tensor $\widetilde{Q}^{\lambda\sigma}$ of Sec. III A naturally appears here in the denominator.

While Eq. (3.29) treats the Coulomb- and phase-screened responses of a charged superconductor in a symmetric fashion, the present form is not totally satisfactory. In particular, gauge invariance is not manifest, and it may be advantageous to recover similar results found in the previous section, as well as the polaritonic resonances of the EM response. To accomplish this, we have to "bias" the above expression towards either a Coulomb-screened type of object or a phase-screened type of object. An analogy from the process of Ref. [25] would be to consider integrating out first either the electrostatic Coulomb field or the phase degree of freedom.

Let us make this procedure more explicit. With a few manipulations, we may explicitly rewrite $Q^{\mu\nu}$ in terms of its Coulomb-screened version $\widetilde{Q}^{\mu\nu}$ so that Eq. (3.29) then has the form

$$\begin{aligned} K^{\mu\nu}_{\text{mf}} &= \widetilde{Q}^{\mu\nu} - \frac{q_\alpha q_\beta}{q_\lambda \widetilde{Q}^{\lambda\sigma}q_\sigma} \begin{pmatrix} Q^{\mu 0} \\ Q^{\mu\beta} \end{pmatrix}^T \\ &\quad\times \begin{pmatrix} \frac{Q^{\alpha 0}Q^{0\beta}}{(V^{-1}-Q^{00})^2} & \frac{Q^{0\beta}}{V^{-1}-Q^{00}} \\ \frac{Q^{\alpha 0}}{V^{-1}-Q^{00}} & 1 \end{pmatrix} \begin{pmatrix} Q^{0\nu} \\ Q^{\alpha\nu} \end{pmatrix}. \end{aligned} \tag{3.30}$$

A natural question to consider is how to best simplify the expression above; how can one extract a concise and manifestly gauge-invariant formula from which the physical phenomena can be deduced? Naively expanding the above expression is an unwieldy endeavor, which would also be increasingly intractable as the number of order-parameter degrees of freedom enlarges. The pivotal realization is that the $2 \times 2$ matrix appearing in the EM response now has zero determinant: it is a singular matrix, which can be expressed using a singular-value decomposition (SVD). Consider the following matrix:

$$M = \begin{pmatrix} ab & a \\ b & 1 \end{pmatrix}. \tag{3.31}$$

Define the matrices $U$, $V$, and $D$ by

$$U = \begin{pmatrix} a & \frac{a}{|a|} \\ 1 & -|a| \end{pmatrix}, \quad V = \begin{pmatrix} b^* & \frac{b^*}{|b|} \\ 1 & -|b| \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.32}$$

The matrix $M$ can then be written as $M = UDV^\dagger$. By matching the coefficients $a$ and $b$ with the coefficients in Eq. (3.30), one obtains

$$K_{\text{mf}}^{\mu\nu} = \widetilde{Q}^{\mu\nu} - \frac{\widetilde{Q}^{\mu\beta} q_\beta q_\alpha \widetilde{Q}^{\alpha\nu}}{q_\lambda \widetilde{Q}^{\lambda\sigma} q_\sigma} \equiv \widetilde{\Pi}^{\mu\nu}. \tag{3.33}$$

Here we have "biased" the matrix expression in Eq. (3.30) into the simpler equation above. It assumes the form of an EM response tensor in the presence of phase fluctuations, as in Eq. (3.28), but now the EM polarization tensors are substituted by their Coulomb-screened versions: $Q^{\mu\nu} \rightarrow \widetilde{Q}^{\mu\nu}$. This expression is manifestly gauge invariant as in Eq. (3.28). Interestingly, this biasing process can be easily done in the reverse manner. In performing similar manipulations to arrive at Eq. (3.30), if we had first exchanged $Q^{\mu\nu}$ for $\Pi^{\mu\nu}$, instead of $\widetilde{Q}^{\mu\nu}$, then it is a simple exercise to show that by an analog SVD the EM response tensor obtained reads

$$K_{\text{mf}}^{\mu\nu} = \Pi^{\mu\nu} + \frac{\Pi^{\mu 0} V \Pi^{0\nu}}{1 - V\Pi^{00}}. \tag{3.34}$$

This expression assumes a Coulomb-screened form, where each tensor participating has been replaced by its phase-screened version: $Q^{\mu\nu} \rightarrow \Pi^{\mu\nu}$. Evidently, since each $\Pi^{\mu\nu}$ is gauge invariant by itself, the whole expression above is gauge invariant again. Naturally, both expressions for $K_{\text{mf}}^{\mu\nu}$ above are equivalent.

Thus, we have introduced a process of folding the effects of each fluctuating field via an SVD of the response tensors. This process clearly biases the form of $K_{\text{mf}}^{\mu\nu}$, although it brings simplification. The denominators of the final form of these response tensors contain the polaritonic resonances of the dielectric functions [28,31,40]. Equating the two denominators to zero

$$q_\lambda \widetilde{Q}^{\lambda\sigma} q_\sigma = 0 = 1 - V\Pi^{00}, \tag{3.35}$$

one obtains the well-known Carlson-Goldman (CG) mode [31,41,42], where plasmons dress the phase fluctuation poles, gapping the phase modes of charged superconductors. At $T = 0$ this results solely in a (double) plasmon mode, whereas in the vicinity of $T \sim T_c$ there is a soft mode (which was originally [41,42] termed the CG mode) and a plasmon mode [31]. Note that the exact relation between the two denominators is: $q_\lambda \widetilde{Q}^{\lambda\sigma} q_\sigma (1 - VQ^{00}) = q_\lambda Q^{\lambda\sigma} q_\sigma (1 - V\Pi^{00})$.

The CG mode arises from the mutual contributions of Coulomb and phase fluctuations to the EM response. The standard approach to consider this, as implemented for example in Refs. [31,36,43–45], involves studying the phase fluctuations using a real-imaginary representation for the superconducting order parameter components. Here we have used the amplitude-phase parametrization, which disentangles the important phase fluctuations from the nonfluctuating amplitude. As discussed in Sec. II, the response formulas are reparametrization covariant, and as demonstrated in Appendix B the collective-mode resonances, which arise from the vanishing of the determinant of the matrix $S_{ab}$ in Eq. (2.6), are also preserved.

### D. EM response for superconductors (with amplitude fluctuations)

Returning to Eq. (3.21), we now include the fluctuations in $\rho(x)$. Contrary to the phase and Coulomb responses, the amplitude part cannot be written solely in terms of the unscreened EM response bubble $Q^{\mu\nu}(q)$. The additional objects which must be defined for calculating the EM response functions read as follows:

$$S^{\rho\rho}(q) \equiv \left. \frac{\delta^2 S_{\text{eff}}[\mathbf{\Delta}, A]}{\delta\rho(-q)\delta\rho(q)} \right|_{A=0,\, \mathbf{\Delta}=\mathbf{\Delta}_{\text{mf}}[0]}, \tag{3.36}$$

$$R^{\rho\mu}(q) \equiv \left. \frac{\delta^2 S_{\text{eff}}[\mathbf{\Delta}, A]}{\delta\rho(-q)\delta A_\mu(q)} \right|_{A=0,\, \mathbf{\Delta}=\mathbf{\Delta}_{\text{mf}}[0]}, \tag{3.37}$$

and similarly

$$S^{\rho\theta}(q) \equiv \left. \frac{\delta^2 S_{\text{eff}}[\mathbf{\Delta}, A]}{\delta\rho(-q)\delta\theta(q)} \right|_{A=0,\, \mathbf{\Delta}=\mathbf{\Delta}_{\text{mf}}[0]} = iR^{\rho\beta}(q) q_\beta, \tag{3.38}$$

$$S^{\rho\varphi}(q) \equiv \left. \frac{\delta^2 S_{\text{eff}}[\mathbf{\Delta}, A]}{\delta\rho(-q)\delta\varphi(q)} \right|_{A=0,\, \mathbf{\Delta}=\mathbf{\Delta}_{\text{mf}}[0]} = iR^{\rho 0}(q). \tag{3.39}$$

Also note that just as $V^{-1}(q)$ contributed to $S^{\varphi\varphi}(q)$ [cf. Eq. (3.9)], the "mass" contribution for $\rho(x)$ in the Hubbard-Stratonovich field in Eq. (3.18) implies that $\widetilde{g}^{-1}$ contributes to $S^{\rho\rho}(q)$.

The EM response tensor now becomes

$$K_{\text{mf}}^{\mu\nu}(q) = Q^{\mu\nu} - \begin{pmatrix} R^{\mu\rho} & iQ^{\mu 0} & iQ^{\mu\beta} q_\beta \end{pmatrix} \begin{pmatrix} S^{\rho\rho} & iR^{\rho 0} & iR^{\rho\beta} q_\beta \\ iR^{0\rho} & V^{-1} - Q^{00} & -Q^{0\beta} q_\beta \\ -iq_\alpha R^{\alpha\rho} & q_\alpha Q^{\alpha 0} & q_\alpha Q^{\alpha\beta} q_\beta \end{pmatrix}^{-1} \begin{pmatrix} R^{\rho\nu} \\ iQ^{0\nu} \\ -iq_\alpha Q^{\alpha\nu} \end{pmatrix}. \tag{3.40}$$

The SVD approach can also be implemented for this situation. First note that, with some manipulation and SVD biasing, the determinant can be reduced to two possible forms. The calculation is outlined in Appendix E and results in

$$\begin{aligned} \det S(q) &= (V^{-1} - \overline{Q}^{00}) S^{\rho\rho} q_\alpha \overline{\widetilde{Q}}^{\alpha\beta} q_\beta \\ &= (V^{-1} - Q^{00}) \widetilde{S}^{\rho\rho} q_\alpha \overline{\widetilde{Q}}^{\alpha\beta} q_\beta, \end{aligned} \tag{3.41}$$

where tilde variables are screened as in Eq. (3.10); for example,

$$\widetilde{S}^{\rho\rho} = S^{\rho\rho} + \frac{R^{\rho 0} V R^{0\rho}}{1 - VQ^{00}}. \tag{3.42}$$

Similarly, the process of "folding" the amplitude fluctuations also leads to "screened" tensors—the ones with a bar on top. Repeating the calculation in Eq. (3.10), now with only the amplitude contributions, one verifies

$$\overline{Q}^{\alpha\beta} \equiv Q^{\alpha\beta} - \frac{R^{\alpha\rho}R^{\rho\beta}}{S^{\rho\rho}}. \tag{3.43}$$

Finally, tensors with both a bar and a tilde are interpreted to mean first evaluate the tensors with respect to the outer screening symbol and then with respect to the inner screening type. To be concrete, as examples we have

$$\overline{\widetilde{Q}}^{\alpha\beta} = \widetilde{Q}^{\alpha\beta} - \frac{\widetilde{R}^{\alpha\rho}\widetilde{R}^{\rho\beta}}{\widetilde{S}^{\rho\rho}}, \tag{3.44}$$

$$\widetilde{\overline{Q}}^{\alpha\beta} = \overline{Q}^{\alpha\beta} + \frac{\overline{Q}^{\alpha 0} V \overline{Q}^{0\beta}}{1 - V\overline{Q}^{00}}. \tag{3.45}$$

From the two ways of writing the determinant above, and noticing that

$$\begin{aligned}(V^{-1} - Q^{00})S^{\rho\rho} + R^{\rho 0}R^{0\rho} &= (V^{-1} - \overline{Q}^{00})S^{\rho\rho} \\ &= (V^{-1} - Q^{00})\widetilde{S}^{\rho\rho},\end{aligned} \tag{3.46}$$

we find an important identity:

$$q_\alpha \widetilde{\overline{Q}}^{\alpha\beta} q_\beta = q_\alpha \overline{\widetilde{Q}}^{\alpha\beta} q_\beta. \tag{3.47}$$

Using these expressions we can now perform the SVD process as in the previous sections, the only requirement is to choose a biasing order in which we want to take into account the influence of each type of fluctuation. For example, taking into account the inversion of the matrix $S(q)$ and the determinant above, we obtain

$$K_{\rm mf}^{\mu\nu}(q) = Q^{\mu\nu} - \frac{q_\alpha q_\beta}{q_\lambda \widetilde{\overline{Q}}^{\lambda\sigma} q_\sigma} \begin{pmatrix} R^{\mu\rho} \\ Q^{\mu 0} \\ Q^{\mu\beta} \end{pmatrix}^T \begin{pmatrix} \frac{(V^{-1}-Q^{00})\widetilde{Q}^{\alpha\beta}}{(V^{-1}-\overline{Q}^{00})S^{\rho\rho}} & \frac{Q^{\alpha\beta}R^{\rho 0} - Q^{\alpha 0}R^{\rho\beta}}{(V^{-1}-\overline{Q}^{00})S^{\rho\rho}} & -\frac{(V^{-1}-Q^{00})\widetilde{R}^{\rho\beta}}{(V^{-1}-\overline{Q}^{00})S^{\rho\rho}} \\ \frac{Q^{\alpha\beta}R^{0\rho} - R^{\alpha\rho}Q^{0\beta}}{(V^{-1}-\overline{Q}^{00})S^{\rho\rho}} & -\frac{\overline{Q}^{\alpha\beta}}{(V^{-1}-\overline{Q}^{00})} & \frac{\overline{Q}^{0\beta}}{(V^{-1}-\overline{Q}^{00})} \\ -\frac{(V^{-1}-Q^{00})\widetilde{R}^{\alpha\rho}}{(V^{-1}-\overline{Q}^{00})S^{\rho\rho}} & \frac{\overline{Q}^{\alpha 0}}{(V^{-1}-\overline{Q}^{00})} & 1 \end{pmatrix} \begin{pmatrix} R^{\rho\nu} \\ Q^{0\nu} \\ Q^{\alpha\nu} \end{pmatrix}. \tag{3.48}$$

Now we focus on the first term $Q^{\mu\nu}$. Introducing the effects of amplitude fluctuations first ("bar" variables) and subsequently the regular screening from Coulomb fluctuations ("tilde" variables), a straightforward calculation and simplification using the relations in Eq. (3.47) results in

$$K_{\rm mf}^{\mu\nu} = \widetilde{\overline{Q}}^{\mu\nu} - \frac{q_\alpha q_\beta}{q_\lambda \widetilde{\overline{Q}}^{\lambda\sigma} q_\sigma} \begin{pmatrix} R^{\mu\rho} \\ Q^{\mu 0} \\ Q^{\mu\beta} \end{pmatrix}^T \begin{pmatrix} \frac{\widetilde{R}^{\alpha\rho}\widetilde{R}^{\rho\beta}}{(\widetilde{S}^{\rho\rho})^2} & -\frac{\overline{Q}^{\alpha 0}\widetilde{R}^{\rho\beta}}{(V^{-1}-\overline{Q}^{00})\widetilde{S}^{\rho\rho}} & -\frac{\widetilde{R}^{\rho\beta}}{\widetilde{S}^{\rho\rho}} \\ -\frac{\widetilde{R}^{\alpha\rho}\overline{Q}^{0\beta}}{(V^{-1}-\overline{Q}^{00})\widetilde{S}^{\rho\rho}} & \frac{\overline{Q}^{\alpha 0}\overline{Q}^{0\beta}}{(V^{-1}-\overline{Q}^{00})^2} & \frac{\overline{Q}^{0\beta}}{(V^{-1}-\overline{Q}^{00})} \\ -\frac{\widetilde{R}^{\alpha\rho}}{\widetilde{S}^{\rho\rho}} & \frac{\overline{Q}^{\alpha 0}}{(V^{-1}-\overline{Q}^{00})} & 1 \end{pmatrix} \begin{pmatrix} R^{\rho\nu} \\ Q^{0\nu} \\ Q^{\alpha\nu} \end{pmatrix}. \tag{3.49}$$

This matrix is now of the form

$$M = \begin{pmatrix} ab & -ad & -a \\ -bc & cd & c \\ -b & d & 1 \end{pmatrix}, \tag{3.50}$$

where

$$a = \frac{\widetilde{R}^{\rho\beta}}{\widetilde{S}^{\rho\rho}}, \quad b = \frac{\widetilde{R}^{\alpha\rho}}{\widetilde{S}^{\rho\rho}},$$
$$c = \frac{\overline{Q}^{0\beta}}{V^{-1} - \overline{Q}^{00}}, \quad d = \frac{\overline{Q}^{\alpha 0}}{V^{-1} - \overline{Q}^{00}}. \tag{3.51}$$

It displays two linearly dependent rows, thus suggesting the singular-value decomposition. Performing the SVD and simplifying the result gives

$$K_{\rm mf}^{\mu\nu} = \widetilde{\overline{Q}}^{\mu\nu} - \frac{\widetilde{\overline{Q}}^{\mu\beta} q_\beta q_\alpha \widetilde{\overline{Q}}^{\alpha\nu}}{q_\lambda \widetilde{\overline{Q}}^{\lambda\sigma} q_\sigma} \equiv \widetilde{\overline{\Pi}}^{\mu\nu}. \tag{3.52}$$

Setting $q_\lambda \widetilde{\overline{Q}}^{\lambda\sigma} q_\sigma = 0$ gives the collective mode dispersion for the polaritons induced by simultaneous Coulomb, phase, and amplitude fluctuations. Again, gauge invariance in the SVD-simplified EM response in Eq. (3.52) is manifest:

$$q_\mu K_{\rm mf}^{\mu\nu} = q_\mu \widetilde{\overline{Q}}^{\mu\nu} - \frac{q_\mu \widetilde{\overline{Q}}^{\mu\beta} q_\beta q_\alpha \widetilde{\overline{Q}}^{\alpha\nu}}{q_\lambda \widetilde{\overline{Q}}^{\lambda\sigma} q_\sigma} = 0. \tag{3.53}$$

As in the previous section, other equivalent forms for the EM response can be obtained by reversing the order in the SVD processes. For example, $K_{\rm mf}^{\mu\nu} = \widetilde{\overline{\Pi}}^{\mu\nu} = \overline{\widetilde{\Pi}}^{\mu\nu}$.

## IV. THE MEISSNER EFFECT IN THE PRESENCE OF COLLECTIVE MODES

### A. Kubo formula

In this section we calculate the superfluid density for superfluid and superconducting systems with amplitude, phase, and Coulomb fluctuations incorporated. It was shown in the previous section that the EM response for a system with all these three types of fluctuations can be compactly written as in Eq. (3.52). Here we will use this formula to study the Meissner response for both $s$-wave and chiral $p + ip$ systems. The Kubo formula for the superfluid density tensor is [27]

$$\frac{e^2}{m} n_s^{xx} = \lim_{\mathbf{q}\to 0} K_{\rm mf}^{ii}(\Omega = 0, \mathbf{q}), \tag{4.1}$$

with no implicit index summation. It is crucial that the static limit $\Omega = 0$ is taken before the long-wavelength limit $\mathbf{q} \to 0$ is considered. This particular order of limits is appropriate for a thermodynamic quantity, whereas the converse procedure is apt for the calculation of optical properties, namely the DC electrical conductivity for instance. For nonuniform systems, the limit $\mathbf{q} \to 0$ must also be carefully specified. To ascertain the appropriate definition, recall that in the presence of an external EM vector potential $A_\nu$, the EM current is $J^\mu(x) = \int_{x'} K^{\mu\nu}(x, x')A_\nu(x')$. The continuity equation is $\partial_\mu J^\mu = 0$; this statement enforces conservation of global particle number [global U(1) symmetry] for a neutral superfluid, whereas for a charged system it enforces conservation of charge. In terms of the response kernel, this equation becomes $(\partial_\mu K^{\mu\nu})A_\nu = 0$. The solution to this equation, for an arbitrary $A_\nu$, is to require a gauge-invariant EM response: $\partial_\mu K^{\mu\nu} = 0$, which in momentum space reads $q_\mu K^{\mu\nu} = 0$. As shown in the previous section, the SVD approach enables this to be manifestly satisfied.

To compute $n_s$ it is convenient to work in the gauge where $\partial_\mu A_\mu = 0$ (Lorenz gauge), which reduces to the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$ in the static limit. The momentum-space form of the Coulomb gauge is $\mathbf{q} \cdot \mathbf{A} = 0$. In deriving the superfluid density $n_s^{ii}$, only the $i$th component of the vector field must be nonvanishing: $A_i \neq 0$. The Coulomb-gauge condition then reduces to $q_i A_i = 0$, demanding $q_i = 0$. The other momentum components go to zero only in the limit. Thus, the appropriate Kubo formula for $n_s^{ii}$ is

$$\frac{e^2}{m} n_s^{ii} = \lim_{q_i \neq q_k, q_i = 0, q_k \to 0} K_{\mathrm{mf}}^{ii}(\Omega = 0, \mathbf{q}). \tag{4.2}$$

This Kubo formula explains why the superfluid density is often termed a "transverse" response [27,29]. In the particular case of nonuniform superfluids, however, the appellation transverse loses its significance. The importance of computing the superfluid density in the appropriate limiting fashion was discussed in Ref. [22], where it was shown that for the Fulde-Ferrell superfluid the amplitude collective mode contributes to the superfluid density. A general argument for why collective modes do not need to be considered in the superfluid density response of uniform superfluids is as follows [27]. In the presence of the external vector potential $A$, the order parameter can be expanded to quadratic order in $A$ as

$$\Delta[A] = \Delta[A = 0] + \Delta^{(1)}[A] + O(A^2). \tag{4.3}$$

Since the order parameter $\Delta$ is a scalar, whereas the vector potential $A$ is a vector, $\Delta$ can depend only on scalar-valued functions of $A$. For a uniform superfluid, the only such scalar quantity is $\nabla \cdot \mathbf{A}$. In the Coulomb gauge, where $\nabla \cdot \mathbf{A} = 0$, it follows that $\Delta^{(1)} = 0$. Thus, collective modes do not contribute to the superfluid density in a uniform superfluid. In the case of a nonuniform superfluid, there are potentially other scalar quantities that depend on $\mathbf{A}$ and thus $\Delta^{(1)}$ need not be zero. The next section provides an explicit calculation of the superfluid density for both $s$-wave and chiral $p + ip$ superconductors with amplitude, phase, and Coulomb interactions.

### B. Explicit superfluid density calculation

First consider the case of a uniform $s$-wave superfluid. Without loss of generality, since the system is uniform we only need to study the response in one direction, say $\hat{x}$. Using the formalism developed in the previous sections, the superfluid density is given by

$$\begin{aligned} \frac{e^2}{m} n_s^{xx} &= \lim_{q_x=0, q_y \to 0} \left[ \widetilde{\widetilde{Q}}^{xx} - \frac{\widetilde{\widetilde{Q}}^{xi} q_i q_j \widetilde{\widetilde{Q}}^{jx}}{q_k \widetilde{\widetilde{Q}}^{kl} q_l} \right] \\ &= \lim_{q_x=0, q_y \to 0} \left[ \widetilde{\widetilde{Q}}^{xx} - \frac{\widetilde{\widetilde{Q}}^{xy} \widetilde{\widetilde{Q}}^{yx}}{\widetilde{\widetilde{Q}}^{yy}} \right]. \end{aligned} \tag{4.4}$$

In the small-momentum limit, $R^{\rho j}(0, \mathbf{q} \to 0) = 0$; this is because in this limit the tensor structure requires $R^{\rho j}(0, \mathbf{q} \to 0) \sim q^j \to 0$. Thus, the generalized response functions are

$$\widetilde{\widetilde{Q}}^{xj} = \overline{Q}^{xj} + \frac{\overline{Q}^{x0} \overline{Q}^{0j}}{V^{-1} - \overline{Q}^{00}} = Q^{xj}. \tag{4.5}$$

As a result, the superfluid density is

$$\frac{e^2}{m} n_s^{xx} = \lim_{q_x=0, q_y \to 0} Q^{xx}. \tag{4.6}$$

This proves that without any particular assumptions about particle-hole symmetry, i.e., whether or not the amplitude and Coulomb mode decouple ($R^{\rho 0} \neq 0$) [46], the superfluid density for an $s$-wave system has no contributions from amplitude, phase, or Coulomb collective modes. This is an explicit proof of the argument presented in the previous section.

Now consider a spinless-$(p + ip)$ superfluid in two spatial dimensions. The $x$ and $y$ responses are equivalent, thus we again only need to consider the former. The superfluid density is as given in Eq. (4.4). Again $R^{\rho j}(0, \mathbf{q} \to 0) = 0$ remains true, and thus

$$\frac{e^2}{m} n_s^{xx} = \lim_{q_x=0, q_y \to 0} Q^{xx}. \tag{4.7}$$

This particular limit is computed as shown below. After performing the Matsubara frequency summation, the response function is [25,28]

$$\begin{aligned} Q^{ij}(i\Omega_m, \mathbf{q}) = \frac{e^2}{2} \int \frac{d^2\mathbf{p}}{(2\pi)^2} \frac{\mathbf{p}^i}{m} \frac{\mathbf{p}^j}{m} \Bigg[ &\left( 1 + \frac{\xi_\mathbf{p}^+ \xi_\mathbf{p}^- + \Delta_0^2 \mathbf{p}_+ \cdot \mathbf{p}_- / p_F^2}{E_\mathbf{p}^+ E_\mathbf{p}^-} \right) \frac{E_\mathbf{p}^+ - E_\mathbf{p}^-}{(E_\mathbf{p}^+ - E_\mathbf{p}^-)^2 - (i\Omega_m)^2} [f(E_\mathbf{p}^+) - f(E_\mathbf{p}^-)] \\ - &\left( 1 - \frac{\xi_\mathbf{p}^+ \xi_\mathbf{p}^- + \Delta_0^2 \mathbf{p}_+ \cdot \mathbf{p}_- / p_F^2}{E_\mathbf{p}^+ E_\mathbf{p}^-} \right) \frac{E_\mathbf{p}^+ + E_\mathbf{p}^-}{(E_\mathbf{p}^+ + E_\mathbf{p}^-)^2 - (i\Omega_m)^2} [1 - f(E_\mathbf{p}^-) - f(E_\mathbf{p}^+)] \Bigg] + \frac{ne^2}{m} \delta^{ij}, \end{aligned} \tag{4.8}$$

where $\mathbf{p}_\pm = \mathbf{p} \pm \mathbf{q}/2$, $\xi^\pm_\mathbf{p} \equiv \xi_{\mathbf{p}\pm\mathbf{q}/2}$, $E^\pm_\mathbf{p} \equiv E_{\mathbf{p}\pm\mathbf{q}/2}$, with $\xi_\mathbf{p} = \mathbf{p}^2/(2m) - \mu$, $E_\mathbf{p} = \sqrt{\xi^2_\mathbf{p} + \Delta_0^2\mathbf{p}^2/p_F^2}$, and $n$ is the total number density. Taking the appropriate frequency and momentum limits results in

$$\begin{aligned}\frac{e^2}{m}n_s^{xx} &= e^2\left[\frac{n}{m} + \int \frac{d^2\mathbf{p}}{(2\pi)^2}\left(\frac{p^x}{m}\right)^2 \frac{\partial f(E_\mathbf{p})}{\partial E_\mathbf{p}}\right] \\ &= e^2 \int \frac{d^2\mathbf{p}}{(2\pi)^2}\left(\frac{p^x}{m}\right)^2 \frac{\Delta_0^2\mathbf{p}^2/p_F^2}{E^2_\mathbf{p}} \\ &\quad\times \left[\frac{1-2f(E_\mathbf{p})}{2E_\mathbf{p}} + \frac{\partial f(E_\mathbf{p})}{\partial E_\mathbf{p}}\right]. \end{aligned} \tag{4.9}$$

In general, for a superfluid system with only one external momentum, namely the momentum $\mathbf{q}$ of the external vector potential $\mathbf{A}$, the EM response can be decomposed into terms comprised of $\delta^{ij}$ and $q^iq^j/\mathbf{q}^2$. In the limit $\mathbf{q} \to 0$, as defined above, it follows that the off-diagonal terms vanish and thus the superfluid density reduces to the standard undressed bubble term. Unless there are other *external* vectors that can couple to the vector potential, the superfluid density always reduces to the undressed bubble term. This statement is a generalization of the analysis in the previous section, which considered only uniform superfluids; here we extend the veracity of the previous proof to include all kinds of superfluids without other *external* vectors that couple to the vector potential.

### C. Transverse and longitudinal responses

In Ref. [21] the EM response for nonuniform superfluids without amplitude fluctuations was derived. This particular article highlighted that for such superfluids the collective modes are, in general, no longer solely "longitudinal," and moreover these modes can be important in what are conventionally termed transverse response functions in the case of uniform systems. In this section we show that our generalized formula reproduces the particular case considered in Ref. [21], namely, a neutral system with only phase fluctuations of the order parameter. Using Eq. (3.52), the response function for such a system, in the static limit, is given by

$$K^{ij}_{\mathrm{mf}}(0,\mathbf{q}) = Q^{ij}(0,\mathbf{q}) - \frac{Q^{ia}(0,\mathbf{q})q_aq_bQ^{bj}(0,\mathbf{q})}{q_cQ^{cd}(0,\mathbf{q})q_d}. \tag{4.10}$$

The undressed EM response (for a spin-$\frac{1}{2}$ system with $e = 1$) reads [27,28]

$$\begin{aligned}Q^{ij}(0,\mathbf{q}) = 2\sum_p \left(\frac{p^i}{m}\frac{p^j}{m}\right)[G(i\omega_n,\mathbf{p}_+)G(i\omega_n,\mathbf{p}_-) \\ + F^*(i\omega_n,\mathbf{p}_+)F(i\omega_n,\mathbf{p}_-)] + \frac{n}{m}\delta^{ij}.\end{aligned} \tag{4.11}$$

The nonbold momenta are four-vectors $p^\mu = (i\omega_n, \mathbf{p})$ with $\omega_n$ a fermionic Matsubara frequency. For simplicity, let us focus on a system with a general-momentum and angle-dependent gap $\Delta_\mathbf{p} \equiv \Delta(\hat{\mathbf{p}})$. The single-particle and anomalous Green's functions are [27,28]

$$G(i\omega_n,\mathbf{p}) = -\frac{i\omega_n + \xi_\mathbf{p}}{\omega_n^2 + \xi_\mathbf{p}^2 + |\Delta_\mathbf{p}|^2}, \tag{4.12}$$

$$F(i\omega_n,\mathbf{p}) = \frac{\Delta_\mathbf{p}}{\omega_n^2 + \xi_\mathbf{p}^2 + |\Delta_\mathbf{p}|^2}. \tag{4.13}$$

A generic static correlation function for a uniform system has the form

$$K^{ij}(0,\mathbf{q}) = \chi_L\frac{q^iq^j}{\mathbf{q}^2} + \chi_T\left(\delta^{ij} - \frac{q^iq^j}{\mathbf{q}^2}\right). \tag{4.14}$$

Here $\chi_T$ and $\chi_L$ denote the transverse and longitudinal part of the full response function, respectively. By taking the dot product with $q^i$ and $q^j$, the longitudinal part is

$$\chi_L = \frac{q^iK^{ij}q^j}{q^2}. \tag{4.15}$$

The longitudinal part of the total response gives zero contribution to the Meissner effect: the full response is purely transverse. In the small-momentum limit the collective-mode part of the response [the second term in Eq. (4.10)] is purely longitudinal, and thus it gives zero contribution to the superfluid density.

Let $i = j$ in Eq. (4.14) and take the trace to obtain $\sum_i K^{ii} = \chi_L + 2\chi_T$. Therefore the transverse part is

$$\chi_T = \frac{1}{2}\left(\sum_i K^{ii} - \chi_L\right). \tag{4.16}$$

Let $(m/n)\chi_T \equiv \chi'_T$. Using Eq. (4.11), this becomes

$$\begin{aligned}\chi'_T(q) = \frac{1}{mn}\sum_p p^2\sin^2(\theta)[G(i\omega_n,\mathbf{p}_+)G(i\omega_n,\mathbf{p}_-) \\ + F^*(i\omega_n,\mathbf{p}_+)F(i\omega_n,\mathbf{p}_-)] + 1.\end{aligned} \tag{4.17}$$

We drop the $q$ dependence in the argument of $\chi_T$ from now on. To evaluate this quantity we invoke standard Fermi-liquid theory and assume a constant density of states near the Fermi surface. Using this approximation, the transverse response then becomes [27]

$$\begin{aligned}\chi'_T = 1 + T\frac{3}{4}\sum_{\omega_n}\int_0^\pi d\theta \sin^3(\theta)\int_{-\infty}^{\infty} d\xi \\ \times \frac{(i\omega_n + \xi_+)(i\omega_n + \xi_-) + \Delta_+^*\Delta_-}{(\omega_n^2 + \xi_+^2 + |\Delta_+|^2)(\omega_n^2 + \xi_-^2 + |\Delta_-|^2)}.\end{aligned} \tag{4.18}$$

Here $\xi_\pm \approx \xi_\mathbf{p} \pm \frac{1}{2}qv_F\cos(\theta)$ with $v_F = p_F/m$ the Fermi velocity, and we have also used a constant density of states approximation $k_F^3 = 3\pi^2 n$. Also, the momentum dependence of $\Delta_\mathbf{p}$ is scaled by $p_F$ and therefore its external $q$ dependence can be neglected. As discussed in Ref. [27], the result of performing the Matsubara frequency summation followed by the $\xi$ integration leads to the correct normal-state result. However, performing this procedure in the reverse order leads to a different answer, in contradiction to the absence of a normal-state Meissner effect. To circumvent this problem, the method employed is to add and subtract the normal-state density expression. This enables performing the integration over $\xi$ first, which results in

$$\chi_T' \approx \frac{3\pi}{4} T \sum_{\omega_n} \int_{-1}^{1} \frac{dx}{\sqrt{\omega_n^2 + |\Delta_{\mathbf{p}}|^2}} \frac{(1-x^2)|\Delta_{\mathbf{p}}|^2}{\omega_n^2 + |\Delta_{\mathbf{p}}|^2 + \frac{1}{4}q^2 v_F^2 x^2}, \tag{4.19}$$

where we performed a small-$q$ expansion. For comparison, the EM current given in Ref. [21] reads

$$\mathbf{J}(\mathbf{q}) = \int dS_p R(\hat{\mathbf{p}})\hat{\mathbf{p}}[\hat{\mathbf{p}} \cdot \mathbf{A}(\mathbf{q}) - \hat{\mathbf{p}} \cdot \hat{\mathbf{q}}\phi(q)], \tag{4.20}$$

with the function $R(\hat{\mathbf{p}}) \equiv R(\hat{\mathbf{p}}; 0, \hat{\mathbf{q}})$ given by

$$R(\hat{\mathbf{p}}; 0, \hat{\mathbf{q}}) = T \sum_{\omega_n} \frac{1}{\sqrt{\omega_n^2 + |\Delta_{\mathbf{p}}|^2}} \frac{|\Delta_{\mathbf{p}}|^2}{\omega_n^2 + |\Delta_{\mathbf{p}}|^2 + \frac{1}{4}q^2 v_F^2 x^2} \tag{4.21}$$

and $\phi(q)$ given by

$$\phi(q) = \frac{\int dS_l R(\hat{\mathbf{l}})\hat{\mathbf{l}} \cdot \hat{\mathbf{q}}\hat{\mathbf{l}} \cdot \mathbf{A}(\mathbf{q})}{\int dS_k R(\hat{\mathbf{k}})(\hat{\mathbf{k}} \cdot \hat{\mathbf{q}})^2}. \tag{4.22}$$

It is straightforward to check that this expression conserves particle number: $\mathbf{q} \cdot \mathbf{J} = 0$. The corresponding response kernel is thus

$$K^{ij}(\Omega = 0, \mathbf{q}) = Q^{ij} - \frac{Q^{ia}q_a q_b Q^{bj}}{q_c Q^{cd} q_d}, \tag{4.23}$$

where $Q^{ij}(0, \mathbf{q}) \equiv \int dS_p \hat{p}^i R(\hat{\mathbf{p}}; 0, \mathbf{q})\hat{p}^j$, with $dS_p$ the measure on the Fermi surface. Furthermore, the transverse part of the response is [47]

$$\begin{aligned}\chi_T' &= \frac{3\pi}{4} \int_{-1}^{1} dx(1-x^2)R(\hat{\mathbf{p}}; 0, \mathbf{q}) \\ &= \frac{3\pi}{4} T \sum_{\omega_n} \int_{-1}^{1} \frac{dx}{\sqrt{\omega_n^2 + |\Delta_{\mathbf{p}}|^2}} \frac{(1-x^2)|\Delta_{\mathbf{p}}|^2}{\omega_n^2 + |\Delta_{\mathbf{p}}|^2 + \frac{1}{4}q^2 v^2 x^2}.\end{aligned} \tag{4.24}$$

Therefore, we have shown that Eq. (4.19), which followed from our generalized formula for phase fluctuations, agrees with Eq. (4.24).

To finish, consider a two-dimensional superfluid where the current and vector potential are parallel: $J^x = K^{xx}A_x$, $J^y = K^{yy}A_y$. The ratio of the EM kernels is

$$\frac{\lambda_x^2}{\lambda_y^2} = \frac{K^{xx}}{K^{yy}}. \tag{4.25}$$

In Ref. [21], where the effects from phase collective modes were the focus, it was pointed out that in the case of a dipolar superfluid this quantity is not unity. The analysis in this section shows that, in the static and long-wavelength limit, the full response is purely transverse, and thus there is no collective-mode contribution to the above ratio. The reason for its departure from unity [21] is merely because the undressed bubble contributions are distinct for the dipolar superfluid.

## V. CONCLUSIONS

The rich physics associated with superfluids and superconductors is most perceptible in the collective fluctuations of the order parameter. These modes show that superconductors are more than just gapped fluids of condensed electron-electron pairs. Rather, superconductors are systems replete with collective excitations due to coherent many-particle effects. Historically these modes were first studied in the context of restoring gauge invariance in a superconductor. More recently, however, a bevy of literature has studied these excitations in more general settings, and one particularly important problem has been understanding their role in the Meissner effect.

The antecedent literature to the present work suggested that collective modes may be ignored in $s$-wave systems, but must be accounted for if the order parameter is anisotropic ($p$-wave, $d$-wave, etc). In this paper we have extended this analysis by developing a general method for computing the electromagnetic response in systems with multiple collective modes. We have shown that, in fact, collective modes do not contribute to the Meissner effect in either uniform or nonuniform superconductors. An exception to this scenario comes about when external wave-vector scales exist, as in Fulde-Ferrell finite-momentum paired superconductors. The by-product of our study was to show that through singular-value decompositions, the electromagnetic response in a system with multiple collective modes present can naturally be computed by folding the various response tensors into dressed constituents. With all of the details we provided, we anticipate that this methodology will also prove useful in other contexts such as charge-density waves and quantum magnetism.

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## APPENDIX A: DERIVATION OF THE MEAN-FIELD EM RESPONSE TENSOR

In this excursus we derive in detail the EM response tensor in Eq. (2.3). For concreteness, whenever we write $\delta S_{\text{eff}}[\mathbf{\Delta}, A]/\delta A_\nu(x')$ (no $\mathbf{A}$ dependence in $\mathbf{\Delta}$), we mean the explicit $A$ dependence is being differentiated, with the collective-mode fields fixed. The functional chain rule produces

$$\frac{\delta S_{\text{eff}}[\mathbf{\Delta}_{\text{mf}}[A], A]}{\delta A_\nu(y)} = \left(\frac{\delta S_{\text{eff}}[\mathbf{\Delta}, A]}{\delta A_\nu(y)}\right)_{\mathbf{\Delta}_{\text{mf}}[A]} + \int_z \left(\frac{\delta S_{\text{eff}}[\mathbf{\Delta}, A]}{\delta \mathbf{\Delta}_a(z)}\right)_{\mathbf{\Delta}_{\text{mf}}[A]} \frac{\delta \mathbf{\Delta}_a^{\text{mf}}[A](z)}{\delta A_\nu(y)}. \tag{A1}$$

At the end of the calculation the value of $\mathbf{\Delta}$ is set to its mean-field value $\mathbf{\Delta}_{\rm mf}[A]$. Similarly, the second derivative of the above expression reads

$$\begin{aligned}\frac{\delta^2 S_{\rm eff}[\mathbf{\Delta}_{\rm mf}[A],A]}{\delta A_\mu(x)\delta A_\nu(y)} &= \left(\frac{\delta^2 S_{\rm eff}[\mathbf{\Delta},A]}{\delta A_\mu(x)\delta A_\nu(y)}\right)_{\mathbf{\Delta}=\mathbf{\Delta}_{\rm mf}[A]} + \int_{z,z'} \frac{\delta\mathbf{\Delta}_a^{\rm mf}[A](z)}{\delta A_\mu(x)}\left(\frac{\delta^2 S_{\rm eff}[\mathbf{\Delta},A]}{\delta\mathbf{\Delta}_a(z)\delta\mathbf{\Delta}_b(z')}\right)_{\mathbf{\Delta}=\mathbf{\Delta}_{\rm mf}[A]}\frac{\delta\mathbf{\Delta}_b^{\rm mf}[A](z')}{\delta A_\nu(y)} \\ &+ \int_z \frac{\delta\mathbf{\Delta}_a^{\rm mf}[A](z)}{\delta A_\mu(x)}\left(\frac{\delta^2 S_{\rm eff}[\mathbf{\Delta},A]}{\delta\mathbf{\Delta}_a(z)\delta A_\nu(y)}\right)_{\mathbf{\Delta}=\mathbf{\Delta}_{\rm mf}[A]} + \int_z \left(\frac{\delta^2 S_{\rm eff}[\mathbf{\Delta},A]}{\delta A_\mu(x)\delta\mathbf{\Delta}_a(z)}\right)_{\mathbf{\Delta}=\mathbf{\Delta}_{\rm mf}[A]}\frac{\delta\mathbf{\Delta}_a^{\rm mf}[A](z)}{\delta A_\nu(y)} \\ &+ \int_z \left(\frac{\delta S_{\rm eff}[\mathbf{\Delta},A]}{\delta\mathbf{\Delta}_a(z)}\right)_{\mathbf{\Delta}=\mathbf{\Delta}_{\rm mf}[A]}\frac{\delta^2\mathbf{\Delta}_a^{\rm mf}[A](z)}{\delta A_\mu(x)\delta A_\nu(y)}. \end{aligned} \tag{A2}$$

Since we are interested in the mean-field EM response, we can invoke the saddle-point condition

$$0 = \left.\frac{\delta S_{\rm eff}[\mathbf{\Delta},A]}{\delta\mathbf{\Delta}_a(z)}\right|_{\mathbf{\Delta}=\mathbf{\Delta}_{\rm mf}[A]}; \tag{A3}$$

thus the last term in Eq. (A2) gives zero mean-field contribution and can be dropped. If one were to consider the EM response at the Gaussian order, however, then this term would contribute. It remains to compute the derivatives of the collective-mode fields $\mathbf{\Delta}_a$ with respect to the vector potential. This can be done by considering the saddle-point conditions. Differentiating Eq. (A3) with respect to $A$ gives

$$\begin{aligned} 0 &= \frac{\delta}{\delta A_\nu(y)}\left(\frac{\delta S_{\rm eff}[\mathbf{\Delta},A]}{\delta\mathbf{\Delta}_a(z)}\right)_{\mathbf{\Delta}=\mathbf{\Delta}_{\rm mf}[A]} \\ &= \left(\frac{\delta^2 S_{\rm eff}[\mathbf{\Delta},A]}{\delta A_\nu(y)\delta\mathbf{\Delta}_a(z)}\right)_{\mathbf{\Delta}=\mathbf{\Delta}_{\rm mf}[A]} + \int_{z'}\left(\frac{\delta^2 S_{\rm eff}[\mathbf{\Delta},A]}{\delta\mathbf{\Delta}_a(z)\delta\mathbf{\Delta}_b(z')}\right)_{\mathbf{\Delta}=\mathbf{\Delta}_{\rm mf}[A]}\frac{\delta\mathbf{\Delta}_b^{\rm mf}[A](z')}{\delta A_\nu(y)}. \end{aligned} \tag{A4}$$

Inverting the saddle-point integral equation yields

$$\frac{\delta\mathbf{\Delta}_b^{\rm mf}[A](z')}{\delta A_\nu(y)} = -\int_z \left(\frac{\delta^2 S_{\rm eff}[\mathbf{\Delta},A]}{\delta\mathbf{\Delta}_b(z')\delta\mathbf{\Delta}_a(z)}\right)^{-1}_{\mathbf{\Delta}=\mathbf{\Delta}_{\rm mf}[A]}\left(\frac{\delta^2 S_{\rm eff}[\mathbf{\Delta},A]}{\delta\mathbf{\Delta}_a(z)\delta A_\nu(y)}\right)_{\mathbf{\Delta}=\mathbf{\Delta}_{\rm mf}[A]}. \tag{A5}$$

Substituting this into Eq. (A2) and taking $A \to 0$, we then obtain Eq. (2.3) of the main text:

$$K_{\rm mf}^{\mu\nu}(x,y) = \left(\frac{\delta^2 S_{\rm eff}[\mathbf{\Delta},A]}{\delta A_\mu(x)\delta A_\nu(y)}\right)_{\mathbf{\Delta}_{\rm mf}[0]} - \int_{z,z'}\left(\frac{\delta^2 S_{\rm eff}[\mathbf{\Delta},A]}{\delta A_\mu(x)\delta\mathbf{\Delta}_a(z)}\right)_{\mathbf{\Delta}_{\rm mf}[0]}\left(\frac{\delta^2 S_{\rm eff}[\mathbf{\Delta},A]}{\delta\mathbf{\Delta}_a(z)\delta\mathbf{\Delta}_b(z')}\right)^{-1}_{\mathbf{\Delta}_{\rm mf}[0]}\left(\frac{\delta^2 S_{\rm eff}[\mathbf{\Delta},A]}{\delta\mathbf{\Delta}_b(z')\delta A_\nu(y)}\right)_{\mathbf{\Delta}_{\rm mf}[0]}. \tag{A6}$$

## APPENDIX B: REPARAMETRIZATION COVARIANCE

The expression in Eq. (A6) for the mean-field EM response is general in the sense that it is the same regardless of the parametrization of the order parameter. We now prove this result. Consider a transformation $\Delta \to \widetilde{\Delta}$:

$$\begin{aligned}\frac{\delta S_{\rm eff}}{\delta\Delta_a(y)} &= \int_z \frac{\delta\widetilde{\Delta}_b(z)}{\delta\Delta_a(y)}\frac{\delta S_{\rm eff}}{\delta\widetilde{\Delta}_b(z)} \\ &= \int_z J_{ab}(y-z)\frac{\delta S_{\rm eff}}{\delta\widetilde{\Delta}_b(z)}. \end{aligned} \tag{B1}$$

Here $S_{\rm eff} \equiv S_{\rm eff}[\mathbf{\Delta},A]$. At the mean-field level, the second derivative now produces

$$\left(\frac{\delta^2 S_{\rm eff}}{\delta\Delta_a(y)\delta\Delta_b(y')}\right)_{\mathbf{\Delta}_{\rm mf}[0]} = \int_{z,z'} J_{ac}(y-z)\left(\frac{\delta^2 S_{\rm eff}}{\delta\widetilde{\Delta}_c(z)\delta\widetilde{\Delta}_d(z')}\right)_{\mathbf{\Delta}_{\rm mf}[0]} J_{db}(z'-y'). \tag{B2}$$

The EM response can thus be written as follows:

$$\begin{aligned} K_{\rm mf}^{\mu\nu}(x,x') - \left(\frac{\delta^2 S_{\rm eff}}{\delta A_\mu(x)\delta A_\nu(x')}\right)_{\mathbf{\Delta}_{\rm mf}[0]} &= -\int_{w,w',y,y',z,z'}\left(\frac{\delta^2 S_{\rm eff}}{\delta A_\mu(x)\delta\widetilde{\mathbf{\Delta}}_{a'}(w)}J_{a'a}(w-y)\right)_{\mathbf{\Delta}_{\rm mf}[0]} \\ &\quad\times\left(J_{ab'}(y-z)\frac{\delta^2 S_{\rm eff}}{\delta\widetilde{\mathbf{\Delta}}_{b'}(z)\delta\widetilde{\mathbf{\Delta}}_{c'}(z')}J_{c'b}(z'-y')\right)^{-1}_{\mathbf{\Delta}_{\rm mf}[0]}\left(J_{bd'}(y'-w')\frac{\delta^2 S_{\rm eff}}{\delta\widetilde{\mathbf{\Delta}}_{d'}(w')\delta A_\nu(x')}\right)_{\mathbf{\Delta}_{\rm mf}[0]} \\ &= -\int_{z,z'}\left(\frac{\delta^2 S_{\rm eff}}{\delta A_\mu(x)\delta\widetilde{\mathbf{\Delta}}_{b'}(z)}\right)_{\mathbf{\Delta}_{\rm mf}[0]}\left(\frac{\delta^2 S_{\rm eff}}{\delta\widetilde{\mathbf{\Delta}}_{b'}(z)\delta\widetilde{\mathbf{\Delta}}_{c'}(z')}\right)^{-1}_{\mathbf{\Delta}_{\rm mf}[0]}\left(\frac{\delta^2 S_{\rm eff}}{\delta\widetilde{\mathbf{\Delta}}_{c'}(z')\delta A_\nu(x')}\right)_{\mathbf{\Delta}_{\rm mf}[0]}. \end{aligned} \tag{B3}$$

Since the indices and integration variables are all dummy labels, this is exactly the same expression as written previously in terms of $\Delta$. Thus the generic formula for the mean-field EM response is basis independent.

An important consequence of this result is that the dispersion of the collective modes is also reparametrization invariant. Using the notation in Sec. II for brevity, the collective-mode dispersion is given by the condition $\det S = 0$, where $S$ is the matrix in Eq. (2.6). In terms of the matrix $\widetilde{S}$, defined in terms of $\widetilde{\Delta}$ variables, this condition becomes

$$0 = \det S = [\det (J)]^2 \det \widetilde{S}. \tag{B4}$$

Since $J$ is invertible, $\det J \neq 0$; hence it follows that $\det \widetilde{S} = 0$ if and only if $\det S = 0$.

## APPENDIX C: POLARIZATION TENSOR CALCULATIONS

In this Appendix we provide a short discussion regarding polarization bubbles. If $\{\Phi\}$ collectively describes a set of fields upon which a fermionic system depends (external electromagnetic fields, Hubbard-Stratonovich auxiliary fields, etc), response tensors are computed as an expansion around a reference set of values $\{\overline{\Phi}\}$:

$$\begin{aligned}
Q_{\Phi\Phi'}(x-x') &= \left.\frac{\delta^2 S_{\text{eff}}[\{\Phi\}]}{\delta\Phi(x)\delta\Phi'(x')}\right|_{\{\Phi\}=\{\overline{\Phi}\}} \\
&= Q_{\text{bos},\Phi\Phi'}(x-x') - \frac{1}{2}\left.\frac{\delta^2 \text{Tr}\ln[-\mathcal{G}^{-1}[\{\Phi\}]]}{\delta\Phi(x)\delta\Phi'(x')}\right|_{\{\Phi\}=\{\overline{\Phi}\}} \\
&= Q_{\text{bos},\Phi\Phi'}(x-x') - \frac{1}{2}\int d^D y \text{tr}\langle y|\frac{\delta\mathcal{G}[\{\Phi\}]}{\delta\Phi(x)}\frac{\delta\mathcal{G}^{-1}[\{\Phi\}]}{\delta\Phi'(x')} + \mathcal{G}[\{\Phi\}]\frac{\delta^2\mathcal{G}^{-1}[\{\Phi\}]}{\delta\Phi(x)\delta\Phi'(x')}|y\rangle\bigg|_{\{\Phi\}=\{\overline{\Phi}\}} \\
&= Q_{\text{bos},\Phi\Phi'}(x-x') + \frac{1}{2}\int d^D y \text{tr}\langle y|\mathcal{G}[\{\Phi\}]\frac{\delta\mathcal{G}^{-1}[\{\Phi\}]}{\delta\Phi(x)}\mathcal{G}[\{\Phi\}]\frac{\delta\mathcal{G}^{-1}[\{\Phi\}]}{\delta\Phi'(x')}|y\rangle\bigg|_{\{\Phi\}=\{\overline{\Phi}\}} \\
&\quad - \frac{1}{2}\int d^D y \text{tr}\langle y|\mathcal{G}[\{\Phi\}]\frac{\delta^2\mathcal{G}^{-1}[\{\Phi\}]}{\delta\Phi(x)\delta\Phi'(x')}|y\rangle\bigg|_{\{\Phi\}=\{\overline{\Phi}\}}.
\end{aligned} \tag{C1}$$

Here $Q_{\text{bos},\Phi\Phi'}(x-x')$ is the bosonic part of the response which arises from differentiating the bosonic contribution to the effective action. Define real-space vertices by

$$\hat{V}_\Phi(x, y, x') \equiv \frac{\delta\mathcal{G}^{-1}[\Phi](x, x')}{\delta\Phi(y)}. \tag{C2}$$

The standard Green's function representation of the polarization bubbles then follows:

$$Q_{\Phi\Phi'}(x-x') = Q_{\text{bos},\Phi\Phi'}(x-x') + \frac{1}{2}\int_{y,y',z,z'} \text{tr}[\mathcal{G}(y,z)\hat{V}_\Phi(z,x,z')\mathcal{G}(z',y')\hat{V}_{\Phi'}(y',x',y)]_{\Phi=\overline{\Phi}} + Q_{\text{dia}}(x-x'), \tag{C3}$$

where $\Phi$ is an arbitrary field in the system. The diamagnetic term $Q_{\text{dia}}$, which arises from the third term in Eq. (C1), contributes only to the electromagnetic response and is written out explicitly in Eq. (C9). The bare EM vertices are defined by

$$\gamma^\mu(x, y, x') = \frac{\delta\mathcal{G}_0^{-1}[A](x, x')}{\delta A_\mu(y)}. \tag{C4}$$

For the models of superconductivity with a quadratic free-particle dispersion studied in the main text, the components of the vertices are explicitly given by

$$\gamma^0(x, y, x') = e\tau_3\delta(x-y)\delta(x-x') \tag{C5}$$

$$\boldsymbol{\gamma}(x, y, x') = \frac{ei}{2m}\tau_0\{\nabla[\delta(x-y)\delta(x-x')] + \delta(x-y)\nabla\delta(x-x')\} + \frac{e^2}{m}\tau_3\mathbf{A}(x)\delta(x-y)\delta(x-x'), \tag{C6}$$

and

$$\frac{\delta\gamma^\nu(x, y, x')}{\delta A_\mu(y')} = -\frac{e^2}{m}\tau_3\delta(x-y')\delta(x-y)\delta(x-x')\delta^{\mu i}\delta^{\nu j}\delta_{ij}. \tag{C7}$$

For the electromagnetic response, the reference value for the external field is $A = 0$. The Fourier expansion of the response is

$$Q^{\mu\nu}(x-y) = \int_q e^{-iq\cdot(x-y)}Q^{\mu\nu}(q), \tag{C8}$$

where $\int_q = TL^d \sum_{i\Omega_m} \int \frac{d\mathbf{q}}{(2\pi)^d}$. Using the general expression in Eq. (C3), the undressed polarization response is

$$Q^{\mu\nu}(q) = \frac{1}{2}\int_p \mathrm{tr}[\mathcal{G}(p+q)\gamma^\mu(p+q,p)\mathcal{G}(p)\gamma^\nu(p,p+q)]\Big|_{A=0} + \frac{ne^2}{m}\delta^{\mu i}\delta^{\nu j}\delta_{ij}. \tag{C9}$$

By definition, the momentum-space vertex is defined by [1]

$$\gamma^\mu(x,y,x') = \int_{p,q} e^{iq(x-y)}e^{ip(x-x')}\gamma^\mu(p+q,p). \tag{C10}$$

Therefore, in the limit of zero external field, the momentum-space vertices are [1,28]

$$\gamma^0(p+q,p)\big|_{A=0} = e\tau_3. \tag{C11}$$

$$\boldsymbol{\gamma}(p+q,p)|_{A=0} = \frac{e}{m}\tau_0\left(\mathbf{p}+\frac{\mathbf{q}}{2}\right) = \boldsymbol{\gamma}(p,p+q)|_{A=0}. \tag{C12}$$

For a three-dimensional system with chiral $p+ip$ pairing, the Nambu Green's function is

$$\mathcal{G}(p) = [i\omega_n - \tau_3\xi_\mathbf{p} + \Delta_0(p_x\tau_1 - p_y\tau_2)]^{-1} = \frac{i\omega_n + \tau_3\xi_\mathbf{p} - \Delta_0(p_x\tau_1 - p_y\tau_2)}{(i\omega_n)^2 - E_\mathbf{p}^2}, \tag{C13}$$

where $\xi_\mathbf{p} = \mathbf{p}^2/(2m) - \mu$ and $E_\mathbf{p}^2 = \xi_\mathbf{p}^2 + \Delta_0^2\mathbf{p}^2/p_F^2$. All other bubbles appearing in the main text can be computed in a similar fashion.

## APPENDIX D: SUPERCONDUCTING PAIRING IN RADIAL COORDINATES

Here we transform the mean-field ansatz for the case of spinless, chiral $p+ip$ pairing to center-of-mass and relative coordinate representation as a concrete example of Eqs. (3.18) and (3.19). The coordinate transformation is

$$\mathbf{R} = \frac{\mathbf{x}+\mathbf{x}'}{2}, \quad \mathbf{r} = \mathbf{x}-\mathbf{x}'. \tag{D1}$$

The Jacobian for this transformation is unity. For a spinless fermionic system, the $p+ip$ ansatz reads

$$\Delta(\mathbf{x},\mathbf{x}',\tau) = \left|\Delta\left(\frac{\mathbf{x}+\mathbf{x}'}{2},\tau\right)\right| e^{i\Phi\left(\frac{\mathbf{x}+\mathbf{x}'}{2},\tau\right)}(\partial_x + i\partial_y)\delta(\mathbf{x}-\mathbf{x}'). \tag{D2}$$

Thus,

$$\begin{aligned}\int d^3x d^3x' \psi^\dagger(\mathbf{x},\tau)\Delta(\mathbf{x},\mathbf{x}',\tau)\psi^\dagger(\mathbf{x}',\tau) &= \int d^3R|\Delta(\mathbf{R},\tau)|e^{i\Phi(\mathbf{R},\tau)}\int d^3r\psi^\dagger(\mathbf{R}+\mathbf{r}/2,\tau)\psi^\dagger(\mathbf{R}-\mathbf{r}/2,\tau)(\partial_{r_x}+i\partial_{r_y})\delta(\mathbf{r})\\ &= \int d^3R|\Delta(\mathbf{R},\tau)|e^{i\Phi(\mathbf{R},\tau)}[\psi^\dagger(\mathbf{R},\tau)(\partial_{R_x}+i\partial_{R_y})\psi^\dagger(\mathbf{R},\tau)]\\ &= \int d^3x|\Delta(\mathbf{x},\tau)|e^{i\Phi(\mathbf{x},\tau)}[\psi^\dagger(\mathbf{x},\tau)(\partial_x+i\partial_y)\psi^\dagger(\mathbf{x},\tau)],\end{aligned} \tag{D3}$$

after integrations by parts, identifications of gradients of fermion fields with respect to $\mathbf{R}$ and $\mathbf{r}$ variables, and relabelling of dummy variables.

In general, non $s$-wave pairing demands a spatially dependent interaction coefficient, say $g(\mathbf{x}-\mathbf{x}')$. In this case, the $p$-wave ansatz simplifies the Gaussian part of the identity introduced in the Hubbard-Stratonovich decomposition:

$$\begin{aligned}\int d^3x d^2x' \frac{|\Delta(\mathbf{x},\mathbf{x}',\tau)|^2}{g(\mathbf{x}-\mathbf{x}')} &= \int d^3R d^2r|\Delta(\mathbf{R},\tau)|e^{-i\Phi(\mathbf{R},\tau)}[(\partial_{r_x}-i\partial_{r_y})\delta(\mathbf{r})]g^{-1}(\mathbf{r})|\Delta(\mathbf{R},\tau)|e^{i\Phi(\mathbf{R},\tau)}(\partial_{r_x}+i\partial_{r_y})\delta(\mathbf{r})\\ &= \int d^3R\frac{|\Delta(\mathbf{R},\tau)|^2}{\tilde{g}},\end{aligned} \tag{D4}$$

where we define the renormalized value for the (inverse) mass scale of the amplitude field as

$$\tilde{g}^{-1} = \int d^2r[(\partial_{r_x}-i\partial_{r_y})\delta(\mathbf{r})]g^{-1}(\mathbf{r})(\partial_{r_x}+i\partial_{r_y})\delta(\mathbf{r}). \tag{D5}$$

## APPENDIX E: 3 × 3 RESPONSE MATRIX DETERMINANT CALCULATION

Here we sketch the calculation and simplification of $\det S(q)$ for the response of a charged superconductor in the presence of Coulomb, amplitude, and phase fluctuations. We first consider biasing towards including the amplitude fluctuation effects. An expansion and consideration of the definition in Eq. (3.43) returns

$$\det S(q) = \det\begin{pmatrix} S^{\rho\rho} & iR^{\rho 0} & iR^{\rho\beta}q_\beta \\ iR^{0\rho} & V^{-1}-Q^{00} & -Q^{0\beta}q_\beta \\ -iq_\alpha R^{\alpha\rho} & q_\alpha Q^{\alpha 0} & q_\alpha Q^{\alpha\beta}q_\beta \end{pmatrix}$$
$$= q_\alpha q_\beta S^{\rho\rho}\left[(V^{-1}-Q^{00})\overline{Q}^{\alpha\beta} + Q^{\alpha 0}Q^{0\beta} + Q^{\alpha\beta}\frac{R^{\rho 0}R^{0\rho}}{S^{\rho\rho}} - Q^{0\beta}\frac{R^{\alpha\rho}R^{\rho 0}}{S^{\rho\rho}} - Q^{\alpha 0}\frac{R^{0\rho}R^{\rho\beta}}{S^{\rho\rho}}\right]. \tag{E1}$$

With the singular-value decomposition structure in mind, we can rework the term in square brackets to produce

$$\det S(q) = q_\alpha q_\beta S^{\rho\rho}\left[(V^{-1}-Q^{00})\overline{Q}^{\alpha\beta} + (R^{\rho 0}\quad Q^{\alpha 0})\begin{pmatrix} \frac{Q^{\alpha\beta}}{S^{\rho\rho}} & -\frac{R^{\alpha\rho}}{S^{\rho\rho}} \\ -\frac{R^{\rho\beta}}{S^{\rho\rho}} & 1 \end{pmatrix}\begin{pmatrix} R^{0\rho} \\ Q^{0\beta}\end{pmatrix}\right]$$
$$= q_\alpha q_\beta S^{\rho\rho}\left[(V^{-1}-\overline{Q}^{00})\overline{Q}^{\alpha\beta} + (R^{\rho 0}\quad Q^{\alpha 0})\begin{pmatrix} \frac{R^{\alpha\rho}R^{\rho\beta}}{(S^{\rho\rho})^2} & -\frac{R^{\alpha\rho}}{S^{\rho\rho}} \\ -\frac{R^{\rho\beta}}{S^{\rho\rho}} & 1 \end{pmatrix}\begin{pmatrix} R^{0\rho} \\ Q^{0\beta}\end{pmatrix}\right]. \tag{E2}$$

Continuing with the decomposition, we fold the effects of Coulomb fluctuations into $\overline{Q}^{\alpha\beta}$ to obtain

$$\det S(q) = S^{\rho\rho}(V^{-1}-\overline{Q}^{00})q_\alpha \widetilde{\overline{Q}}^{\alpha\beta} q_\beta. \tag{E3}$$

A reversed order of the fluctuation considerations allows writing

$$\det S(q) = q_\alpha q_\beta (V^{-1}-Q^{00})\left[S^{\rho\rho}\widetilde{Q}^{\alpha\beta} - R^{\alpha\rho}R^{\rho\beta} - \frac{R^{\alpha\rho}R^{\rho 0}Q^{0\beta}}{(V^{-1}-Q^{00})} - \frac{Q^{\alpha 0}R^{0\rho}R^{\rho\beta}}{(V^{-1}-Q^{00})} + \frac{Q^{\alpha\beta}R^{\rho 0}R^{0\rho}}{(V^{-1}-Q^{00})}\right]. \tag{E4}$$

Proceeding with a similar analysis, this leads to

$$\det S(q) = \widetilde{S}^{\rho\rho}(V^{-1}-Q^{00})q_\alpha \overline{\widetilde{Q}}^{\alpha\beta} q_\beta, \tag{E5}$$

proving Eq. (3.41) in the main text.

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