Time dynamics of Bethe ansatz solvable models

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We develop a method for finding the time evolution of exactly solvable models by Bethe ansatz, with Hilbert space linear in excitation number. The dynamical Bethe wave function takes the same form as the stationary Bethe wave function, except for time-varying Bethe parameters and a complex phase prefactor. From this, we derive a set of first-order nonlinear coupled differential equations for the Bethe parameters, called the dynamical Bethe equations. We find that this gives the exact solution to particular types of exactly solvable models, including the Bose-Hubbard dimer. The developed formalism allows us to demonstrate analytically that time dynamics of the detuning-quenched Bose-Hubbard dimer occurs in the subspace which dimensionality is less than that of the physical Hilbert space. This allows for calculation of the time dynamics within a reduced problem dimensionality.

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I. INTRODUCTION

Exact methods of mathematical physics have substantially increased our understanding of many paramount nonlinear phenomena. One such method is the quantum inverse method (QIM), which was developed almost 40 years ago by Sklyanin *et al.* and others [1–3]. The QIM, together with the algebraic version of the Bethe ansatz [4–6], has been successfully applied to various problems from different areas of physics such as one-dimensional BECs [7,8], spin chains [9–12], $(1+1)$ models of quantum field theory [13], the $(2+1)$ model of classical statistical physics [14], conformal field theory and string theory [15], quantum optics [16], and quantum dots [17].

Obtaining the time dynamics of quantum many-body systems remains an important but very challenging problem due to the high computational and calculational demands. In the case of the QIM, the dynamics of the system after a quench of one or several parameters has been successfully studied for some cases [18,19]. However, in general, the QIM without modifications cannot be applied to the system with time-dependent parameters. Recently, several exact methods for time-dependent Hamiltonians were proposed. In Ref. [20], a set of conditions under which the Schrodinger equation can be solved exactly was presented. It was also shown in Ref. [20] that among Hamiltonians satisfying these conditions are the multistate Landau-Zener model and the generalized Tavis-Cummings model. Earlier in Ref. [21], Barmettler *et al.* proposed a generalization of the Bethe wave function for the dynamical case and presented its explicit form for the detuning-driven Tavis-Cummings model. There has also been progress in studies of the exact dynamics of periodically driven systems [22]. However, a general formulation of how to perform the time evolution of an integrable system has not been shown.

In this paper, we study the generalization of the Bethe wave function for the time-dependent case. Specifically, consider that we are dealing with an integrable system with Bethe wave function

$$\prod_{j=1}^{M} \mathbf{B}(\lambda_j)|\mathrm{vac}\rangle, \tag{1}$$

where $\mathbf{B}(\lambda)$ is an operator which depends on complex parameter λ and $|\mathrm{vac}\rangle$ is the pseudovacuum reference state, specific to the model being considered. For a model which Hilbert space is small enough, we show that its time evolution can be described exactly using the dynamical Bethe wave function:

$$e^{ip(t)}\prod_{j=1}^{M} \mathbf{B}(\lambda_j(t))|\mathrm{vac}\rangle, \tag{2}$$

where $p(t)$ is a complex phase. The time-dependent wave function has exactly the same structure as the Bethe wave

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function, but its parameters are functions of time and it has a time-varying prefactor. One of the most important features of the Bethe vectors is that it allows for the determinant representation for computing local observables [23], which is widely used in calculations of the Bethe ansatz [24,25]. The fact that the time-dependent wave function (2) has the structure of a Bethe vector allows us to transfer all the Bethe ansatz machinery to the time-dependent case.

When a system is exactly solvable by the QIM, one can always make (1) an eigenfunction by choosing special values of the parameters λ_j, which satisfy the Bethe equations. In this paper, we formulate a set of conditions for when the dynamical Bethe wave function (2) satisfies the time-dependent Schrodinger equation. The set of conditions is a set of nonlinear coupled differential equations, which we call the dynamical Bethe equations. The time-dependent wave function can always be represented by the dynamical Bethe wave function (2) for an arbitrary smooth time dependence of the model parameters if the Hilbert space of the model under consideration is small enough.

We provide an explicit example of the dynamical Bethe equations for a detuning-driven Bose-Hubbard dimer. We show analytically that the evolution of the detuning-quenched Bose-Hubbard dimer happens in the subspace which dimensionality is smaller than that of the original Hilbert space. This effect is preserved in the thermodynamic limit for an arbitrary long time, which is an important feature of such nonergodic phenomena.

The form of the wave function (2) first appeared in Ref. [21] for the Tavis-Cummings model, where the set of dynamical Bethe equations for $\lambda_j(t)$ was found, and its connection of trajectories $\lambda_j(t)$ with classical motion in a potential was established. So far, all the examples of the dynamically integrable models considered in [20,21,26] belong to the Gaudin class [27] of integrable models or models with a classical R matrix. Here we show that (2) can be applied to a wider class of integrable models, which goes beyond the Gaudin class. The Bose-Hubbard dimer example that we show here belongs to the so-called rational *XXX* R-matrix class.

II. DYNAMICAL BETHE EQUATIONS

We first discuss the general method of finding the dynamical Bethe wave function. The model under consideration is assumed to be solvable by an algebraic Bethe ansatz. We also assume that the set of these Bethe vectors forms a complete orthogonal set. This condition should be checked for every specific model separately, but for the vast majority of physically relevant models, it is known to be satisfied. Also for simplicity, we restrict the considered models to be those with a rational R matrix and *XXX*- or *XXZ*-like R matrices. In practice, these three classes cover most physically relevant models.

A central quantity in integrable models is the trace of the monodromy matrix $\tau(\lambda)$. For more details on this and the Bethe ansatz technique in general, see the Appendices, where the essential points have been summarized, or the extensive review of Ref. [28]. This operator has many useful algebraic properties provided by the integrability of the model; its specific form should be defined for each model separately. By construction, $\tau(\lambda)$ is explicitly connected with the Hamiltonian $\hat{\mathcal{H}}$ of the model under consideration. Usually the Hamiltonian $\hat{\mathcal{H}}$ can be expressed as some elementary function or a residue of $\tau(\lambda)$ at some certain point λ_0. Because of the connection between $\hat{\mathcal{H}}$ and $\tau(\lambda)$, we will see that it is beneficial to consider the following Schrodinger-like equation:

$$i\frac{d}{dt}|\Psi(t)\rangle = \tau(\lambda)|\Psi(t)\rangle. \quad (3)$$

This will allow us to learn the complete information about the time dynamics of the system.

We look for the solution of (3) of the form

$$\left|\Psi_M^\sigma(t)\right\rangle = e^{ip^\sigma(t)}\prod_{j=1}^M \mathbf{B}\big(\lambda_j^\sigma(t)\big)|\text{vac}\rangle, \quad (4)$$

where M is the number of excitations in the system and σ enumerates the eigenstates. At $t=0$, the vectors (4) are eigenvectors which form a complete orthogonal set and the set of parameters $\Lambda_M^\sigma(t) = \{\lambda_1^\sigma(t), \lambda_2^\sigma(t), \dots, \lambda_M^\sigma(t)\}$ satisfies the stationary Bethe equations for each σ. We demand time-dependent wave functions to also form a complete set,

$$\sum_\sigma |\Psi(\{\lambda^\sigma(t)\})\rangle\langle\Psi(\{\lambda^\sigma(t)\})| \propto \hat{I}, \quad (5)$$

where we have a proportionality because the wave functions are not normalized. The expansion of the Bethe vectors (4) over a convenient basis is a difficult problem and, in general, is not solvable because of the complex structure of (4). For example, $\mathbf{B}(\lambda)$ can be represented as a series of exponential length in M.

Thus, instead of studying the Schrodinger equation (3) directly, we demand that

$$\langle\Psi(\{\lambda^{\sigma'}(t)\})|\Psi(\{\lambda^\sigma(t)\})\rangle = 0, \quad (6)$$

for $\sigma < \sigma'$, where $\sigma \in [1, L]$ and L is the dimensionality of the Hilbert space under consideration. This states that the Bethe vectors are mutually orthogonal for all t. We also demand that

$$\langle\dot{\Psi}(\{\lambda^{\sigma'}(t)\})|\Psi(\{\lambda^\sigma(t)\})\rangle + \langle\Psi(\{\lambda^{\sigma'}(t)\})|\dot{\Psi}(\{\lambda^\sigma(t)\})\rangle = 0, \quad (7)$$

which must be satisfied for any solution of (3).

We now would like to write (6) and (7) as a set of coupled differential equations. Equation (6) may be reexpressed in this form by writing $|\Psi(\{\lambda^\sigma(t)\})\rangle$ in terms of its derivative, which for Bethe vectors always take a special form. To show this, we start by finding the result of operator $\tau(\mu)$ on the Bethe wave function (4), giving the well-known result

$$\tau(\mu)\prod_{j=1}^M \mathbf{B}\big(\lambda_j^\sigma\big)|\text{vac}\rangle = \Theta\big(\mu, \{\lambda_j^\sigma\}\big)\prod_{j=1}^M \mathbf{B}\big(\lambda_j^\sigma\big)|\text{vac}\rangle + \sum_{n=1}^M \phi_n\big(\mu, \{\lambda_j^\sigma\}\big)\mathbf{B}(\mu)\prod_{\substack{j=1\\ j\neq n}}^M \mathbf{B}\big(\lambda_j^\sigma\big)|\text{vac}\rangle, \quad (8)$$

where $\Theta(\mu, \{\lambda_j^\sigma\})$ and $\phi_n(\mu, \{\lambda_j^\sigma\})$ are eigenvalues and the off-shell functions defined in (A14) and (A15) correspondingly. By combining (3), (4), and (8), we obtain

$$i\left(i\frac{dp}{dt}-\Theta\big(\mu,\{\lambda_j^\sigma(t)\}\big)\right)\prod_{j=1}^{M}\mathbf{B}\big(\lambda_j^\sigma(t)\big)|\text{vac}\rangle$$
$$=-i\frac{d}{dt}\prod_{j=1}^{M}\mathbf{B}\big(\lambda_j^\sigma(t)\big)|\text{vac}\rangle$$
$$+\sum_{n=1}^{M}\phi_n\big(\mu,\{\lambda_j^\sigma(t)\}\big)\mathbf{B}(\mu)\prod_{\substack{j=1\\ j\neq n}}^{M}\mathbf{B}\big(\lambda_j^\sigma(t)\big)|\text{vac}\rangle. \quad (9)$$

Demanding that the right-hand side is proportional to the left-hand side,

$$f\big(\{\lambda_j^\sigma(t)\}\big)\prod_{j=1}^{M}\mathbf{B}\big(\lambda_j^\sigma(t)\big)|\text{vac}\rangle=-i\frac{d}{dt}\prod_{j=1}^{M}\mathbf{B}\big(\lambda_j^\sigma(t)\big)|\text{vac}\rangle$$
$$+\sum_{n=1}^{M}\phi_n\big(\mu,\{\lambda_j^\sigma(t)\}\big)\mathbf{B}(\mu)\prod_{\substack{j=1\\ j\neq n}}^{M}\mathbf{B}\big(\lambda_j^\sigma(t)\big)|\text{vac}\rangle, \quad (10)$$

where $f(\{\lambda_j^\sigma(t)\})$ is a smooth function. If (10) is satisfied, we can solve (3) with (4) by choosing a special form of phase factor $p^\sigma(t)$,

$$p^\sigma(t)=-\int_0^t\big[i\Theta\big(\mu,\{\lambda_j^\sigma(t')\}\big)+f\big(\{\lambda_j^\sigma(t')\}\big)\big]dt'. \quad (11)$$

Although it is possible to explicitly find both $f(\{\lambda_j^\sigma(t)\})$ and $p^\sigma(t)$, in practice this is not necessary because the phase factor $e^{ip^\sigma(t)}$ cancels for any observable due to normalization.

After substitution of (10) into (6), the conditions (6) transfer to the set of differential equations:

$$i\langle\Psi(\{\lambda^{\sigma'}(t)\})|\dot{\Psi}(\{\lambda^\sigma(t)\})\rangle$$
$$=\langle\Psi(\{\lambda^{\sigma'}(t)\})|\sum_{n=1}^{M}\phi_n\big(\mu,\{\lambda_j^\sigma(t)\}\big)\mathbf{B}(\mu)$$
$$\times\prod_{\substack{j=1\\ j\neq n}}^{M}\mathbf{B}\big(\lambda_j^\sigma(t)\big)|\text{vac}\rangle. \quad (12)$$

Now conditions (7) and (12) are a set of L^2-L nonlinear differential equations, with ML variables, where M is the number of parameters which parametrize Bethe wave function (4). The solution of (7) and (12) is a set of trajectories $\Lambda^\sigma(t)=\{\lambda_1^\sigma(t),\dots,\lambda_M^\sigma(t)\}$, for each wave function enumerated by σ. When $M=L-1$, the number of equations coincides with the number of variables and (6) and (7) always have a solution. The dynamical Bethe wave function can always be constructed if the Hilbert space of the system under consideration is small enough, for arbitrary smooth time dependence of the parameters of the model.

In a Bethe ansatz, it is typical for the Bethe wave function to be parameterized by a number of parameters which is much smaller than the size of the Hilbert space. For example, while the Hilbert space of the *XXZ* Heisenberg magnet has an exponentially large dimension, its Bethe wave function is parametrized by a number of parameters linearly proportional to the number of excitations, which provides a great advantage in terms of computational complexity. The equations (7) and (12), however, become overdetermined if the dimensionality of Hilbert space $L>M+1$. The existence of solutions for (7) and (12) when it is overdetermined is not prohibited because the equations are nonlinear. A trivial example of such a solution is adiabatic evolution when $\{\lambda_1^\sigma(t),\dots,\lambda_M^\sigma(t)\}$ is a solution of static Bethe equations at every moment.

III. DETUNING-DRIVEN BOSE-HUBBARD DIMER

Let us now illustrate the above technique by calculating the time dynamics of the interacting Bose-Hubbard dimer. This is equivalent to a two-site Bose-Hubbard model and can be described by the following Hamiltonian [29,30] (note that longer Bose-Hubbard chains are proven to be nonintegrable [31]):

$$\hat{H}=\Delta b^\dagger b+a^\dagger b+ab^\dagger+c^2a^\dagger ab^\dagger b, \quad (13)$$

where a, b are bosonic annihilation operators for the two sites, Δ is the detuning, and c^2 is the interaction strength. This model provides both a simple and nontrivial example of a dynamically integrable model from the *XXX* class.

For the case that the coupling constant c is time independent whereas the detuning $\Delta(t)$ continuously depends on time, the dynamical Bethe equations can be written in a particularly simple form. The time-dependent Bethe wave function can then be written following (4) using the above definitions. The dynamical Bethe equations then are found to be

$$i\left(\frac{\dot{\Delta}}{c}-\dot{\lambda}_n^\sigma(t)\right)=\varphi_n(\{\lambda^\sigma\})\lambda_n^\sigma(t),\ \forall n=1,\dots,N, \quad (14)$$

where $\varphi_n(\{\lambda^\sigma\})$ are the so-called off-shell functions defined as

$$\varphi_n(\{\lambda^\sigma\})=\left(\frac{\Delta}{c}-\lambda_n^\sigma\right)\prod_{\substack{j=1\\ j\neq n}}^{N}\left(1-\frac{c}{\lambda_n^\sigma-\lambda_j^\sigma}\right)$$
$$+\frac{1}{c\lambda_n^\sigma}\prod_{\substack{j=1\\ j\neq n}}^{N}\left(1-\frac{c}{\lambda_j^\sigma-\lambda_n^\sigma}\right). \quad (15)$$

When $\varphi_n(\{\lambda\})=0,\ \forall n=1,\dots,N$, where N is the number of particles, these reduce to the static Bethe equations. The dynamical Bethe equations are a set of first-order coupled ordinary differential equations. As the initial condition for (14), we need to pick a set of parameters $\{\lambda(0)\}=\{\lambda_1(0),\dots,\lambda_N(0)\}$, which parametrizes the initial state $|\Psi_N(0)\rangle$. For example, if the initial state is an eigenstate, the set $\{\lambda(0)\}$ should satisfy the static Bethe equations. Also note that such initial states can be connected with some product states [32], which can provide additional physical interpretation.

We numerically solve the set of equations (14) for a detuning with time dependence $\Delta(t)=\Delta_0+t\cos(t^2)$, which has a rather nontrivial, nonlinear, and aperiodic dependence. The initial condition was chosen to be the solution of static Bethe equations which corresponds to the ground state of (13). As the observable, we calculate the intersite coherence $|\langle a^\dagger b\rangle|/N$; Fig. 1(a) shows our results. We find that the method perfectly reproduces the time dynamics calculated by

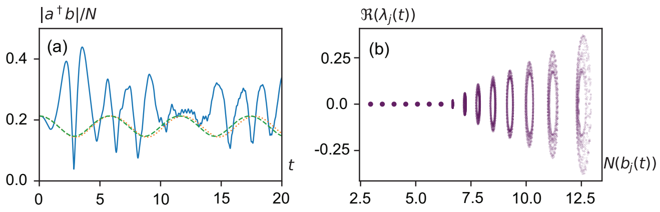

FIG. 1. (a) The intersite coherence $|a^\dagger b|/N$ for the case of driven $\Delta(t) = \Delta_0 + t\cos(t^2)$ (solid line) and quenched from Δ_0 to Δ detuning; the dotted line is obtained from the exact solution of (16), and the dashed line is obtained from the approximate solution when seven equations are truncated. (b) Stroboscopic maps for the case of quenched detuning using (16), where each set of N points represents a solution of the dynamical Bethe equations at some certain moment of time $t \in [0, 20]$. $N = 15$, $c = 0.531$, $\Delta_0 = 0.697$, $\Delta' = 1.569$ throughout.

exact diagonalization, giving identical curves. The method is computationally efficient in the sense that the the solution requires the evolution of N coupled equations. In Fig. 1(c), we show the stroboscopic maps of the solutions of dynamical Bethe equations (14); the point of certain color corresponds to the value of the component of the solution of (14) $\lambda_j(t_k)$ at the moment t_k. Instead of solving (14), one may solve the more general system of Eqs. (7) and (12), which is applicable for an arbitrary time dependence of both Δ and c^2. We have verified that the solutions of (7) and (12) perfectly coincide with the solutions of (14), where only Δ has a time dependence.

IV. DIMENSIONALITY REDUCTION OF HILBERT SPACE

In the above example, we have reproduced the exact time evolution of the Bose-Hubbard dimer by solving N coupled dynamical Bethe equations. Since the dimension of the Hilbert space is $N + 1$, this has the equivalent computational overhead as solving for the wave function using the Schrodinger equation directly. We show here that for some types of evolution, only a subset of the Bethe roots are time varying to a good approximation, and hence the time evolution can be calculated with reduced computational overhead.

We consider the case of a quench, where the parameters are suddenly changed from c, Δ to c', Δ'. In this case, the parameters of the model are constant and hence the set of equations (14) becomes

$$-i\frac{\dot{\lambda}_n(t)}{\lambda_n(t)} = \varphi_n(\{\lambda\}) \quad \forall n = 1, \dots, N. \tag{17}$$

The set of equations (16) describes the evolution of an initial state $|\Psi_N(0)\rangle$ with a static Hamiltonian (13). The initial state can be parameterized by a Bethe vector with the set of parameters $\{\lambda^0\}$ satisfying static Bethe equations for the initial parameters c, Δ. After the quench is performed, the Hamiltonian parameters change to c', Δ', and hence we need to establish the connection between the old wave function expressed in terms of c, Δ, and the new one expressed in terms of c', Δ'.

For the case that only the detuning is quenched $c' = c$, the initial conditions for (16) can be found to be

$$\lambda_j(0) = \lambda_j^0 + \frac{\Delta' - \Delta}{c}. \tag{18}$$

It has been shown in Ref. [30] that for many eigenstates, the first several roots of the static Bethe equations can be approximated equidistantly. In particular, for the attractive case $c^2 > 0$ in the thermodynamic limit when $N \to \infty$, the ground state is a Fermi sea of Bethe roots, $\lambda_j^0 = jc + \Delta/c$, $\forall j = 0, \dots, N-1$. When N is finite, there are $k+1$ roots λ_k of the static Bethe equations which can be approximated equidistantly with some precision ϵ such that $|\lambda_k - \frac{\Delta}{c} - kc| < \epsilon$. For the case of quenched detuning, the first several roots $\lambda_j(0)$ of the initial conditions are also distributed equidistantly since (17) causes just a constant shift of all the roots λ_j^0. It is clear from the form of (15) that as long as the equidistant approximation is applicable, such a constant shift may not drive the off-shell function away from zero, $\varphi_j(\{\lambda\}) \simeq 0$. So, from (16), we can conclude that there are k roots, $\lambda_j(t) \simeq jc + \Delta'/c$ for $j = 0, \dots, k$, where k is the number of equidistantly distributed roots. This fact allows us to reduce the number of equations we have to solve from N to $N - k$. In other words, with quenching of the detuning, the surface of the Fermi sea remains still and all the dynamics occurs in the deep. The depth at which the disturbance of the sea becomes noticeable defines the dimensionality of the dynamics.

We have constructed an example of the system with a dynamical wave function $|\Psi(t)\rangle$ that is locked inside of a subspace of dimension that is lower than that of physical Hilbert space. To illustrate this, we plot stroboscopic maps in Fig. 1(b) for the solution of dynamical Bethe equations (16) which can be interpreted as a trajectory in some phase space. It has been shown in Ref. [30] that the equidistant approximation works better for bigger N; for example, the error for ground-state energy decreases faster than linearly with N, which ensures that the described reduction of dimensionality is not related to any finite-size effects. The above result gives a rigorous basis for the heuristic result that in a quench, the dynamics takes place predominantly within a reduced Hilbert space. For example, in a direct time evolution of the Schrodinger equation for a quenched Bose-Hubbard dimer, one finds that it is possible to truncate the Hilbert space in terms of a small number of eigenstates with initial parameters c, Δ. While this can be numerically verified for small systems easily, it is difficult to claim this is generally true in the thermodynamic limit. The use of our formalism allows one to clearly show that this will remain true regardless of the dimensionality due to the equidistant approximation becoming exact in the limit $N \to \infty$.

V. OUTLOOK AND CONCLUSIONS

We have described a method for evaluating the time dynamics of systems that are exactly solvable by the Bethe ansatz. The method is based on the dynamical Bethe wave function (2) which is a straightforward generalization of the Bethe ansatz for the dynamical case, where the Bethe parameters are time dependent and there is a time-varying complex phase. The main advantage of the dynamical Bethe wave functions (2) is that they are mathematically manageable thanks to the well-developed Bethe ansatz results which are directly applicable.

The set of differential dynamical Bethe equations (7) and (12) can be applied to any Bethe ansatz solvable model from the *XXX*, *XXZ*, or Gaudin class which has a dimensionality not bigger than $N+1$, where N is the number of parameters in the Bethe wave functions. What would be interesting is if the dynamical Bethe wave function, being considered as a variational ansatz, could describe, not necessarily exactly, the diabatic evolution of a nontrivial model with a larger Hilbert space than $N+1$. We have shown explicitly in the case of the Bose-Hubbard dimer that in addition to reproducing the exact dynamics, such an approximate time evolution within a reduced problem dimensionality is possible within the framework. Importantly, it has provided a tool to analytically justify that such reduction does not vanish in the large-N limit.

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APPENDIX A: ESSENTIAL NOTES ON ALGEBRAIC BETHE ANSATZ

Here we briefly sketch the main aspects of the algebraic Bethe ansatz technique which are necessary for the understanding of the mathematical details of the present paper. For an extensive review of the Bethe ansatz, we refer the reader to Refs. [6,28,33].

The cornerstone of any integrable model is the R matrix, which in this paper always takes the form

$$\mathbf{R}(\lambda,\mu)=\begin{pmatrix} f(\mu,\lambda) & 0 & 0 & 0\\ 0 & g(\mu,\lambda) & 1 & 0\\ 0 & 1 & g(\mu,\lambda) & 0\\ 0 & 0 & 0 & f(\mu,\lambda)\end{pmatrix}. \tag{A1}$$

Here the entries $f(\mu,\lambda)$ and $g(\mu,\lambda)$ are specified for each model separately. In general, the R matrix is the solution of the Yang-Baxter equation [28] and can take many different forms. The specific form of the R matrix generates a family of integrable models.

In order to construct an integrable model, we need to define the monodromy matrix,

$$T(\lambda)=\begin{pmatrix} A(\lambda) & B(\lambda)\\ C(\lambda) & D(\lambda)\end{pmatrix}, \tag{A2}$$

which depends on the complex spectral parameter λ. Here, $A(\lambda)$, $B(\lambda)$, $C(\lambda)$, and $D(\lambda)$ are operators acting in the Hilbert space of the model under consideration, and their explicit representation depends on the model. The monodromy matrix should also satisfy the Yang-Baxter equation

$$R(\lambda,\mu)\otimes T(\lambda)\otimes T(\mu)=T(\mu)\otimes T(\lambda)\otimes R(\lambda,\mu). \tag{A3}$$

To construct the Hamiltonian of a particular integrable model, we define the trace of the monodromy matrix,

$$\tau(\lambda)=\mathrm{Tr}T(\lambda)=A(\lambda)+D(\lambda). \tag{A4}$$

The Hamiltonian $\hat{\mathcal{H}}$ of the model may be expressed via the trace of the monodromy matrix $\tau(\lambda)$, or its derivative at some specified $\lambda=\lambda_0$. Usually it can be expressed as some elementary function of $\tau(\lambda_0)$ or as a residue of the $\tau(\lambda)$ at a particular point λ_0.

The pseudovacuum state $|\mathrm{vac}\rangle$ is a state from the Hilbert space of the model, which is annihilated by the operator $C(\lambda)|\mathrm{vac}\rangle=0$. The conjugated operator also satisfies $\langle\mathrm{vac}|B(\lambda)=0$. We also define two eigenvalue functions $a(\lambda)$ and $d(\lambda)$ according to

$$\begin{aligned} A(\lambda)|\mathrm{vac}\rangle&=a(\lambda)|\mathrm{vac}\rangle,\\ D(\lambda)|\mathrm{vac}\rangle&=d(\lambda)|\mathrm{vac}\rangle.\end{aligned} \tag{A5}$$

The Bethe wave function is then defined as

$$\big|\Psi\big(\{\lambda_j^\sigma\}\big)\big\rangle=\prod_{j=1}^{M}B\big(\lambda_j^\sigma\big)|\mathrm{vac}\rangle, \tag{A6}$$

where $\{\lambda_j^\sigma\}$ is the set of complex parameters $\{\lambda_j^\sigma\}=\{\lambda_1^\sigma,\lambda_2^\sigma,\dots,\lambda_M^\sigma\}$, M is the number of excitations in the system, and σ labels the wave function. The wave function (A6) is an eigenfunction of the trace of monodromy matrix $\tau(\lambda)$,

$$\tau(\lambda)\big|\Psi\big(\{\lambda_j^\sigma\}\big)\big\rangle=\Theta\big(\lambda,\{\lambda_j^\sigma\}\big)\big|\Psi\big(\{\lambda_j^\sigma\}\big)\big\rangle, \tag{A7}$$

if the set $\{\lambda_j^\sigma\}$ satisfies to the set of Bethe equations,

$$\frac{a\big(\lambda_j^\sigma\big)}{d\big(\lambda_j^\sigma\big)}\prod_{\substack{n=1\\ n\neq j}}^{M}\frac{f\big(\lambda_j^\sigma,\lambda_n^\sigma\big)}{f\big(\lambda_n^\sigma,\lambda_j^\sigma\big)}=1,\qquad j=1,2,\dots,M. \tag{A8}$$

All the roots within one solution $\{\lambda_j^\sigma\}$ should be different; otherwise, $|\Psi(\{\lambda_j^\sigma\})\rangle$ cannot be an eigenfunction. The Bethe equations (A8) are a set of coupled nonlinear algebraic equations. It has M equations and N solutions, where N is equal to the amount of eigenstates in the system.

For our purposes, it is important to know the effect of the transfer matrix $\tau(\lambda)$ acting on the Bethe vector (A6). For notational simplicity, we omit the index σ henceforth, such that $\{\lambda_j\}$ denotes the set $\{\lambda_1^\sigma,\lambda_2^\sigma,\dots,\lambda_M^\sigma\}$. From Ref. [28], it is known that

$$\begin{aligned} A(\lambda)\prod_{j=1}^{M}B(\lambda_j)|\mathrm{vac}\rangle&=a(\lambda)\Lambda(\lambda,\{\lambda_j\})\prod_{j=1}^{M}B(\lambda_j)|\mathrm{vac}\rangle\\ &\quad+\sum_{n=1}^{M}a(\lambda_n)\Lambda_n(\lambda,\{\lambda_j\})B(\lambda)\prod_{\substack{j=1\\ j\neq n}}^{M}B(\lambda_j)|\mathrm{vac}\rangle.\end{aligned} \tag{A9}$$

Here we defined the functions

$$\begin{aligned} \Lambda(\lambda,\{\lambda_j\})&=\prod_{j=1}^{M}f(\lambda,\lambda_j),\\ \Lambda_n(\lambda,\{\lambda_j\})&=g(\lambda_n,\lambda)\prod_{\substack{j=1\\ j\neq n}}^{M}f(\lambda_n,\lambda_j).\end{aligned} \tag{A10}$$

For the $D(\lambda)$ operator, we have the similar expressions

$$D(\lambda)\prod_{j=1}^{M}B(\lambda_j)|\mathrm{vac}\rangle = d(\lambda)\bar{\Lambda}(\lambda,\{\lambda_j\})\prod_{j=1}^{M}B(\lambda_j)|\mathrm{vac}\rangle + \sum_{n=1}^{M}d(\lambda_n)\bar{\Lambda}_n(\lambda,\{\lambda_j\})B(\lambda)\prod_{\substack{j=1\\ j\neq n}}^{M}B(\lambda_j)|\mathrm{vac}\rangle, \quad (A11)$$

where we defined

$$\bar{\Lambda}(\lambda,\{\lambda_j\}) = \prod_{j=1}^{M} f(\lambda_j,\lambda),$$
$$\bar{\Lambda}_n(\lambda,\{\lambda_j\}) = g(\lambda,\lambda_n)\prod_{\substack{j=1\\ j\neq n}}^{M} f(\lambda_j,\lambda_n). \quad (A12)$$

Combining these results, we can find the effect of acting $\tau(\lambda)$ on the Bethe wave function, given by

$$\tau(\lambda)\prod_{j=1}^{M}B(\lambda_j)|\mathrm{vac}\rangle = \Theta(\lambda,\{\lambda_j\})\prod_{j=1}^{M}B(\lambda_j)|\mathrm{vac}\rangle + \sum_{n=1}^{M}\phi_n(\lambda,\{\lambda_j\})B(\lambda)\prod_{\substack{j=1\\ j\neq n}}^{M}B(\lambda_j)|\mathrm{vac}\rangle. \quad (A13)$$

Here we defined the eigenenergy,

$$\Theta(\lambda,\{\lambda_j\}) = a(\lambda)\Lambda(\lambda,\{\lambda_j\}) + d(\lambda)\bar{\Lambda}(\lambda,\{\lambda_j\}), \quad (A14)$$

and the off-shell function,

$$\phi_n(\lambda,\{\lambda_j\}) = a(\lambda_n)\Lambda_n(\lambda,\{\lambda_j\}) + d(\lambda_n)\bar{\Lambda}_n(\lambda,\{\lambda_j\}). \quad (A15)$$

If we now demand that the off-shell function (A15) is zero, it is evident that the wave function (A6) is an eigenfunction for $\tau(\lambda)$. The roots of off-shell functions (A15) coincide with the roots of Bethe equations (A8), but we should distinguish between these since later we will encounter cases where the off-shell function is not zero.

Finally, we mention several important properties of Bethe wave functions. The dual Bethe wave functions are defined as

$$\langle\Psi(\{\lambda_j^{\sigma}\})| = \langle\mathrm{vac}|\prod_{j=1}^{M}C(\lambda_j^{\sigma}). \quad (A16)$$

In general, despite the notation, the wave function (A16) does not coincide with the Hermitian conjugate of the function (A6), i.e., $\langle\Psi(\{\lambda_j^{\sigma}\})| \neq |\Psi(\{\lambda_j^{\sigma}\})\rangle^{\dagger}$. Dual vectors like this must be introduced in order to evaluate scalar products and averages of observables. Generally, in the literature devoted to the Bethe ansatz, the left bracket $\langle\Psi|$ implies the dual vector (A16).

For most of the integrable models, it has been proven that Bethe vectors form a complete set [6,28],

$$\sum_{\sigma=1}^{N}|\Psi(\{\lambda_j^{\sigma}\})\rangle\langle\Psi(\{\lambda_j^{\sigma}\})| \propto \hat{I}, \quad (A17)$$

where $\hat{I}$ is the identity operator and N is the size of the Hilbert space. In general, Bethe wave functions are not normalized.

One of the most important properties of Bethe wave functions is that for many models, it is possible to evaluate the scalar product of the Bethe wave functions and averages of the operators by applying Slavnov's formula [23]. This allows one to express the scalar product as a determinant. We do not reproduce the general form of the Slavnov's formula here because of its complexity, and it not very useful to consider it without specifying the model. Application of Slavnov's formula to the models considered in this paper has been studied in Refs. [34,35].

APPENDIX B: DYNAMICAL BETHE EQUATIONS FOR THE BOSE-HUBBARD DIMER

Here we give more details of the derivation of the dynamical Bethe equations for the detuning-driven Bose-Hubbard dimer. A more detailed description regarding the Bethe ansatz solution of this model can be found in Ref. [34], and we use the same notations as this paper.

The Hamiltonian of the Bose-Hubbard dimer is

$$\hat{H} = \Delta b^{\dagger}b + a^{\dagger}b + ab^{\dagger} + c^2a^{\dagger}ab^{\dagger}b. \quad (B1)$$

The diagonal elements of the monodromy matrix are, in this case,

$$A(\lambda) = \lambda^2 - \lambda\left(ca^{\dagger}a + cb^{\dagger}b + \frac{\Delta}{c}\right) + \Delta b^{\dagger}b + a^{\dagger}b + c^2a^{\dagger}ab^{\dagger}b, \quad (B2)$$

$$D(\lambda) = ab^{\dagger} + c^{-2}. \quad (B3)$$

The Hamiltonian (B1) can then be expressed via trace of the monodromy matrix (A4) according to

$$\hat{H} = \tau(0) - c^{-2}. \quad (B4)$$

According to the definitions (A5), the eigenvalue functions are then

$$a(\lambda) = \lambda\left(\lambda - \frac{\Delta}{c}\right), \quad (B5)$$
$$d(\lambda) = c^{-2}. \quad (B6)$$

The elements of the R matrix are defined as

$$f(\mu,\lambda) = 1 - \frac{c}{\mu-\lambda}, \quad (B7)$$
$$g(\mu,\lambda) = -\frac{c}{\mu-\lambda}. \quad (B8)$$

We now wish to look for Bethe eigenfunctions of the form

$$|\Psi_N^{\sigma}\rangle = \prod_{j=1}^{N}\mathbf{B}(\lambda_j^{\sigma})|\mathrm{vac}\rangle, \quad (B9)$$

where the pseudovacuum state is $|\mathrm{vac}\rangle = |0\rangle_a \otimes |0\rangle_b$. σ is the index which labels the energy levels of the system; for the sake of notational simplicity, we omit this below. We introduce the generalized creation operator $\mathbf{B}(\lambda)$ which depends on a complex parameter λ [34],

$$\mathbf{B}(\lambda) = \left(\lambda - \frac{\Delta}{c}\right)b^{\dagger} - ca^{\dagger}ab^{\dagger} - c^{-1}a^{\dagger}. \quad (B10)$$

The eigenvector depends on N complex parameters $\{\lambda\} = \{\lambda_1, \lambda_2, \dots, \lambda_N\}$. By applying the Bethe ansatz machinery, we can evaluate

$$\hat{H}|\Psi_N\rangle = E_N(\{\lambda\})\prod_{j=1}^{N}\mathbf{B}(\lambda_j)|\text{vac}\rangle - \sum_{n=1}^{N}\varphi_n(\{\lambda\})\mathbf{X}\prod_{\substack{j=1\\ j\neq n}}^{N}\mathbf{B}(\lambda_j)|\text{vac}\rangle, \tag{B11}$$

where we have defined

$$E_N(\{\lambda\}) = -c^{-2} + c^{-2}\prod_{j=1}^{N}\left(1-\frac{c}{\lambda_j}\right), \tag{B12}$$

$$\varphi_n(\{\lambda\}) = \left(\frac{\Delta}{c}-\lambda_n\right)\prod_{\substack{j=1\\ j\neq n}}^{N}\left(1-\frac{c}{\lambda_n-\lambda_j}\right) + \frac{1}{c\lambda_n}\prod_{\substack{j=1\\ j\neq n}}^{N}\left(1-\frac{c}{\lambda_j-\lambda_n}\right). \tag{B13}$$

Here, $E_N(\{\lambda\})$ is the energy and $\varphi_n(\{\lambda\})$ is the off-shell function. From (B11), we can see that when set $\{\lambda\}$ satisfies

$$\varphi_n(\{\lambda\}) = 0, \quad \forall n = 1, \dots, N, \tag{B14}$$

the wave function (B9) becomes an eigenfunction of the Hamiltonian (B1). The set of equations (B14) are known as the Bethe equations.

We now look for a time-dependent wave function of the form

$$|\Psi_N(t)\rangle = e^{ip(t)}\prod_{j=1}^{N}\mathbf{B}(\lambda_j(t))|\text{vac}\rangle. \tag{B15}$$

If only the detuning Δ is time dependent, it is easy to see that $[\frac{d}{dt}\mathbf{B}, \mathbf{B}] = 0$, and the derivative of (B15) can be taken easily. Substituting (B15) into the time-dependent Schrodinger equation, we obtain

$$[p'(t) + E_N(\{\lambda\})]\prod_{j=1}^{N}\mathbf{B}(\lambda_j)|\text{vac}\rangle = \sum_{n=1}^{N}\left[i\left(\dot{\lambda}_n - \frac{\dot{\Delta}}{c}\right)b^\dagger + \varphi_n(\{\lambda\})\mathbf{X}\right]\prod_{\substack{j=1\\ j\neq n}}^{N}\mathbf{B}(\lambda_j)|\text{vac}\rangle. \tag{B16}$$

If we demand now that

$$i\left(\frac{\dot{\Delta}}{c} - \dot{\lambda}_n(t)\right) = \varphi_n(\{\lambda\})\lambda_n(t), \quad \forall n = 1, \dots, N, \tag{B17}$$

the wave function (B15) will satisfy the time-dependent Schrodinger equation. We call the set of conditions (14) the dynamical Bethe equations. The dynamical Bethe equations are a set of first-order coupled ordinary differential equations. For the initial condition of (14), we need to pick a set $\{\lambda(0)\} = \{\lambda_1(0), \dots, \lambda_N(0)\}$, which parametrizes the initial state $|\Psi_N(0)\rangle$. For example, if the initial state is an eigenstate, the set $\{\lambda(0)\}$ should satisfy the static Bethe equations (B14). The phase factor $p(t)$ is given by

$$p(t) = \int_0^t dt'\left[-E_N(\{\lambda\}) + \sum_{n=1}^{N}\frac{i}{\lambda_n}\left(\dot{\lambda}_n - \frac{\dot{\Delta}}{c}\right)\right]. \tag{B18}$$

To evaluate observables, one may use the determinant representation as a general approach [23,34]. A more convenient approach is to use the expansion of Bethe vectors (B9) over the Fock space, which was developed in [30]:

$$|\Psi_N(\{\lambda\})\rangle = \sum_{m=0}^{N}\sum_{l=0}^{N-m}\sum_{k=0}^{l}(-1)^m\sqrt{k!}\sqrt{(N-k)!}D(l,k)\binom{N-m}{l}\Gamma_{lmk}|k\rangle_a\otimes|N-k\rangle_a,$$

$$\langle\Psi_N(\{\lambda\})| = \sum_{m=0}^{N}\sum_{k=0}^{N-m}(-1)^m\langle N-k|_a\otimes\langle k|_b\sqrt{k!}\sqrt{(N-k)!}c^{-2k-m+N}D(N-m,k)e_m, \tag{B19}$$

where the coefficient Γ_{lmk} is defined as

$$\Gamma_{lmk} = \Delta^{N-m-l}c^{-N+m+2l-2k}e_m, \tag{B20}$$

and $D(M,k)$ are coefficients defined by the following recurrence relation:

$$D(M,k) = kD(M-1,k) + D(M-1,k-1), \tag{B21}$$

with the conditions $D(1,1) = 1$ and $D(M,k) = 0$ if $k > M$. This coefficient possesses the obvious property $D(M,1) = D(n,n) = 1$. The general expression for $D(M,k)$ is given by

$$D(M,k) = \sum_{n_1=0}^{M-k}\sum_{n_2=0}^{M-k-n_1}\sum_{n_3=0}^{M-k-n_1-n_2}\cdots \cdots\times\sum_{n_{k-1}=0}^{M-k-n_1-\cdots-n_{k-1}}k^{n_1}(k-1)^{n_2}\dots2^{n_{k-1}}. \tag{B22}$$

[1] E. K. Sklyanin, L. A. Takhtadzhyan, and L. D. Faddeev, Theor. Math. Phys. **40**, 688 (1979).

[2] E. K. Sklyanin, J. Sov. Math. **19**, 1546 (1982).

[3] P. P. Kulish and E. K. Sklyanin, in *Integrable Quantum Field Theories* (Springer, New York, 1982), pp. 61–119.

[4] N. Slavnov, arXiv:1804.07350.

[5] F. Levkovich-Maslyuk, J. Phys. A: Math. Theor. **49**, 323004 (2016).

[6] V. E. Korepin, N. M. Bogoliubov, and A. G. Izergin, *Quantum Inverse Scattering Method and Correlation Functions*, Vol. 3 (Cambridge University Press, Cambridge, 1997).

[7] E. H. Lieb and W. Liniger, Phys. Rev. **130**, 1605 (1963).

[8] M. Knap, C. J. M. Mathy, M. Ganahl, M. B. Zvonarev, and E. Demler, Phys. Rev. Lett. **112**, 015302 (2014).
[9] J.-M. Maillet, *Quantum Spaces* (2007), pp. 161–201.
[10] N. Kitanine, J. Maillet, and V. Terras, Nucl. Phys. B **554**, 647 (1999).
[11] G. Kato, M. Shiroishi, M. Takahashi, and K. Sakai, J. Phys. A: Math. Gen. **36**, L337 (2003).
[12] M. Bortz and F. Göhmann, Eur. Phys. J. B **46**, 399 (2005).
[13] L. Faddeev, Integrable models in 1+ 1 dimensional quantum field theory. No. CEA-CONF–6565. CEA Centre d'Etudes Nucleaires de Saclay, 1982, https://inis.iaea.org/collection/NCLCollectionStore/_Public/14/762/14762434.pdf.
[14] T. Thiery and P. Le Doussal, J. Phys. A: Math. Theor. **50**, 045001 (2016).
[15] G. Arutyunov, S. Frolov, and M. Staudacher, J. High Energy Phys. 10 (2004) 016.
[16] N. M. Bogoliubov and P. P. Kulish, Zap. Nauchn. Semin. POMI **398**, 26 (2012).
[17] M. Bortz and J. Stolze, Phys. Rev. B **76**, 014304 (2007).
[18] A. Faribault, P. Calabrese, and J.-S. Caux, J. Math. Phys. **50**, 095212 (2009).
[19] J. Zill, T. Wright, K. Kheruntsyan, T. Gasenzer, and M. Davis, SciPost Phys. **4**, 011 (2018).
[20] N. A. Sinitsyn, E. A. Yuzbashyan, V. Y. Chernyak, A. Patra, and C. Sun, Phys. Rev. Lett. **120**, 190402 (2018).
[21] P. Barmettler, D. Fioretto, and V. Gritsev, Europhys. Lett. **104**, 10004 (2013).
[22] V. Gritsev and A. Polkovnikov, SciPost Phys. **2**, 021 (2017).
[23] N. Slavnov, Theor. Math. Phys. **79**, 502 (1989).
[24] O. Gamayun, O. Lychkovskiy, E. Burovski, M. Malcomson, V. V. Cheianov, and M. B. Zvonarev, Phys. Rev. Lett. **120**, 220605 (2018).
[25] V. B. Bulchandani, R. Vasseur, C. Karrasch, and J. E. Moore, Phys. Rev. B **97**, 045407 (2018).
[26] D. Fioretto, J.-S. Caux, and V. Gritsev, New J. Phys. **16**, 043024 (2014).
[27] M. Gaudin, *La Fonction d'Onde de Bethe* (Masson, Paris, 1983).
[28] N. Slavnov, arXiv:1804.07350.
[29] G. J. Milburn, J. Corney, E. M. Wright, and D. F. Walls, Phys. Rev. A **55**, 4318 (1997).
[30] I. Ermakov, T. Byrnes, and N. Bogoliubov, Phys. Rev. A **97**, 023626 (2018).
[31] F. Haldane, Phys. Lett. A **80**, 281 (1980).
[32] Z. Mei and C. J. Bolech, Phys. Rev. E **95**, 032127 (2017).
[33] F. H. Essler, H. Frahm, F. Göhmann, A. Klümper, and V. E. Korepin, *The One-dimensional Hubbard Model* (Cambridge University Press, Cambridge, 2005).
[34] N. Bogoliubov, J. Math. Sci. **213**, 662 (2016).
[35] N. Bogoliubov, I. Ermakov, and A. Rybin, J. Phys. A: Math. Theor. **50**, 464003 (2017).