Boltzmann relaxation dynamics of strongly interacting spinless fermions on a lattice

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Motivated by the recent interest in nonequilibrium phenomena in quantum many-body systems, we study strongly interacting fermions on a lattice by deriving and numerically solving quantum Boltzmann equations that describe their relaxation to thermodynamic equilibrium. The derivation is carried out by inspecting the hierarchy of correlations within the framework of the $1/Z$ expansion. Applying the Markov approximation, we obtain the dynamic equations for the distribution functions. Interestingly, we find that in the strong-coupling limit, collisions between particles and holes dominate over particle-particle and hole-hole collisions—in stark contrast to weakly interacting systems. As a consequence, our numerical simulations show that the relaxation timescales strongly depend on the type of excitations (particles or holes or both) that are initially present.

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I. INTRODUCTION

In interacting quantum many-body systems, the nature of the excitations and their relaxation to the thermodynamic equilibrium state [1,2] display a large diversity, and may vary from system to system, depending both on the dimensionality and on the specific interactions [3–13]. Here, we are interested in quantum systems in which the excitations can be described as quasiparticles. This raises the question of whether the quasiparticles interact in such a way that their incoherent scattering processes finally lead to their equilibration. In other words, we ask if, and under which conditions, it is possible to construct a quantum Boltzmann equation that describes the collisions between quasiparticles. For weak interactions, a famous example is of course the Fermi liquid [14,15] of electrons subject to Coulomb interaction in three spatial dimensions. The quasiparticles are electrons and holes, possibly with a renormalized mass, that interact via two-particle scattering in a screened Coulomb potential [16]. However, for strongly interacting electrons, as typically found in transition-metal oxides or nitrides, already the electronic ground state may differ strongly from the weakly interacting case (for a review, see, e.g., Ref. [17]), and the relaxation kinetics of quasiparticle excitations remains elusive. Hence, the general principles of the nonequilibrium relaxation are an active research topic up to date [18]. In the following, we restrict our discussion to closed quantum lattice systems without disorder and dissipation. This means that equilibration is supposed to proceed solely by intrinsic interactions. Yet, there is rich physics to be found: While the lifetime of excitations in a Fermi liquid follows a generic law, it turns out that relaxation in a quantum system with strong interactions may proceed via several intermediate stages and thus on widely different timescales, see, e.g., Refs. [19–23].

In this work, we show for a particular example that even in the strongly interacting limit the kinetic equation describing thermalization still has the mathematical structure of a quantum Boltzmann equation, albeit with a different physical interpretation of the collision term, see also Ref. [24]. Specifically, we study a lattice model of spinless fermions with interactions between neighboring lattice sites. In the limit of strong interactions, giving rise to a gapped excitation spectrum, we find that electron-hole scattering is the dominant relaxation mechanism, in striking contrast to the conventional Fermi liquid, where hole-hole and electron-electron interactions contribute on equal footing with electron-hole interactions to the overall relaxation rate.

Spinless fermions are considered as a very simple model epitomizing the features of a metal-insulator transition [25,26]. In applications to the electronic structure of materials, the model may be applicable to crystalline solids with partial band filling in the independent electron approximation, but strong on-site Coulomb repulsion, which guarantees that each lattice site will be occupied only once, and double occupancy by electrons of opposite spin can be ignored at sufficiently low excitation energies. In addition, in a solid with a less than half-filled band, sufficiently strong on-site Coulomb interaction gives rise to a ferromagnetic ground state [27]; i.e., all spins are aligned and, as a first approximation, the spin degree of freedom can be neglected. It is noteworthy that ultra-cold atoms in an optical lattice offer another possibility to realize the model studied here, provided that the repulsion between atoms at the same lattice site is sufficiently strong to preclude multiple occupation. In this case, the trapped atoms may be considered *effectively* as spinless fermions independent of their actual spin. In addition, we note that, in the context of ultracold atoms, spinless fermions interacting indirectly via additional bosons have been considered [28,29].

In the center of our interest are lattice systems with a high coordination number Z (which means in practice, lattices in high dimensions). This is in contrast to the physics in one-dimensional systems, where the quasiparticle picture is often not suitable as a starting point for further analysis. The peculiar thermalization behavior of one-dimensional

systems has been extensively studied in recent work, see, e.g., Refs. [30–36]. In the opposite limit of high dimensionality, several authors used nonequilibrium Green functions on the two-time Keldysh contour as the starting point of their description. The Kadanoff-Baym equation [37] for these Green functions is then solved either numerically or with the help of some approximations, see, e.g. Ref. [38]. In popular approaches, the thermalization behavior is described by a self-energy that is taken from dynamic mean-field theory [39]. While this is appropriate in the limit of very high dimensionality, it usually neglects momentum conservation and the dependence of the scattering rates, and hence the self-energy, on the momentum transfer in the collision [21]. Finally, schemes based on the usual BBGKY hierarchy can only treat weak and moderate interactions, see, e.g., Ref. [24], while we are mainly interested in strong interactions. Thus derivations of Boltzmann equations for strongly interacting many-body systems on higher-dimensional lattices, as considered here, would be highly nontrivial using other methods.

The structure of the paper is as follows. After defining the model of spinless fermions, we briefly recapitulate the derivation of the Boltzmann equation [24] in the weakly interacting case, making use of the Born-Markov approximation [40,41]. Next, we introduce correlation functions in the spirit of the BBGKY hierarchy of nonequilibrium statistical mechanics [42–44], but with the important difference that we consider the correlations between lattice sites instead of those between particles. We show that an expansion in $1/Z$ allows us to define the spectrum of quasiparticle excitations at the Z^{-1} level, while the higher orders of the expansion give rise to interactions among quasiparticles and offer a natural way to close the BBGKY-like hierarchy of equations. We discuss the solutions for translationally invariant systems, in particular for the nontrivial example of the correlated ground state on a bipartite lattice. Kinetic equations are worked out explicitly for the limit of strong interaction. If, in addition, the interaction is also short-ranged and extends to nearest-neighbor lattice sites only, the interaction between quasiparticles is found to be strongly anisotropic. We illustrate the consequences of the anisotropic interactions by numerically solving the kinetic equation. In contrast to the well-known Fermi liquid with isotropic Coulomb interaction, thermalization of quasiparticles displays several timescales due to the dependence of the collisions on momentum transfer. In particular, this behavior is observed when the initial distributions of quasiparticles and quasiholes differ strongly from each other.

II. THE MODEL

We consider spinless fermions [17,25] moving on a lattice given by the hopping matrix $J_{\mu\nu}$ and repelling each other via the Coulomb matrix $V_{\mu\nu}$

$$\hat{H} = -\frac{1}{Z}\sum_{\mu,\nu} J_{\mu\nu}\hat{c}_\mu^\dagger \hat{c}_\nu + \frac{1}{2Z}\sum_{\mu,\nu} V_{\mu\nu}\hat{n}_\mu\hat{n}_\nu\,. \tag{1}$$

As usual, $\hat{c}_\mu^\dagger$ and $\hat{c}_\nu$ are the fermionic creation and annihilation operators for the lattice sites μ and ν with the corresponding number operators $\hat{n}_\mu = \hat{c}_\mu^\dagger\hat{c}_\mu$. Furthermore, Z denotes the coordination number of the translationally invariant lattice, i.e., the number of nearest neighbors. In the following, we consider nearest-neighbor interaction and tunneling for simplicity, but our results can be generalized in a straight forward manner.

In the limit of small interactions $V_{\mu\nu}$, the ground state of (1) can be described by a Fermi gas and is thus metallic for $0 < \langle\hat{n}_\mu\rangle < 1$. For large interactions $V_{\mu\nu}$, however, the structure of the ground state changes. Assuming half filling and a bipartite lattice, we have a spontaneous breaking of the translational symmetry where one sublattice is occupied while the other sublattice is empty (up to small virtual tunneling corrections), which is usually referred to as a charge density wave—quite analogous to the famous Mott insulator state in the Fermi-Hubbard model, see, e.g., Refs. [45].

III. WEAK-INTERACTION LIMIT

Let us start by briefly recapitulating the conventional derivation of the Boltzmann equation in the limit of weak interactions, see, e.g., Ref. [41]. After a spatial Fourier transform $\hat{c}_\mu \to \hat{c}_{\mathbf{k}}$, the relevant distribution functions $f_{\mathbf{k}}$ are just the occupation numbers per mode $\mathbf{k}$ and their time derivative reads according to (1)

$$\begin{aligned} i\partial_t f_{\mathbf{k}} = i\partial_t\langle\hat{c}_{\mathbf{k}}^\dagger\hat{c}_{\mathbf{k}}\rangle = -\int_{\mathbf{p}}\int_{\mathbf{q}} V_{\mathbf{q}}(&\langle\hat{c}_{\mathbf{k}}^\dagger\hat{c}_{\mathbf{p}}^\dagger\hat{c}_{\mathbf{p}+\mathbf{q}}\hat{c}_{\mathbf{k}-\mathbf{q}}\rangle^{\mathrm{corr}} \\ &- \langle\hat{c}_{\mathbf{k}-\mathbf{q}}^\dagger\hat{c}_{\mathbf{p}+\mathbf{q}}^\dagger\hat{c}_{\mathbf{p}}\hat{c}_{\mathbf{k}}\rangle^{\mathrm{corr}})\,, \end{aligned} \tag{2}$$

where we have defined the four-momentum correlators via $\langle\hat{c}_{\mathbf{k}}^\dagger\hat{c}_{\mathbf{p}}^\dagger\hat{c}_{\mathbf{k}'}\hat{c}_{\mathbf{p}'}\rangle^{\mathrm{corr}} = \langle\hat{c}_{\mathbf{k}}^\dagger\hat{c}_{\mathbf{p}}^\dagger\hat{c}_{\mathbf{k}'}\hat{c}_{\mathbf{p}'}\rangle + \langle\hat{c}_{\mathbf{k}}^\dagger\hat{c}_{\mathbf{k}'}\rangle\langle\hat{c}_{\mathbf{p}}^\dagger\hat{c}_{\mathbf{p}'}\rangle - \langle\hat{c}_{\mathbf{k}}^\dagger\hat{c}_{\mathbf{p}'}\rangle\langle\hat{c}_{\mathbf{p}}^\dagger\hat{c}_{\mathbf{k}'}\rangle$. To first order in the interaction strength $V_{\mathbf{q}}$, their time derivative reads

$$\begin{aligned} &i\partial_t\langle\hat{c}_{\mathbf{k}}^\dagger c_{\mathbf{p}}^\dagger c_{\mathbf{p}+\mathbf{q}}c_{\mathbf{k}-\mathbf{q}}\rangle^{\mathrm{corr}} \\ &\quad= (J_{\mathbf{k}} + J_{\mathbf{p}} - J_{\mathbf{k}-\mathbf{q}} - J_{\mathbf{p}+\mathbf{q}})\langle\hat{c}_{\mathbf{k}}^\dagger c_{\mathbf{p}}^\dagger c_{\mathbf{p}+\mathbf{q}}c_{\mathbf{k}-\mathbf{q}}\rangle^{\mathrm{corr}} \\ &\qquad- (V_{\mathbf{q}} - V_{\mathbf{k}-\mathbf{p}-\mathbf{q}}) \\ &\qquad\times[f_{\mathbf{k}}f_{\mathbf{p}}(1-f_{\mathbf{k}-\mathbf{q}})(1-f_{\mathbf{p}+\mathbf{q}}) - (f_{\mathbf{k}}f_{\mathbf{p}} \leftrightarrow f_{\mathbf{k}-\mathbf{q}}f_{\mathbf{p}+\mathbf{q}})]. \end{aligned} \tag{3}$$

Abbreviating these four-momentum correlators by $C_{\mathbf{kpq}}$, the above equation can be cast into the simple form $i\partial_t C_{\mathbf{kpq}} = \Omega_{\mathbf{kpq}}C_{\mathbf{kpq}} - S_{\mathbf{kpq}}$ with the source term $S_{\mathbf{kpq}}$ containing the distribution functions $f_{\mathbf{k}}$. Formally, this linear equation has the retarded solution

$$C_{\mathbf{kpq}}(t) = i\int_{-\infty}^{t} dt'\, S_{\mathbf{kpq}}(t')\exp\{-i\Omega_{\mathbf{kpq}}(t-t')\}\,. \tag{4}$$

In order to arrive at the Boltzmann equation which is local in time, we now employ the Markov approximation $S_{\mathbf{kpq}}(t') \approx S_{\mathbf{kpq}}(t)$ in the above integrand, which can be motivated by the fact that the distribution functions are slowly varying. Then (4) can be solved approximately

$$C_{\mathbf{kpq}}(t) \approx \frac{S_{\mathbf{kpq}}(t)}{\Omega_{\mathbf{kpq}} - i\varepsilon}\,, \tag{5}$$

where the infinitesimal convergence factor $\varepsilon > 0$ is inserted in order to pick out the retarded solution. As usual, the limit $\varepsilon \downarrow 0$ yields the principal value plus a delta distribution. The principal value corresponds to the adiabatic solution of $i\partial_t C_{\mathbf{kpq}} = \Omega_{\mathbf{kpq}}C_{\mathbf{kpq}} - S_{\mathbf{kpq}} \approx 0$, while the delta distribution

contributes at $\Omega_{\mathbf{kpq}} = 0$ where adiabaticity breaks down. This is the term which generates the Boltzmann collision term, where $\Omega_{\mathbf{kpq}} = 0$ corresponds to energy conservation. Inserting the approximate solution (5) back into (2) yields the well-known Boltzmann equation (see, e.g., Ref. [41])

$$\begin{aligned}\partial_t f_{\mathbf{k}} = -2\pi \int_{\mathbf{p}} \int_{\mathbf{q}} & V_{\mathbf{q}}(V_{\mathbf{q}} - V_{\mathbf{k}-\mathbf{p}-\mathbf{q}}) \\ & \times \delta(J_{\mathbf{k}} + J_{\mathbf{p}} - J_{\mathbf{k}-\mathbf{q}} - J_{\mathbf{p}+\mathbf{q}}) \\ & \times [f_{\mathbf{k}} f_{\mathbf{p}}(1 - f_{\mathbf{k}-\mathbf{q}})(1 - f_{\mathbf{p}+\mathbf{q}}) - (f_{\mathbf{k}} f_{\mathbf{p}} \leftrightarrow f_{\mathbf{k}-\mathbf{q}} f_{\mathbf{p}+\mathbf{q}})]. \end{aligned} \tag{6}$$

Here, $\mathbf{q}$ denotes the momentum transfer, i.e., particles with initial momenta $\mathbf{k}$ and $\mathbf{p}$ collide and are scattered to the final momenta $\mathbf{k} - \mathbf{q}$ and $\mathbf{p} + \mathbf{q}$ or vice versa. The delta distribution in the second line represents energy conservation in such a collision process and the factor in the first line yields the differential cross section. As is well known, this equation respects the conservations laws of energy, momentum and probability, as well as consistency conditions (such as the crossing relation) and has far reaching consequences such as the H theorem, see, e.g., Ref. [46].

IV. HIERARCHY OF CORRELATIONS

In the above derivation, we exploited the assumption of weak interaction in two ways: first, by employing a perturbative expansion in $V_{\mu\nu}$ in Eq. (3), and, second, by applying the Markov approximation (5). This approximation is based on the separation of timescales, i.e., the distribution functions $f_{\mathbf{k}}$ are slowly varying (on a timescale set by $V_{\mu\nu}$) in comparison to the rapid oscillations in (4) with the frequencies $\Omega_{\mathbf{kpq}}$ which are set by $J_{\mu\nu}$. For strong interactions, this procedure is no longer applicable. However, we will show in the following that the coordination number of lattice sites in high dimensions can be used in a similar way to establish a systematic expansion.

The framework for deriving this expansion is provided by the hierarchy of correlations [47–54]. In this approach, one considers the reduced density matrices $\hat{\rho}_\mu$ for one site, $\hat{\rho}_{\mu\nu}$ for two sites, and so on. Multisite density matrices will in general not be simple products of the single-site quantities. We therefore split off the difference due to correlations between sites, i.e., we write $\hat{\rho}^{\text{corr}}_{\mu\nu} = \hat{\rho}_{\mu\nu} - \hat{\rho}_\mu \hat{\rho}_\nu$, and analogously for multisite correlations. The time-dependence of correlations can be cast into the following hierarchy of evolution equations:

$$\partial_t \hat{\rho}_\mu = f_1\big(\hat{\rho}_\nu, \hat{\rho}^{\text{corr}}_{\mu\nu}\big), \tag{7}$$

$$\partial_t \hat{\rho}^{\text{corr}}_{\mu\nu} = f_2\big(\hat{\rho}_\nu, \hat{\rho}^{\text{corr}}_{\mu\nu}, \hat{\rho}^{\text{corr}}_{\mu\nu\sigma}\big), \tag{8}$$

$$\partial_t \hat{\rho}^{\text{corr}}_{\mu\nu\sigma} = f_3\big(\hat{\rho}_\nu, \hat{\rho}^{\text{corr}}_{\mu\nu}, \hat{\rho}^{\text{corr}}_{\mu\nu\sigma}, \hat{\rho}^{\text{corr}}_{\mu\nu\sigma\lambda}\big), \tag{9}$$

$$\partial_t \hat{\rho}^{\text{corr}}_{\mu\nu\sigma\lambda} = f_4\big(\hat{\rho}_\nu, \hat{\rho}^{\text{corr}}_{\mu\nu}, \hat{\rho}^{\text{corr}}_{\mu\nu\sigma}, \hat{\rho}^{\text{corr}}_{\mu\nu\sigma\lambda}, \hat{\rho}^{\text{corr}}_{\mu\nu\sigma\lambda\zeta}\big), \tag{10}$$

and in complete analogy for the higher correlators [47].

To derive a systematic expansion, we consider the hierarchy of correlations in the formal limit of large coordination numbers $Z \to \infty$. Following Ref. [47], it can be shown that the n-site correlators are by a factor $1/Z$ smaller than $n-1$-site correlators. For instance, starting from the on-site density matrix $\hat{\rho}_\mu = \mathcal{O}(Z^0)$ as the zeroth order, two-site correlators are smaller, $\hat{\rho}^{\text{corr}}_{\mu\nu} = \mathcal{O}(1/Z)$. Furthermore, the three-site correlators are suppressed even stronger via $\hat{\rho}^{\text{corr}}_{\mu\nu\sigma} = \mathcal{O}(1/Z^2)$, and so on. The decreasing role of higher-order correlators justifies an approximative scheme based on a truncation of the hierarchy at some specific level even without having to invoke any separation of timescales, ergodicity, or other supportive arguments. In physics language, an iterative approximation scheme can be described as follows: We start from a mean-field solution $\hat{\rho}^0_\mu$ which is obtained to zeroth order in $1/Z$ by neglecting $\hat{\rho}^{\text{corr}}_{\mu\nu}$ on the right-hand side of (7) and equating $\partial_t \hat{\rho}_\mu \approx f_1(\hat{\rho}_\nu, 0)$. Next, we insert this solution $\hat{\rho}^0_\mu$ into (8) and obtain, to first order in $1/Z$, the approximation $\partial_t \hat{\rho}^{\text{corr}}_{\mu\nu} \approx f_2(\hat{\rho}^0_\nu, \hat{\rho}^{\text{corr}}_{\mu\nu}, 0)$ This provides us with a set of inhomogeneous linear differential equations for the two-point correlations $\hat{\rho}^{\text{corr}}_{\mu\nu}$. The stationary solutions of this set can be considered as the quasiparticle modes; and in this way, the quasiparticle energy spectrum is obtained.

However, aiming at the derivation of a quantum Boltzmann equation, it is clear that we have to go further. This can be understood from the following considerations. The quasiparticles resulting from a truncation at the level of (8) are noninteracting; hence (8) is insufficient to derive a Boltzmann collision term to first order in $1/Z$. In other words, a set of differential equations linear in the variable $\hat{\rho}^{\text{corr}}_{\mu\nu}$, such as $\partial_t \hat{\rho}^{\text{corr}}_{\mu\nu} \approx f_2(\hat{\rho}^0_\nu, \hat{\rho}^{\text{corr}}_{\mu\nu}, 0)$ can not describe collisions. Therefore we need to study higher orders in $1/Z$ and and interpret the interactions between the quasiparticles arising on this level as collision terms. The above derivation of a quantum Boltzmann equation for weak interactions already suggests that one should not stop on the level of the three-point correlators $\hat{\rho}^{\text{corr}}_{\mu\nu\sigma}$ that enter into the right-hand side of (8). Due to the structure of the Coulomb interactions, we have to include the four-point correlators, too, in order to derive the Boltzmann equation (see below).

To arrive at a consistent treatment up to order $1/Z^2$, one should also insert the solution for $\hat{\rho}^{\text{corr}}_{\mu\nu}$, once it has been obtained, back into equation (7). A similar argument can be applied to $\hat{\rho}^{\text{corr}}_{\mu\nu\sigma}$ which should be inserted into (8) to obtain an improved quasiparticle spectrum. We speak of this procedure as taking into account the back-reactions. In physical terms, this amounts to a renormalization of the mean-field description by $\hat{\rho}^0_\mu$ due to the quasiparticle fluctuations. For the case considered in the following application, a small perturbation around the charge-density wave state at half filling, the back-reactions play a minor role, and they will be omitted in the following.

A. Translation-invariant systems

In this section, we specialize to spatially homogeneous systems. Consequently, we re-formulate the equations in the Fourier space of wave vectors. As starting point of the hierarchy, we first need to specify the on-site density matrix $\hat{\rho}_\mu$ or its zeroth-order (mean-field) approximation $\hat{\rho}^0_\mu$. Further specializing to the case of a half-filled band, Eq. (7) has the

simple solution

$$\hat{\rho}_\mu = \tfrac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) = \tfrac{1}{2}\,\mathbf{1}_\mu\,. \tag{11}$$

Due to the assumed spatial homogeneity and particle number conservation, this solution is actually unique and hence there is no back-reaction; i.e., $\hat{\rho}_\mu = \hat{\rho}^0_\mu$.

As in Eq. (2), the distribution functions $f_\mathbf{k}$ are given by the relevant two-point correlations $\langle \hat{c}^\dagger_\mu \hat{c}_\nu\rangle^{\rm corr}$ via

$$\langle \hat{c}^\dagger_\mu \hat{c}_\nu\rangle = \langle \hat{c}^\dagger_\mu \hat{c}_\nu\rangle^{\rm corr} + \delta_{\mu\nu}\langle \hat{n}_\mu\rangle = \int_\mathbf{k} f_\mathbf{k} e^{i\mathbf{k}\cdot(\mathbf{x}_\nu - \mathbf{x}_\mu)}\,. \tag{12}$$

Then, Eq. (8) implies

$$i\partial_t f_\mathbf{k} = \int_\mathbf{q} V_{\mathbf{k}+\mathbf{q}}(g_{\mathbf{q}\mathbf{k}} - g_{\mathbf{k}\mathbf{q}})\,, \tag{13}$$

where the $g_{\mathbf{q}\mathbf{k}}$ denote the Fourier components of the relevant three-point correlations

$$\langle \hat{n}_\alpha \hat{c}^\dagger_\beta \hat{c}_\gamma\rangle^{\rm corr} = \int_\mathbf{p}\int_\mathbf{q} g_{\mathbf{p}\mathbf{q}} e^{i\mathbf{p}\cdot(\mathbf{x}_\beta - \mathbf{x}_\alpha) + i\mathbf{q}\cdot(\mathbf{x}_\gamma - \mathbf{x}_\alpha)}\,. \tag{14}$$

The time-derivatives of the three-point correlators can be obtained from (9) and have a form similar to (3)

$$i\partial_t g_{\mathbf{q}\mathbf{k}} = (J_\mathbf{q} - J_\mathbf{k}) g_{\mathbf{q}\mathbf{k}} + S^{(3)}_{\mathbf{q}\mathbf{k}}\,, \tag{15}$$

but, in contrast to (3), the source term $S^{(3)}_{\mathbf{q},\mathbf{k}}$ contains four-point correlations (instead of distribution functions)

$$\langle \hat{c}^\dagger_\alpha \hat{c}_\beta \hat{c}^\dagger_\mu \hat{c}_\nu\rangle^{\rm corr} = \int_\mathbf{k}\int_\mathbf{p}\int_\mathbf{q} h_{\mathbf{k}\mathbf{p}\mathbf{q}} \times e^{i\mathbf{k}\cdot(\mathbf{x}_\alpha - \mathbf{x}_\nu) + i\mathbf{p}\cdot(\mathbf{x}_\beta - \mathbf{x}_\nu) + i\mathbf{q}\cdot(\mathbf{x}_\mu - \mathbf{x}_\nu)}\,. \tag{16}$$

Finally, their time-derivative reads according to (10)

$$i\partial_t h_{\mathbf{k}\mathbf{p}\mathbf{q}} = (J_\mathbf{k} - J_\mathbf{p} + J_\mathbf{q} - J_{\mathbf{k}+\mathbf{p}+\mathbf{q}}) h_{\mathbf{k}\mathbf{p}\mathbf{q}} + S^{(4)}_{\mathbf{k}\mathbf{p}\mathbf{q}}\,, \tag{17}$$

where the source term $S^{(4)}_{\mathbf{k}\mathbf{p}\mathbf{q}}$ contains products of two-point correlators, somewhat similar to (3).

Now we may integrate the evolution equations (15) and (17) in the same way as in (4), which yields a double time integral. In order to approximate this integral, we again use the Markov approximation: Since the two-point correlations $\langle \hat{c}^\dagger_\mu \hat{c}_\nu\rangle^{\rm corr}$ scale with $1/Z$ but the three-point correlators $\langle \hat{n}_\alpha \hat{c}^\dagger_\beta \hat{c}_\gamma\rangle^{\rm corr}$ scale with $1/Z^2$, the distribution functions $f_\mathbf{k}$ are slowly varying according to (13), because the right-hand side is suppressed by an additional factor of $1/Z$. (To first order in $1/Z$, the distribution functions $f_\mathbf{k}$ are constant.) In contrast, the Fourier components of the three-point $g_{\mathbf{q}\mathbf{k}}$ and four-point $h_{\mathbf{k}\mathbf{p}\mathbf{q}}$ contributions are rapidly oscillating with the frequencies $\Omega_{\mathbf{q}\mathbf{k}} = J_\mathbf{q} - J_\mathbf{k}$ as well as $\Omega_{\mathbf{k}\mathbf{p}\mathbf{q}} = J_\mathbf{k} - J_\mathbf{p} + J_\mathbf{q} - J_{\mathbf{k}+\mathbf{p}+\mathbf{q}}$ according to Eqs. (15) and (17). Using this separation of timescales, the double time integral can be evaluated within Markov approximation in analogy to (5) by simplifying the integrand according to $f_\mathbf{k}(t) \approx f_\mathbf{k}(t')$.

Inserting this solution of the double time integral back into Eq. (13), we obtain a Boltzmann equation which has exactly the same form as in (6). This is perhaps not too surprising since we did not assume that the interactions $V_{\mu\nu}$ are strong. In fact, the on-site state (11) could represent free (or weakly interacting) fermions in their ground state (or in a thermal state). As a crucial difference, however, the above derivation of the Boltzmann equation is based on an expansion into powers of $1/Z$ instead of $V_{\mu\nu}$. Thus the above $1/Z$-derivation can also be applied to the strongly interacting case.

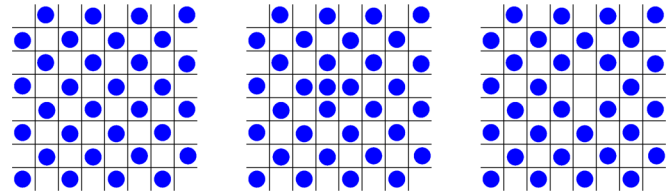

FIG. 1. Sketch of a square lattice with a checker-board pattern as an example for a charge-density wave state (left) with a quasiparticle (middle) and quasihole (right) excitation. By definition of the model, the original fermions (blue dots) can only move to the nearest neighboring lattice sites, i.e., one step in horizontal or vertical direction, but not along the diagonal. Due to the strong repulsion V, a quasiparticle (middle) and quasihole (right) can only move to next-to-nearest neighboring lattice sites, which involves second-order tunneling processes such as co-tunneling of two fermions (middle) or sequential tunneling of one fermion (right). We also see that neither two quasiparticles (middle) nor two quasiholes (right) can occupy nearest neighboring lattice sites.

B. Mean-field solutions with broken symmetry

Let us now consider the limit of strong interactions $V_{\mu\nu}$. Next, we choose as a reference a mean-field solution that necessarily depends on the type of lattice and the filling factor. Assuming a bipartite lattice at half filling, the ground state is a Mott-type insulator [17] since the fermions mainly occupy one sublattice, and tunneling to the other sublattice is suppressed by the repulsion $V_{\mu\nu}$. Thus we start with the mean-field ansatz

$$\hat{\varrho}^0_\mu = \begin{cases} |0\rangle\langle 0| = \mathbf{1}_\mu - \hat{n}_\mu & \text{for} \quad \mu \in \mathcal{A} \\ |1\rangle\langle 1| = \hat{n}_\mu & \text{for} \quad \mu \in \mathcal{B} \end{cases}, \tag{18}$$

where $\mathcal{A}$ and $\mathcal{B}$ denote the two sublattices. This ansatz asserts different occupation of each sublattice and thus breaks the translational symmetry of the original problem. Physically, this would correspond to a charge density wave. In a square lattice or in the two-dimensional principal lattice planes of cubic or hypercubic lattices, for example, the fermions would form a checker-board pattern, see Fig. 1.

In this case, the proper treatment of the correlations $\langle \hat{c}^\dagger_\mu \hat{c}_\nu\rangle^{\rm corr}$ requires a case distinction. One needs to distinguish which of the two sublattices μ and ν belong to. In the following, we denote these sublattices by calligraphic superscripts, e.g., for $\mu \in \mathcal{A}$ and $\nu \in \mathcal{B}$, the expectation value $\langle \hat{c}^\dagger_\mu \hat{c}_\nu\rangle$ is given by the Fourier transform of $f^{\mathcal{A}\mathcal{B}}_\mathbf{k}$, and analogously for other combinations of superscripts. The on-site equation (7) then determines the back-reaction of the correlations onto the mean field via $i\partial_t\langle \hat{n}^\mathcal{A}\rangle = -i\partial_t\langle \hat{n}^\mathcal{B}\rangle = \int_\mathbf{k} J_\mathbf{k}(f^{\mathcal{B}\mathcal{A}}_\mathbf{k} - f^{\mathcal{A}\mathcal{B}}_\mathbf{k})$, but we shall not consider this small small correction in the following.

Since we have four functions $f^{\mathcal{A}\mathcal{A}}_\mathbf{k}$, $f^{\mathcal{A}\mathcal{B}}_\mathbf{k}$, $f^{\mathcal{B}\mathcal{A}}_\mathbf{k}$, and $f^{\mathcal{B}\mathcal{B}}_\mathbf{k}$, we denote the two sublattices by capital superscripts such as $X \in \{\mathcal{A}, \mathcal{B}\}$. Then Eq. (8) becomes

$$i\partial_t f^{XY}_\mathbf{k} = J_\mathbf{k}\big(f^{\bar{X}Y}_\mathbf{k} - f^{X\bar{Y}}_\mathbf{k}\big) - (V^{\bar{X}} - V^{\bar{Y}}) f^{XY}_\mathbf{k} + S^{XY}_\mathbf{k}\,, \tag{19}$$

where $\bar{X}$ denotes the sublattice opposite to X, i.e., if $X = \mathcal{A}$ then $\bar{X} = \mathcal{B}$ and vice versa. Furthermore, $V^{\mathcal{A}}$ denotes the interaction energy associated to sublattice $\mathcal{A}$, i.e., $V^{\mathcal{A}} = \sum_\alpha V_{\alpha\beta}\langle\hat{n}_\alpha\rangle/Z$ for any $\alpha \in \mathcal{A}$. For all interactions equal, this simplifies to $V^{\mathcal{A}} = V\langle\hat{n}_\alpha\rangle$ Again, the source terms $S^{XY}_{\mathbf{k}}$ also contain three-point correlations.

Before continuing, let us diagonalize the above linear set of equations (with source terms $S^{XY}_{\mathbf{k}}$) because the $f^{XY}_{\mathbf{k}}$ are rapidly oscillating instead of slowly varying. This can be achieved via a rotation in the X-Y subspace with an orthogonal 2×2 transformation matrix $O^a_X(\mathbf{k})$ via $f^{ab}_{\mathbf{k}} = \sum_{XY} O^a_X(\mathbf{k}) f^{XY}_{\mathbf{k}} O^b_Y(\mathbf{k})$, see Appendix C 2. In terms of the rotated functions $f^{ab}_{\mathbf{k}}$, the evolution equation (19) simplifies to

$$i\partial_t f^{ab}_{\mathbf{k}} = \left(E^b_{\mathbf{k}} - E^a_{\mathbf{k}}\right) f^{ab}_{\mathbf{k}} + S^{ab}_{\mathbf{k}}\,, \tag{20}$$

with the quasiparticle ($a = +$) and hole ($a = -$) energies

$$E^{\pm}_{\mathbf{k}} = \frac{1}{2}\left(V \pm \sqrt{(V^{\mathcal{A}} - V^{\mathcal{B}})^2 + 4J^2_{\mathbf{k}}}\right), \tag{21}$$

where we have used $V^{\mathcal{A}} + V^{\mathcal{B}} = V$ due to $\langle\hat{n}^{\mathcal{A}}\rangle + \langle\hat{n}^{\mathcal{B}}\rangle = 1$. This formula with its two solutions is reminiscent of the two solutions of the Fermi-Hubbard model in the Mott insulator phase [55] where a lower and an upper Hubbard band are formed. In the following, we speak of a quasiparticle band and a quasihole band referring to the $+$ and $-$ sign in (21). Apart from the gap $V^{\mathcal{A}} - V^{\mathcal{B}}$ which is basically the repulsion energy, the quadratic dependence on the hopping $J^2_{\mathbf{k}}$ indicates that (quasi) particles and quasiholes can only move via second-order tunneling processes such as co-tunneling, cf. Fig. 1.

We see that the functions $f^{ab}_{\mathbf{k}}$ in (20) are rapidly oscillating for $a \neq b$ but slowly varying for $a = b$. Hence the latter two are the quasiparticle ($a = b = +$) and quasihole ($a = b = -$) distribution functions, which we denote by $f^+_{\mathbf{k}}$ and $f^-_{\mathbf{k}}$, respectively. Their dynamics can be derived in complete analogy to the previous case, cf. Eqs. (13)–(17), the only differences are the additional particle/hole indices on the correlation functions $f^{ab}_{\mathbf{k}}$, $g^{abc}_{\mathbf{qk}}$, and $h^{abcd}_{\mathbf{kpq}}$, as well as the source terms $S^{ab}_{\mathbf{k}}$, $S^{abc}_{\mathbf{qk}}$, and $S^{abcd}_{\mathbf{kpq}}$. Apart from these additional indices, the derivation of the Boltzmann equation is completely analogous to the previous case, where we finally arrive at [see Eq. (C31)]

$$\begin{aligned}\partial_t f^d_{\mathbf{k}} = -2\pi \int_{\mathbf{p}}\int_{\mathbf{q}} \sum_{a,b,c} M^{abcd}_{\mathbf{p+q,p,k-q,k}} \\ \times \delta\left(E^a_{\mathbf{p+q}} - E^b_{\mathbf{p}} + E^c_{\mathbf{k-q}} - E^d_{\mathbf{k}}\right) \\ \times \left[f^d_{\mathbf{k}} f^b_{\mathbf{p}}\left(1 - f^c_{\mathbf{k-q}}\right)\left(1 - f^a_{\mathbf{p+q}}\right) - \left(f^d_{\mathbf{k}} f^b_{\mathbf{p}} \leftrightarrow f^c_{\mathbf{k-q}} f^a_{\mathbf{p+q}}\right)\right].\end{aligned} \tag{22}$$

The matrix elements $M^{abcd}_{\mathbf{p+q,p,k-q,k}}$ contain different processes, such as collision of two quasiparticles $M^{++++}_{\mathbf{p+q,p,k-q,k}}$ or two quasiholes $M^{----}_{\mathbf{p+q,p,k-q,k}}$, but also pair-creation processes, e.g., with one incoming quasiparticle and two outgoing particles plus one quasihole, as long as they are allowed by energy conservation—which is enforced by the second line of (22).

C. Other geometries

The mean-field background sketched in Fig. 1 is based on a square or hypercubic lattice at half filling. It might be illuminating to discuss other geometries. Since a graphene-type honeycomb lattice is also bipartite, one would get an analogous Mott-type insulator state at half filling (although no longer with a checkerboard structure, of course) by placing the fermions in one sublattice and keeping the other one empty. In this case, the dispersion relations $J_{\mathbf{k}}$ would change, but apart from that the energies $E^{\pm}_{\mathbf{k}}$, are still given by the same functional form (21).

A triangular lattice, on the other hand, is not bipartite, and thus does not feature such a Mott-type insulator state at half filling. However, at a filling factor of one third, one can obtain an analogous state by occupying one sublattice (also triangular) while keeping the other two sublattices empty (because the triangular lattice is tripartite). In this case, each occupied lattice site would be surrounded by empty sites.

Another interesting point is a departure from half-filling. As long as this deviation is small enough such that it does not destroy the global checkerboard structure, one could take it into account by an imbalance of the initial conditions for the distribution functions $f^+_{\mathbf{k}}$ and $f^-_{\mathbf{k}}$. Increasing the $f^+_{\mathbf{k}}$ or decreasing $f^-_{\mathbf{k}}$ (i.e., increasing the quasihole occupation $1 - f^-_{\mathbf{k}}$) corresponds to a filling factor a bit above or below one half, respectively.

Even more possibilities may arise if we do not restrict ourselves to a simple nearest-neighbor interaction. Depending on the interaction matrix $V_{\mu\nu}$, one obtains a plethora of geometric phases. In this context, it is also interesting to note an analogy to ultra-cold atoms in optical lattices (see, e.g., Refs. [28,29]). In this setup, it is possible to realize a scenario where the nearest-neighbor interaction of the spinless fermions considered here is replaced by an effective interaction via additional bosonic atoms. This effectively nonlocal interaction may support new phases such as a supersolid phase, which displays some similarities to the checkerboard structure considered here (although it is not a precise one-to-one correspondence).

V. STRONG-INTERACTION LIMIT

Let us now consider the strongly interacting limit $V_{\mathbf{k}} \gg J_{\mathbf{k}}$ in order to simplify the complicated expressions of the various matrix elements $M^{abcd}_{\mathbf{p+q,p,k-q,k}}$. In this regime, the effective band width of the two bands described by Eq. (21) is small compared to the gap between the two because the former scales with J^2/V as compared to the latter scaling with V. Collisions between quasiparticles, quasiholes, or quasiparticle and -hole can, therefore, not provide the energy needed to overcome the gap and create additional particle-hole pairs. The dominant process in this limit is the particle-hole scattering which is determined by the matrix elements

$$\begin{aligned}M^{0011}_{\mathbf{p+q,p,k-q,k}} &= M^{1100}_{\mathbf{p+q,p,k-q,k}} \\ &\approx V_{\mathbf{q}}\left[V_{\mathbf{q}} + \frac{V_{\mathbf{k-q-p}}}{V^2(n^{\mathcal{A}} - n^{\mathcal{B}})^2}\left(J_{\mathbf{p}}J_{\mathbf{k}} + J_{\mathbf{p+q}}J_{\mathbf{k-q}}\right)\right].\end{aligned} \tag{23}$$

The seconds term in (23) is suppressed by a factor of J^2/V^2 since this particle-hole exchange term requires two hopping events. The particle-particle and hole-hole scattering demands at least four hopping events and is given by

$$M^{0000}_{\mathbf{p+q},\mathbf{p},\mathbf{k-q},\mathbf{k}} = M^{1111}_{\mathbf{p+q},\mathbf{p},\mathbf{k-q},\mathbf{k}} \approx \frac{V_\mathbf{q}(J_{\mathbf{p+q}}J_\mathbf{p} + J_{\mathbf{k-q}}J_\mathbf{k})}{V^4(n^\mathcal{A} - n^\mathcal{B})^4} \times [V_\mathbf{q}(J_{\mathbf{p+q}}J_\mathbf{p} + J_{\mathbf{k-q}}J_\mathbf{k}) - V_{\mathbf{k-p-q}}(J_{\mathbf{p+q}}J_\mathbf{k} + J_{\mathbf{k-q}}J_\mathbf{p})]\,. \tag{24}$$

In addition, there is particle-hole scattering which involves to leading order the exchange-term $\sim V_\mathbf{q}V_{\mathbf{k-p-q}}$,

$$M^{0110}_{\mathbf{p+q},\mathbf{p},\mathbf{k-q},\mathbf{k}} = M^{1001}_{\mathbf{p+q},\mathbf{p},\mathbf{k-q},\mathbf{k}} \approx \frac{V_\mathbf{q}(J_{\mathbf{p+q}}J_{\mathbf{k-q}} + J_\mathbf{p}J_\mathbf{k})}{V^2(n^\mathcal{A} - n^\mathcal{B})^2} \times \left[\frac{V_\mathbf{q}(J_{\mathbf{p+q}}J_{\mathbf{k-q}} + J_\mathbf{p}J_\mathbf{k})}{V^2(n^\mathcal{A} - n^\mathcal{B})^2} + V_{\mathbf{k-p-q}}\right]. \tag{25}$$

Note that here the contribution $\sim V_\mathbf{q}^2$ is of higher order in contrast to the particle-hole scattering channel given by (23).

Keeping the lowest terms only implies a considerable simplification of the quantum Boltzmann equation (22), which now reads

$$\partial_t f^+_\mathbf{k} = -2\pi \int_\mathbf{p}\int_\mathbf{q} V_\mathbf{q}^2\, \delta(E^-_{\mathbf{p+q}} - E^-_\mathbf{p} + E^+_{\mathbf{k-q}} - E^+_\mathbf{k}) \times [f^+_\mathbf{k} f^-_\mathbf{p}(1 - f^+_{\mathbf{k-q}})(1 - f^-_{\mathbf{p+q}}) - (f^+_\mathbf{k} f^-_\mathbf{p} \leftrightarrow f^+_{\mathbf{k-q}} f^-_{\mathbf{p+q}})]. \tag{26}$$

In this limit of strong interactions, the scattering cross section only depends on the momentum transfer $\mathbf{q}$.

Quasiparticles and quasiholes have to be considered two distinct classes rather than a pair of particle and antiparticle as in the weakly interacting case. This becomes clear from the absence of the second term $V_\mathbf{q}V_{\mathbf{k-p-q}}$ of Eq. (6) from Eq. (26). This term is usually interpreted as interference term between processes with exchanged collision partners. In the present case, where quasiparticle and quasihole are independent, it does appear but is strongly suppressed by the denominator $V^2(n^\mathcal{A} - n^\mathcal{B})^2$ in the first factor on the right hand side of Eq. (25).

Quite intuitively, the suppression of particle-particle (or hole-hole) collisions can be understood by the observation that two particles cannot come close enough to interact directly (same for two holes): they can only interact via higher-order virtual hopping processes, see Fig. 1. In contrast, a quasiparticle and a quasihole can occupy neighboring lattice sites and thus they can interact directly via $V_{\mu\nu}$.

As expected, the Boltzmann equations (22) and (26) respect the standard conservations laws (e.g., energy, momentum and probability) and satisfy the usual consistency conditions (e.g., the crossing relation). Note that the quasiparticle $f^+_\mathbf{k}$ and quasihole $f^-_\mathbf{k}$ excitations obey fermionic statistics, consistent with the structure in Eqs. (22) and (26), i.e., the presence of terms of the type $(1 - f^+_{\mathbf{k-q}})$, etc. As another analogy to the weakly interacting limit (6), the quasihole distribution function $f^-_\mathbf{k}$ approaches unity in the strongly interacting ground state, i.e., the hole excitations are properly described by $1 - f^-_\mathbf{k}$, as in (6).

It is also possible to construct a quantity

$$H = -\sum_a \int_\mathbf{k} \left(f^a_\mathbf{k} \ln f^a_\mathbf{k} + \left(1 - f^a_\mathbf{k}\right) \ln\left(1 - f^a_\mathbf{k}\right)\right) \tag{27}$$

that is nondecreasing under collisions and thus to derive an H theorem [24,56]. As a consequence, the populations of both particles and holes will finally reach stationary distributions, i.e., the system reaches thermalization.

VI. TIMESCALE ANALYSIS

Since quasiparticles and quasiholes are considered to be independent, the relaxation described by the quantum Boltzmann equation take place on different timescales depending on the preparation of initial conditions. To explore this possibility, we perform numerical studies on the basis of a specific model: the spinless fermions move on a two-dimensional square lattice, Coulomb interactions V are limited to nearest-neighbor sites, and $V \gg J$. For our calculations, we use a ratio of $J/V = 10^{-3}$ and set V to one. The initial condition is taken as a small perturbation δf of the charge-density wave that could be realized, e.g., by photodoping [23]. For this particular choice, the energies entering into the model are given by

$$E^\pm_\mathbf{k} = \frac{V}{2}\left(1 \pm \sqrt{1 + \frac{J^2}{V^2}\,(\cos(k_x) + \cos(k_y))^2}\right). \tag{28}$$

Since differences of these energies scale with J^2/V (i.e., the effective bandwidth) while the scattering cross section in the Boltzmann equation (26) scales with V^2, the typical order of magnitude of the relaxation rate scales with V^3/J^2. A look at the second line of Eq. (26), however, shows that the rates are also affected by the distribution functions. To probe this dependence, we vary the initial values for the $f^\pm_\mathbf{k}$ over a few orders of magnitude and see from our calculations that the relaxation rates scale linearly with the initial perturbation δf. This behavior can be understood if we look again at Eq. (26): for a given small perturbation $\delta f \ll 1$ from the charge-density wave ground state of the system, we have $f^+_\mathbf{k} = \delta f$ and $f^-_\mathbf{k} = 1 - \delta f \approx 1$ for the perturbed states. Inserting into the rate equation (26), we find that the four distribution functions in the rate can be approximated by a total factor of $\delta f\, f^+_\mathbf{k}$, which proves the linearity in δf.

Figure 2 shows a graphical representation of the band structure. The upper (quasiparticle) and the lower (quasihole) band are mirror-symmetric with respect to the center plane of the gap. The quasiparticle band has a maximum at the center of the Brillouin zone and minima at the zone boundaries whereas the lower band is maximal at these k-points and minimal at the zone center. For simplicity sake, we will refer to the set of k-points whose eigen-energies are closest to the gap as the *diamond*.

Note that the Coulomb matrix element $V_\mathbf{q}$ has a strong $\mathbf{q}$ dependence that is equivalent to the $\mathbf{k}$-dependent term under the square root of $E_\mathbf{k}$ in the case of nearest-neighbor interactions. Due to this structure of $V_\mathbf{q}$, we have high scattering rates for transitions along the diamond (with both initial and final states inside the diamond) as well as for transitions between the centers of adjacent Brillouin zones. Contrarily, the scattering

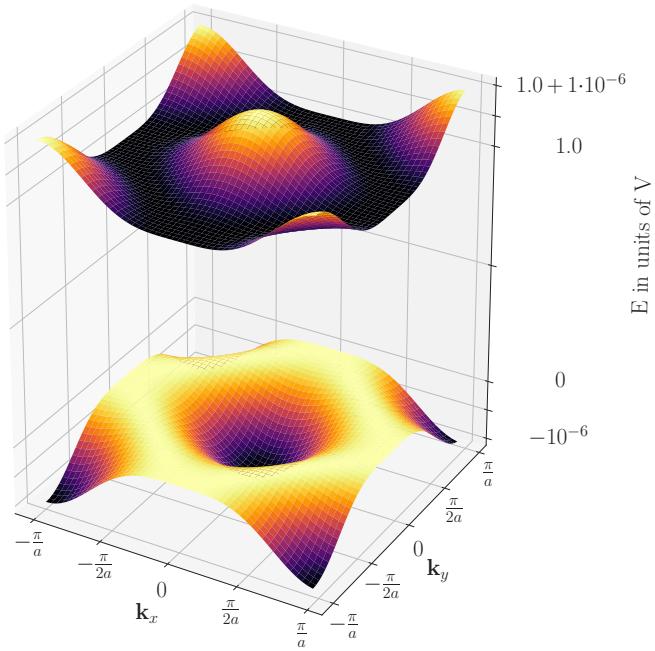

FIG. 2. Band structure for a two-dimensional square lattice for $J/V = 10^{-3}$. We can see the Brillouin zone and parts of the adjacent ones. The gap is shrunk by six orders of magnitude to show the **k** dependence of both bands in one plot.

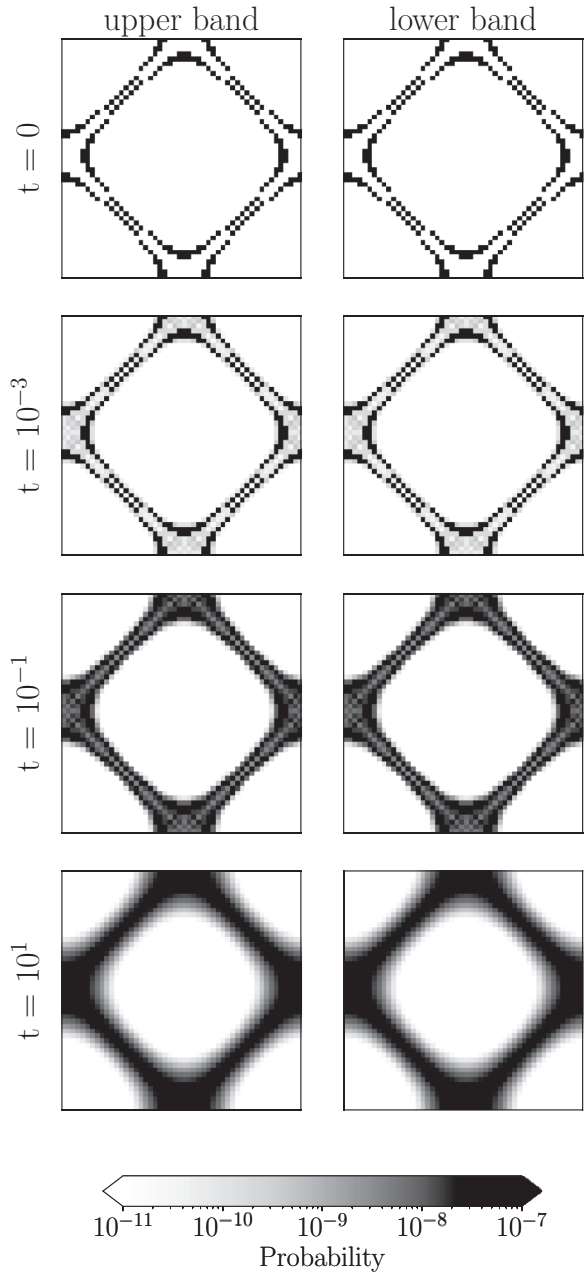

FIG. 3. Time series (in units of J^2/V^3) of the evolution of a low-energy excitation with symmetric initial conditions. For both bands at $t = 0$, a set of states that is close to but not at the gap is perturbed.

rates between the zone center and the diamond are much reduced and even tend to zero towards the diamond corners.

As a first test case, we use initial conditions that are mirror-symmetric with respect to the gap and represent a low-energy input: quasiparticles and quasiholes initially occupy the same few **k** states close to but slightly away from the gap with a low probability of $\delta f = 10^{-7}$ per state. Given these initial conditions, we then integrate the quantum Boltzmann equation on a numerical grid of 50×50 k points in the Brillouin zone using an adaptive time step method.

A selected part from the resulting time-series is depicted in Fig. 3 and shows how the distributions evolve. We observe that the scattering among the quasiparticles and quasiholes leads to a spread of their initial distribution over the whole diamond. At the end, the states with the lowest (highest for holes) energies have the highest occupation probabilities and we see as expected for a thermalized distribution that the probabilities decrease towards higher (lower) energies (cf. Fig. 4).

To explore the consequences of unequal initial populations in the respective bands, we choose initial conditions where quasiparticles are again located close to the gap while now the holes are located close to the Brillouin zone center. Analyzing the time series in Fig. 5, we notice that the relaxation of the initial population proceeds on different timescales.

In the early stages at $t = 10^{-1}J^2/V^3$, scattering among the holes at the center with the particles in the diamond results in a localized broadening around the center and the flanks of the diamond. At the time $t = J^2/V^3$, the unoccupied corner states of the diamond in the quasiparticle band start to fill. Simultaneously, the occupation probabilities of the quasihole states close to the gap start to increase such that the diamond becomes visible whereas for the quasiparticles in turn the center becomes populated. At later times, the distributions of quasiparticles and quasiholes within the Brillouin zone come to resemble each other more and more closely until they eventually become equal at around $t = 10^2J^2/V^3$: Now both have their population maximum at the zone center, but the diamond is also still populated.

For the interpretation of these results, we have to keep in mind that in the limit $V \gg J$, quasiparticles and quasiholes scatter with each other, but not among themselves. A quasiparticle and a quasihole are able to exchange both energy and momentum in the scattering. However, the momentum transfer is governed by the Coulomb matrix element $V_{\mathbf{q}}$, assigning different probability to different momentum transfers.

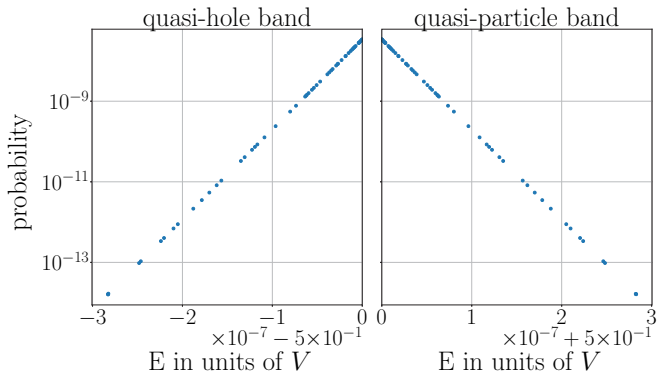

FIG. 4. Probability distribution at $t = 10^2 J^2/V^3$ for the case of the symmetric initial conditions. Note that the occupation probabilities for the quasiholes are plotted as $1 - f^-$.

For this reason, we observe in the early stage transitions for which the energy is almost conserved separately for quasielectrons and quasiholes (scattering events from the zone center to an adjacent one or within the diamond by one lattice vector) since the momentum dependence of $V_{\mathbf{q}}$ favors these transitions. At the same time, scattering from the center to the diamond flank with a momentum transfer of half a lattice vector contributes as the energy exchange is a good match and the Coulomb matrix element is still sizable.

In the later stages, scattering with arbitrary energy and momentum transfer start to play its role. Due to these processes, quasiparticle and -hole occupation probabilities become more and more similar. The equilibrium configuration for the quasiparticles (quasiholes) shows a higher (lower) population in the high-energy region compared to the diamond region (cf. last panel of Fig. 5). It corresponds to an inverted Fermi-Dirac distribution characteristic of a system at negative temperature as can be seen in the log plot of the distributions in Fig. 6. The same applies analogously for the quasihole distribution. Note that such a negative temperature is facilitated by the upper limit for the energy (bounded spectrum) and the large initial value for the total energy of the system.

VII. OUTLOOK: BACK-REACTION

In the strongly interacting limit $V \gg J$ considered here, the role of the microscopic parameters J and V reduces to a simple overall scaling V^3/J^2 of the relaxation rate (see above), the rest is determined by purely geometrical dimensionless quantities. Going away from this limit, however, the situation becomes more complex.

One the one hand, the eigenenergies $E^{\pm}_{\mathbf{k}}$ and the matrix elements $M^{abcd}_{\mathbf{p}+\mathbf{q},\mathbf{p},\mathbf{k}-\mathbf{q},\mathbf{k}}$ entering the Boltzmann equation (22) depend nontrivially on the dimensionless ratio J/V. On the other hand, by including back-reaction via $i\partial_t \langle \hat{n}^{\mathcal{A}} \rangle = -i\partial_t \langle \hat{n}^{\mathcal{B}} \rangle = \int_{\mathbf{k}} J_{\mathbf{k}} (f^{\mathcal{B}\mathcal{A}}_{\mathbf{k}} - f^{\mathcal{A}\mathcal{B}}_{\mathbf{k}})$, the time-dependence of the distribution functions $f^{\pm}_{\mathbf{k}}$ stemming from the Boltzmann equation (22) does also entail a time-dependence of the sublattice fillings $\langle \hat{n}^{\mathcal{A}} \rangle$ and $\langle \hat{n}^{\mathcal{B}} \rangle$, which in turn modify the eigenenergies $E^{\pm}_{\mathbf{k}}$ and the matrix elements $M^{abcd}_{\mathbf{p}+\mathbf{q},\mathbf{p},\mathbf{k}-\mathbf{q},\mathbf{k}}$ entering the Boltzmann equation (22) in a time-dependent manner.

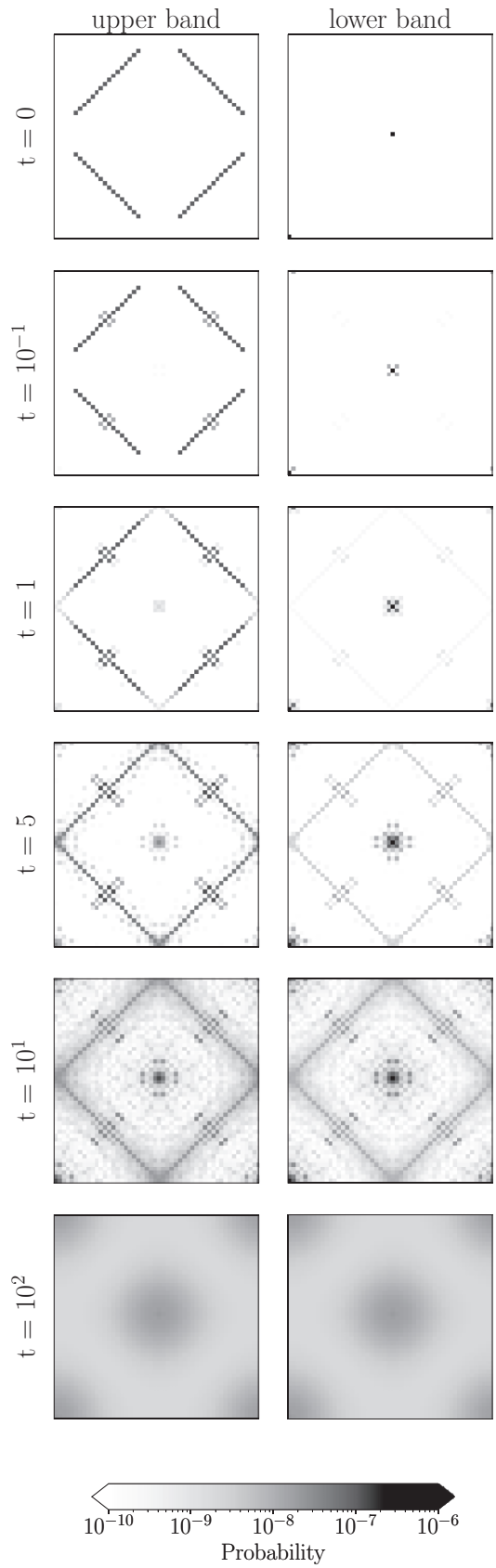

FIG. 5. Times series (in units of J^2/V^3) of the evolution of a high energy excitation. The upper (quasiparticle) band is initially perturbed in the diamond shaped minimum, the lower (quasihole) band in the zone center.

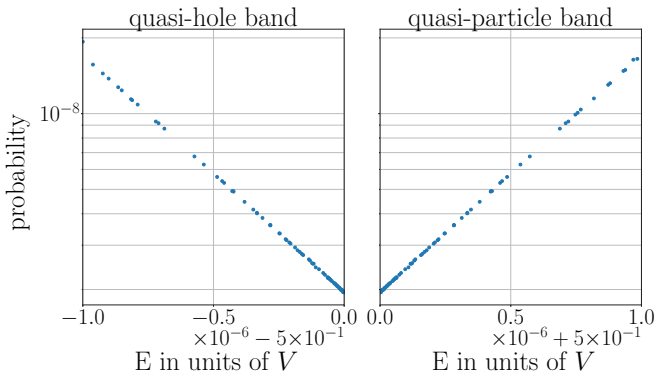

FIG. 6. Probability distribution at $t = 10^2 J^2/V^3$ for the case of the asymmetric initial conditions. Note that the occupation probabilities for the quasiholes are plotted as $1 - f^-$.

In principle, these intricate interdependencies can be taken in account self-consistently. Fortunately, in the limit of small $J/V \ll 1$ and small populations $f^+_{\mathbf{k}} \ll 1$ and $f^-_{\mathbf{k}} \approx 1$ considered here, all these modifications are tiny and can be neglected.

VIII. CONCLUSIONS

For strongly interacting spinless fermions on a general regular bipartite lattice in higher dimensions, we employ the hierarchy of correlations in order to derive the quasiparticle and quasihole excitations, their spectrum as well as their mutual interactions, which allows us to obtain a quantum Boltzmann equation [54]. In the strong-coupling limit, the ground state (at half filling) is given by the charge-density wave state, quite analogous to the Mott insulator phase in the Fermi-Hubbard model. In this limit, we find that collisions between quasiparticles and quasiholes dominate over particle-particle and hole-hole scattering events.

As a result, the relaxation and thermalization dynamics strongly depends both on the absolute magnitude and on the initial distribution of the excitations (quasiparticles or quasiholes) in the Brillouin zone. For small initial quasiparticle populations, their lifetime is inversely proportional to the initial occupation probability, i.e., the strength of the excitation. Due to the varying efficiency of momentum transfer, relaxation proceeds in two stages if the distributions of quasiparticles and holes in the Brillouin zone are initially very different. Only in the second stage, the distributions of quasiparticles and holes begin to resemble each other. The H theorem for the quantum Boltzmann equation ensures that a unique equilibrium state is finally reached.

In summary, we demonstrated that thermalization even in a strongly correlated system can still be described in high-dimensional systems within the well-known framework of a quantum Boltzmann equation, but the solutions of this equation fall into different classes depending on the initial conditions chosen. As possible directions for future work, one could study initial conditions departing from half-filling (corresponding to doped Mott insulators) or with parameters close to the metal-insulator transition.

ACKNOWLEDGMENTS

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APPENDIX A: CORRELATORS AND DEFINITIONS

Before we present the details of our calculation, we give here the explicit form of the correlation functions and their Fourier representations. For spinless fermions, the Heisenberg equations for the annihilation and creation operators are

$$i\partial_t \hat{c}_\alpha = -\frac{1}{Z}\sum_\mu J_{\mu\alpha}\hat{c}_\mu + \frac{1}{2Z}\sum_\mu V_{\mu\alpha}(\hat{c}_\alpha \hat{n}_\mu + \hat{n}_\mu \hat{c}_\alpha), \quad \text{(A1)}$$

$$i\partial_t \hat{c}^\dagger_\alpha = \frac{1}{Z}\sum_\mu J_{\mu\alpha}\hat{c}^\dagger_\mu - \frac{1}{2Z}\sum_\mu V_{\mu\alpha}(\hat{c}^\dagger_\alpha \hat{n}_\mu + \hat{n}_\mu \hat{c}^\dagger_\alpha). \quad \text{(A2)}$$

Using (A1), we can deduce the equation of motion for arbitrary n-point expectation values. Since the hierarchy is based on the correlations among lattice sites, we need in addition the relation between n-point correlators and n-point expectation values. Up to first order in $1/Z$, we have for $\mu \neq \nu$ the two-point correlations

$$\langle \hat{c}^\dagger_\mu \hat{c}_\nu \rangle^{\mathrm{corr}} = \int_{\mathbf{k}} f^{\mathrm{corr}}_{\mathbf{k}} e^{i\mathbf{k}\cdot(\mathbf{x}_\mu - \mathbf{x}_\nu)} \quad \text{(A3)}$$

and the particle-number correlations $\langle \hat{n}_\mu \hat{n}_\nu \rangle^{\mathrm{corr}} = \langle \hat{n}_\mu \hat{n}_\nu \rangle - \langle \hat{n}_\mu \rangle \langle \hat{n}_\nu \rangle$ which will be omitted in the following since they do not contribute to the Boltzmann collision terms in leading order. The relevant three-point correlators in second order of the hierarchical expansion are given for $\alpha \neq \mu \neq \nu$ by

$$\langle \hat{n}_\alpha \hat{c}^\dagger_\mu \hat{c}_\nu \rangle^{\mathrm{corr}} = \langle \hat{n}_\alpha \hat{c}^\dagger_\mu \hat{c}_\nu \rangle - \langle \hat{n}_\alpha \rangle \langle \hat{c}^\dagger_\mu \hat{c}_\nu \rangle^{\mathrm{corr}} \quad \text{(A4)}$$

and have the Fourier decomposition

$$\langle \hat{n}_\alpha \hat{c}^\dagger_\mu \hat{c}_\nu \rangle^{\mathrm{corr}} = \int_{\mathbf{p}_1,\mathbf{p}_2} g_{\mathbf{p}_1,\mathbf{p}_2} e^{i\mathbf{p}_1\cdot(\mathbf{x}_\mu - \mathbf{x}_\alpha) + i\mathbf{p}_2\cdot(\mathbf{x}_\nu - \mathbf{x}_\alpha)}. \quad \text{(A5)}$$

Furthermore, the 4-point correlators for $\alpha \neq \beta \neq \mu \neq \nu$ are defined as

$$\begin{aligned}\langle \hat{c}^\dagger_\alpha \hat{c}_\beta \hat{c}^\dagger_\mu \hat{c}_\nu \rangle^{\mathrm{corr}} = {} & \langle \hat{c}^\dagger_\alpha \hat{c}_\beta \hat{c}^\dagger_\mu \hat{c}_\nu \rangle - \langle \hat{c}^\dagger_\alpha \hat{c}_\beta \rangle^{\mathrm{corr}} \langle \hat{c}^\dagger_\mu \hat{c}_\nu \rangle^{\mathrm{corr}} \\ & + \langle \hat{c}^\dagger_\alpha \hat{c}_\nu \rangle^{\mathrm{corr}} \langle \hat{c}^\dagger_\mu \hat{c}_\beta \rangle^{\mathrm{corr}} \end{aligned} \quad \text{(A6)}$$

and we define their Fourier components via

$$\begin{aligned}\langle \hat{c}^\dagger_\alpha \hat{c}_\beta \hat{c}^\dagger_\mu \hat{c}_\nu \rangle^{\mathrm{corr}} = {} & \int_{\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3} h_{\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3} \\ & \times e^{i\mathbf{p}_1\cdot(\mathbf{x}_\alpha - \mathbf{x}_\nu) + i\mathbf{p}_2\cdot(\mathbf{x}_\beta - \mathbf{x}_\nu) + i\mathbf{p}_3\cdot(\mathbf{x}_\mu - \mathbf{x}_\nu)}. \end{aligned} \quad \text{(A7)}$$

APPENDIX B: HOMOGENEOUS LATTICE

It is instructive to derive with the hierarchical method the well-known Boltzmann equations for a homogeneous lattice at half filling, see Eq. (6). The homogeneity of the fermion distribution enforces time-independence of the on-site occupation number which translates to the zeroth-order equation $\partial_t \langle \hat{n}_\mu \rangle = 0$. The two-point correlators remain constant in order $1/Z$ but their equations of motion have an inhomogeneity of order $1/Z^2$ which is determined by the three-point

correlators,

$$i\partial_t \langle \hat{c}_\mu^\dagger \hat{c}_\nu \rangle^{\rm corr} = S^{(2)}_{\mu\nu} = -\frac{1}{Z}\sum_\alpha (V_{\alpha\mu} - V_{\alpha\nu}) \langle \hat{n}_\alpha \hat{c}_\mu^\dagger \hat{c}_\nu \rangle^{\rm corr}, \tag{B1}$$

or, translated to Fourier space,

$$i\partial_t f^{\rm corr}_{\mathbf{k}} = S^{(2)}_{\mathbf{k}} = -\int_{\mathbf{q}} V_{\mathbf{k}+\mathbf{q}} (g_{\mathbf{q},\mathbf{k}} - g_{\mathbf{k},\mathbf{q}}). \tag{B2}$$

From the hierarchy of correlations follows the evolution equation for the three-point correlators which contains the two-point correlator $\langle \hat{c}_\mu^\dagger \hat{c}_\nu \rangle^{\rm corr}$ and the particle number correlator $\langle \hat{n}_\mu \hat{n}_\nu \rangle^{\rm corr}$. As mentioned in the previous section, the latter do not contribute to Boltzmann collisions terms in order $1/Z^3$. We find

$$\begin{aligned} &i\partial_t \langle \hat{n}_\alpha \hat{c}_\mu^\dagger \hat{c}_\nu \rangle^{\rm corr} \\ &\quad = \frac{1}{Z}\sum_\gamma J_{\gamma\mu} \langle \hat{n}_\alpha \hat{c}_\gamma^\dagger \hat{c}_\nu \rangle^{\rm corr} \\ &\quad\quad - \frac{1}{Z}\sum_\gamma J_{\gamma\nu} \langle \hat{n}_\alpha \hat{c}_\mu^\dagger \hat{c}_\gamma \rangle^{\rm corr} + S^{(3)}_{\alpha\mu\nu,1/Z^2} + S^{(3)}_{\alpha\mu\nu,1/Z^3} \end{aligned} \tag{B3}$$

with the source terms

$$\begin{aligned} S^{(3)}_{\alpha\mu\nu,1/Z^2} = &\frac{1}{Z}\sum_\gamma J_{\gamma\alpha} [\langle \hat{c}_\alpha^\dagger \hat{c}_\nu \rangle^{\rm corr} \langle \hat{c}_\mu^\dagger \hat{c}_\gamma \rangle^{\rm corr} \\ &- \langle \hat{c}_\gamma^\dagger \hat{c}_\nu \rangle^{\rm corr} \langle \hat{c}_\mu^\dagger \hat{c}_\alpha \rangle^{\rm corr}] \\ &- \frac{1}{4}\left(\frac{V_{\mu\alpha}}{Z} - \frac{V_{\nu\alpha}}{Z}\right) \langle \hat{c}_\mu^\dagger \hat{c}_\nu \rangle^{\rm corr} + \dots \end{aligned} \tag{B4}$$

and

$$S^{(3)}_{\alpha\mu\nu,1/Z^3} = \frac{1}{Z}\sum_\gamma J_{\gamma\alpha} [\langle \hat{c}_\gamma^\dagger \hat{c}_\alpha \hat{c}_\mu^\dagger \hat{c}_\nu \rangle^{\rm corr} - \langle \hat{c}_\alpha^\dagger \hat{c}_\gamma \hat{c}_\mu^\dagger \hat{c}_\nu \rangle^{\rm corr}] + \dots \tag{B5}$$

In (B4), we suppressed the particle-number correlations and in (B5) we suppressed all terms except the four-point correlators. As will be shown below, the latter are relevant for the Boltzmann dynamics in leading order. After the Fourier transformation of (B3)–(B5), we obtain

$$i\partial_t g_{\mathbf{q},\mathbf{k}} = (J_{\mathbf{q}} - J_{\mathbf{k}}) g_{\mathbf{q},\mathbf{k}} + S^{(3)}_{\mathbf{q},\mathbf{k},1/Z^2} + S^{(3)}_{\mathbf{q},\mathbf{k},1/Z^3} \tag{B6}$$

with

$$S^{(3)}_{\mathbf{q},\mathbf{k},1/Z^2} = (J_{\mathbf{q}} - J_{\mathbf{k}}) f^{\rm corr}_{\mathbf{q}} f^{\rm corr}_{\mathbf{k}} - \tfrac{1}{4} V_{\mathbf{q}+\mathbf{k}} \left(f^{\rm corr}_{\mathbf{k}} - f^{\rm corr}_{\mathbf{q}}\right) \tag{B7}$$

and

$$S^{(3)}_{\mathbf{q},\mathbf{k},1/Z^3} = \int_{\mathbf{p}} (J_{\mathbf{p}} - J_{\mathbf{k}+\mathbf{q}+\mathbf{p}}) h_{\mathbf{p},-\mathbf{k}-\mathbf{q}-\mathbf{p},\mathbf{q}}. \tag{B8}$$

We integrate the evolution equation (B6) within the Markov approximation and obtain

$$g_{\mathbf{q},\mathbf{k}} = \frac{i\left(S^{(3)}_{\mathbf{q},\mathbf{k},1/Z^2} + S^{(3)}_{\mathbf{q},\mathbf{k},1/Z^3}\right)}{i(J_{\mathbf{k}} - J_{\mathbf{q}}) - \epsilon}. \tag{B9}$$

Finally, we have to consider the dynamics of the four-point correlators which is given in real space by

$$\begin{aligned} &i\partial_t \langle \hat{c}_\alpha^\dagger \hat{c}_\beta \hat{c}_\mu^\dagger \hat{c}_\nu \rangle^{\rm corr} \\ &\quad = \frac{1}{Z}\sum_\gamma J_{\gamma\alpha} \langle \hat{c}_\gamma^\dagger \hat{c}_\beta \hat{c}_\mu^\dagger \hat{c}_\nu \rangle^{\rm corr} - \frac{1}{Z}\sum_\gamma J_{\gamma\beta} \langle \hat{c}_\alpha^\dagger \hat{c}_\gamma \hat{c}_\mu^\dagger \hat{c}_\nu \rangle^{\rm corr} \\ &\quad\quad + \frac{1}{Z}\sum_\gamma J_{\gamma\mu} \langle \hat{c}_\alpha^\dagger \hat{c}_\beta \hat{c}_\gamma^\dagger \hat{c}_\nu \rangle^{\rm corr} \\ &\quad\quad - \frac{1}{Z}\sum_\gamma J_{\gamma\nu} \langle \hat{c}_\alpha^\dagger \hat{c}_\beta \hat{c}_\mu^\dagger \hat{c}_\gamma \rangle^{\rm corr} + S^{(4)}_{\alpha\beta\mu\nu,1/Z^3} + \mathcal{O}(1/Z^4) \end{aligned} \tag{B10}$$

with the source term

$$\begin{aligned} S^{(4)}_{\alpha\beta\mu\nu,1/Z^3} = &\frac{J_{\alpha\beta}}{Z} [\langle \hat{n}_\beta \hat{c}_\mu^\dagger \hat{c}_\nu \rangle^{\rm corr} - \langle \hat{n}_\alpha \hat{c}_\mu^\dagger \hat{c}_\nu \rangle^{\rm corr} + \langle \hat{c}_\mu^\dagger \hat{c}_\beta \rangle^{\rm corr} \langle \hat{c}_\beta^\dagger \hat{c}_\nu \rangle^{\rm corr} - \langle \hat{c}_\mu^\dagger \hat{c}_\alpha \rangle^{\rm corr} \langle \hat{c}_\alpha^\dagger \hat{c}_\nu \rangle^{\rm corr}] \\ &+ \frac{J_{\alpha\nu}}{Z} [\langle \hat{n}_\alpha \hat{c}_\mu^\dagger \hat{c}_\beta \rangle^{\rm corr} - \langle \hat{n}_\nu \hat{c}_\mu^\dagger \hat{c}_\beta \rangle^{\rm corr} + \langle \hat{c}_\mu^\dagger \hat{c}_\alpha \rangle^{\rm corr} \langle \hat{c}_\alpha^\dagger \hat{c}_\beta \rangle^{\rm corr} - \langle \hat{c}_\mu^\dagger \hat{c}_\nu \rangle^{\rm corr} \langle \hat{c}_\nu^\dagger \hat{c}_\beta \rangle^{\rm corr}] \\ &+ \frac{J_{\beta\mu}}{Z} [\langle n_\mu \hat{c}_\alpha^\dagger \hat{c}_\nu \rangle^{\rm corr} - \langle \hat{n}_\beta \hat{c}_\alpha^\dagger \hat{c}_\nu \rangle^{\rm corr} + \langle \hat{c}_\alpha^\dagger \hat{c}_\mu \rangle^{\rm corr} \langle \hat{c}_\mu^\dagger \hat{c}_\nu \rangle^{\rm corr} - \langle \hat{c}_\alpha^\dagger \hat{c}_\beta \rangle^{\rm corr} \langle \hat{c}_\beta^\dagger \hat{c}_\nu \rangle^{\rm corr}] \\ &+ \frac{J_{\mu\nu}}{Z} [\langle \hat{n}_\nu \hat{c}_\alpha^\dagger \hat{c}_\beta \rangle^{\rm corr} - \langle \hat{n}_\mu \hat{c}_\alpha^\dagger \hat{c}_\beta \rangle^{\rm corr} + \langle \hat{c}_\alpha^\dagger \hat{c}_\nu \rangle^{\rm corr} \langle \hat{c}_\nu^\dagger \hat{c}_\beta \rangle^{\rm corr} - \langle \hat{c}_\alpha^\dagger \hat{c}_\mu \rangle^{\rm corr} \langle \hat{c}_\mu^\dagger \hat{c}_\beta \rangle^{\rm corr}] \\ &- \frac{1}{Z}\sum_\gamma (V_{\alpha\gamma} - V_{\beta\gamma}) \langle \hat{n}_\gamma \hat{c}_\mu^\dagger \hat{c}_\nu \rangle^{\rm corr} \langle \hat{c}_\alpha^\dagger \hat{c}_\beta \rangle^{\rm corr} - \frac{1}{Z}\sum_\gamma (V_{\nu\gamma} - V_{\alpha\gamma}) \langle \hat{n}_\gamma \hat{c}_\mu^\dagger \hat{c}_\beta \rangle^{\rm corr} \langle \hat{c}_\alpha^\dagger \hat{c}_\nu \rangle^{\rm corr} \\ &- \frac{1}{Z}\sum_\gamma (V_{\beta\gamma} - V_{\mu\gamma}) \langle \hat{n}_\gamma \hat{c}_\alpha^\dagger \hat{c}_\nu \rangle^{\rm corr} \langle \hat{c}_\mu^\dagger \hat{c}_\beta \rangle^{\rm corr} - \frac{1}{Z}\sum_\gamma (V_{\mu\gamma} - V_{\nu\gamma}) \langle \hat{n}_\gamma \hat{c}_\alpha^\dagger \hat{c}_\beta \rangle^{\rm corr} \langle \hat{c}_\mu^\dagger \hat{c}_\nu \rangle^{\rm corr}. \end{aligned} \tag{B11}$$

The dynamics of the Fourier components is then governed through

$$i\partial_t h_{\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3} = (J_{\mathbf{p}_1} - J_{\mathbf{p}_2} + J_{\mathbf{p}_3} - J_{\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3}) h_{\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3} + S^{(4)}_{\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,1/Z^3} \tag{B12}$$

with

$$S^{(4)}_{\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,1/Z^3} = (J_{\mathbf{p}_1} - J_{\mathbf{p}_2}) f^{\mathrm{corr}}_{\mathbf{p}_3} f^{\mathrm{corr}}_{\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3} + \left[J_{\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3} - J_{\mathbf{p}_1} - V_{\mathbf{p}_2+\mathbf{p}_3}\left(f^{\mathrm{corr}}_{\mathbf{p}_1} - f^{\mathrm{corr}}_{\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3}\right)\right] g_{\mathbf{p}_3,\mathbf{p}_2}$$
$$- (\mathbf{p}_1 \leftrightarrow \mathbf{p}_3) + (\mathbf{p}_1 \leftrightarrow \mathbf{p}_3, \mathbf{p}_2 \leftrightarrow -\mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3) - (\mathbf{p}_2 \leftrightarrow -\mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3)\,. \tag{B13}$$

After solving (B12) within Markov approximation and plugging the result back into (B9), the evolution equation (B2) takes the form

$$i\partial_t f^{\mathrm{corr}}_{\mathbf{k}} = -\int_{\mathbf{q}} \frac{iV_{\mathbf{k}+\mathbf{q}}}{i(J_{\mathbf{k}} - J_{\mathbf{q}}) - \epsilon}\left[S^{(3)}_{\mathbf{q},\mathbf{k},1/Z^2} + \int_{\mathbf{p}} \frac{(J_{\mathbf{p}} - J_{\mathbf{k}+\mathbf{q}+\mathbf{p}}) iS^{(4)}_{\mathbf{p},-\mathbf{k}-\mathbf{q}-\mathbf{p},\mathbf{q},1/Z^3}}{i(J_{\mathbf{k}} - J_{\mathbf{q}} + J_{\mathbf{k}+\mathbf{q}+\mathbf{p}} - J_{\mathbf{p}}) - \epsilon}\right] - \mathrm{c.c.} \tag{B14}$$

After some algebra and using the identity $\pi\delta(x) = \lim_{\epsilon\to 0} \epsilon/(\epsilon^2 + x^2)$, we find in the continuum limit from (B14) the Boltzmann dynamics

$$\partial_t f_{\mathbf{k}} = -2\pi \int_{\mathbf{q},\mathbf{p}} \delta(J_{\mathbf{k}} + J_{\mathbf{p}} - J_{\mathbf{k}-\mathbf{q}} - J_{\mathbf{p}+\mathbf{q}}) V_{\mathbf{q}}(V_{\mathbf{q}} - V_{\mathbf{k}-\mathbf{p}-\mathbf{q}})[f_{\mathbf{k}} f_{\mathbf{p}}(1 - f_{\mathbf{k}-\mathbf{q}})(1 - f_{\mathbf{p}+\mathbf{q}}) - f_{\mathbf{k}-\mathbf{q}} f_{\mathbf{p}+\mathbf{q}}(1 - f_{\mathbf{k}})(1 - f_{\mathbf{p}})]\,, \tag{B15}$$

where we introduced the electron distribution functions $f_{\mathbf{k}}$ which are the Fourier components of $\langle \hat{c}^\dagger_\mu \hat{c}_\nu\rangle = \langle \hat{c}^\dagger_\mu \hat{c}_\nu\rangle^{\mathrm{corr}} + \delta_{\mu\nu}\langle \hat{n}_\mu\rangle$, i.e., $f_{\mathbf{k}} = 1/2 + f^{\mathrm{corr}}_{\mathbf{k}}$. Finally, we want to remark that in the evaluation of (B14) all terms which do not contribute to the collision terms cancel each other. In order to see this, it is necessary to include beside the particle-number correlators also the four-point correlators $\langle \hat{n}_\alpha \hat{n}_\beta \hat{c}^\dagger_\mu \hat{c}_\nu\rangle^{\mathrm{corr}}$ (which were not considered in the calculation above) and several local terms which ensure that the correlators vanish identically if two or more lattice sites are equal.

APPENDIX C: CHARGE-DENSITY WAVE

1. Single-site evolution

We consider a bipartite lattice at half filling such that the fermion densities add up to unity, $n^{\mathcal{A}} + n^{\mathcal{B}} = 1$. For labeling the sublattice we use the capital superscripts such as $X \in \{\mathcal{A}, \mathcal{B}\}$. The time evolution of the on-site occupation number is given by

$$i\partial_t \langle \hat{n}_\mu\rangle = \frac{1}{Z}\sum_\alpha J_{\alpha\mu}[\langle \hat{c}^\dagger_\alpha \hat{c}_\mu\rangle^{\mathrm{corr}} - \langle \hat{c}^\dagger_\mu \hat{c}_\alpha\rangle^{\mathrm{corr}}] \tag{C1}$$

which translates after a Fourier transformation to

$$i\partial_t n^X = \int_{\mathbf{q}} J_{\mathbf{q}}\left[f^{\mathrm{corr},\bar{X}X}_{\mathbf{q}} - f^{\mathrm{corr},X\bar{X}}_{\mathbf{q}}\right]. \tag{C2}$$

The superscript $\bar{X}$ denotes the sublattice opposite to X.

2. Quasiparticle and hole distribution functions.

For the two-point correlations, we generalize the evolution equation (B1) for the charge density background and find

$$i\partial_t \langle \hat{c}^\dagger_\mu \hat{c}_\nu\rangle^{\mathrm{corr}} = \frac{1}{Z}\sum_\alpha J_{\alpha\mu}\langle \hat{c}^\dagger_\alpha \hat{c}_\nu\rangle^{\mathrm{corr}} - \frac{1}{Z}\sum_\alpha J_{\alpha\nu}\langle \hat{c}^\dagger_\mu \hat{c}_\alpha\rangle^{\mathrm{corr}} - \frac{1}{Z}\sum_\alpha (V_{\alpha\mu} - V_{\alpha\nu})\langle \hat{n}_\alpha\rangle\langle \hat{c}^\dagger_\mu \hat{c}_\nu\rangle^{\mathrm{corr}} + S_{\mu\nu,1/Z} + S_{\mu\nu,1/Z^2} \tag{C3}$$

where we separated the source terms according to their order $1/Z$,

$$S_{\mu\nu,1/Z} = \frac{J_{\mu\nu}}{Z}(\langle \hat{n}_\nu\rangle - \langle \hat{n}_\mu\rangle) - \delta_{\mu\nu}\frac{1}{Z}\sum_\alpha J_{\alpha\mu}[\langle \hat{c}^\dagger_\alpha \hat{c}_\mu\rangle^{\mathrm{corr}} - \langle \hat{c}^\dagger_\mu \hat{c}_\alpha\rangle^{\mathrm{corr}}]\,, \tag{C4}$$

$$S_{\mu\nu,1/Z^2} = -\frac{1}{Z}\sum_\alpha (V_{\alpha\mu} - V_{\alpha\nu})\langle \hat{n}_\alpha \hat{c}^\dagger_\mu \hat{c}_\nu\rangle^{\mathrm{corr}} - \frac{1}{Z}V_{\mu\nu}(\langle \hat{n}_\mu\rangle - \langle \hat{n}_\nu\rangle)\langle \hat{c}^\dagger_\mu \hat{c}_\nu\rangle^{\mathrm{corr}}\,. \tag{C5}$$

The second term in (C4) was added such that the evolution equation (C3) is also valid for $\mu = \nu$ and the Fourier summation can be performed over all lattice sites. From (C3) we find for the evolution of the Fourier components

$$i\partial_t\left(f^{\mathrm{corr},XY}_{\mathbf{k}} + \delta^{XY} n^X\right) = J_{\mathbf{k}}\left(f^{\mathrm{corr},\bar{X}Y}_{\mathbf{k}} - f^{\mathrm{corr},X\bar{Y}}_{\mathbf{k}}\right) - (V^{\bar{X}} - V^{\bar{Y}}) f^{\mathrm{corr},XY}_{\mathbf{k}} + S^{XY}_{\mathbf{k},1/Z} + S^{XY}_{\mathbf{k},1/Z^2}\,, \tag{C6}$$

where Eq. (C2) was used. We can rewrite (C6) using the variables $f^{XY}_{\mathbf{k}} = f^{\mathrm{corr},XY}_{\mathbf{k}} + \delta^{XY} n^X$ which are the Fourier components of the two-site expectation value $\langle \hat{c}^\dagger_\mu \hat{c}_\nu\rangle$,

$$i\partial_t f^{XY}_{\mathbf{k}} = J_{\mathbf{k}}\left(f^{\bar{X}Y}_{\mathbf{k}} - f^{X\bar{Y}}_{\mathbf{k}}\right) - (V^{\bar{X}} - V^{\bar{Y}}) f^{XY}_{\mathbf{k}} + S^{XY}_{\mathbf{k},1/Z} + S^{XY}_{\mathbf{k},1/Z^2}\,. \tag{C7}$$

The relation (C7) can be diagonalized via a rotation in the X-Y subspace by means of $f^{ab}_{\mathbf{k}} = \sum_{XY} O^a_X(\mathbf{k}) O^b_Y(\mathbf{k}) f^{XY}_{\mathbf{k}}$ with the momentum-dependent rotation matrix

$$O^a_X(\mathbf{k}) = \begin{pmatrix} \cos\alpha_{\mathbf{k}} & \sin\alpha_{\mathbf{k}} \\ -\sin\alpha_{\mathbf{k}} & \cos\alpha_{\mathbf{k}} \end{pmatrix}. \tag{C8}$$

The entries of this matrix are given by

$$\cos\alpha_{\mathbf{k}} = \frac{J_{\mathbf{k}}}{|J_{\mathbf{k}}|} \frac{\sqrt{\omega_{\mathbf{k}} + (V^{\mathcal{A}} - V^{\mathcal{B}})}}{\sqrt{2\omega_{\mathbf{k}}}} \tag{C9}$$

and

$$\sin\alpha_{\mathbf{k}} = \frac{\sqrt{\omega_{\mathbf{k}} - (V^{\mathcal{A}} - V^{\mathcal{B}})}}{\sqrt{2\omega_{\mathbf{k}}}} \tag{C10}$$

with the eigenfrequency $\omega_{\mathbf{k}} = \sqrt{(V^{\mathcal{A}} - V^{\mathcal{B}})^2 + 4J^2_{\mathbf{k}}}$. For a slowly varying charge-density background we can assume $[\partial_t, O^a_X(\mathbf{k})] \approx 0$ such that the diagonalization of (C7) leads to

$$i\partial_t f^{ab}_{\mathbf{k}} = \big(-E^a_{\mathbf{k}} + E^b_{\mathbf{k}}\big) f^{ab}_{\mathbf{k}} + S^{ab}_{\mathbf{k},1/Z} + S^{ab}_{\mathbf{k},1/Z^2} \tag{C11}$$

with the quasiparticle ($a = +$) and hole ($a = -$) energies $E^{\pm}_{\mathbf{k}} = [V \pm \omega_{\mathbf{k}}]/2$. For $a = b$, the variables are the distribution functions for quasiparticles and holes, namely,

$$f^{aa}_{\mathbf{k}} = f^{\mathrm{corr},aa}_{\mathbf{k}} + \sum_X O^a_X(\mathbf{k}) O^a_X(\mathbf{k}) n^X \equiv f^a_{\mathbf{k}}\,. \tag{C12}$$

For the slowly varying distribution functions $f^a_{\mathbf{k}}$, the $1/Z$ source term in (C11) is vanishing. Thus their time evolution is governed by terms which are at least of order $1/Z^2$:

$$i\partial_t f^a_{\mathbf{k}} = S^{aa}_{\mathbf{k},1/Z^2}\,. \tag{C13}$$

There are two important identities which are useful for the transformation from the sublattice space to the particle-hole space. The first one is the inversion of Eq. (C12)

$$f^{\mathrm{corr},XY}_{\mathbf{k}} = \sum_a O^a_X(\mathbf{k}) O^a_X(\mathbf{k}) f^a_{\mathbf{k}} - \delta^{XY} n^X + \mathcal{O}(1/Z^2)\,. \tag{C14}$$

which can be derived from the fact that the off-diagonal correlations approach their prethermalized value to lowest order, i.e.,

$$f^{\mathrm{corr},a\bar{a}}_{\mathbf{k}} = -\sum_X O^X_a(\mathbf{k}) O^X_{\bar{a}}(\mathbf{k}) n^X + \mathcal{O}(1/Z^2)\,. \tag{C15}$$

The second identity is the eigenvalue equation for rotation matrix

$$J_{\mathbf{k}} O^X_a(\mathbf{k}) = \big(-E^a_{\mathbf{k}} - V^{\bar{X}}\big) O^{\bar{X}}_a(\mathbf{k})\,. \tag{C16}$$

3. Three-point correlators

The Boltzmann collisions are contained in the $1/Z^2$ term in Eq. (C7) which have the form

$$S^{XY}_{\mathbf{k},1/Z^2} = -\int_{\mathbf{q}} V_{\mathbf{k}+\mathbf{q}} \big(g^{\bar{X}XY}_{\mathbf{q},\mathbf{k}} - g^{\bar{Y}XY}_{\mathbf{k},\mathbf{q}}\big)\,. \tag{C17}$$

Transforming this source term to particle-hole space, we find from (C13) the generalization of (B2) to be

$$i\partial_t f^a_{\mathbf{k}} = -\int_{\mathbf{q}} \sum_{b,X} V_{\mathbf{k}+\mathbf{q}} \big(O^a_X(\mathbf{k}) O^b_X(\mathbf{q}) g^{\bar{X},ba}_{\mathbf{q},\mathbf{k}} - \mathrm{c.c.}\big)\,. \tag{C18}$$

Here we rotated the three-point correlations according to $g^{Z,ab}_{\mathbf{q},\mathbf{k}} = \sum_{X,Y} O^a_X(\mathbf{q}) O^b_Y(\mathbf{k}) g^{ZXY}_{\mathbf{q},\mathbf{k}}$. Their dynamics is determined by the real-space equation

$$i\partial_t \langle \hat{n}_\alpha \hat{c}^\dagger_\mu \hat{c}_\nu \rangle^{\mathrm{corr}} = \frac{1}{Z}\sum_\gamma J_{\gamma\mu} \langle \hat{n}_\alpha \hat{c}^\dagger_\gamma \hat{c}_\nu \rangle^{\mathrm{corr}} - \frac{1}{Z}\sum_\gamma J_{\gamma\nu} \langle \hat{n}_\alpha \hat{c}^\dagger_\mu \hat{c}_\gamma \rangle^{\mathrm{corr}} - \frac{1}{Z}\sum_\gamma (V_{\gamma\mu} - V_{\gamma\nu}) \langle \hat{n}_\gamma \rangle \langle \hat{n}_\alpha \hat{c}^\dagger_\mu \hat{c}_\nu \rangle^{\mathrm{corr}} + S_{\alpha\mu\nu,1/Z^2} + S_{\alpha\mu\nu,1/Z^3} \tag{C19}$$

which is a generalization of (B3). Again, the particle-number correlators can be omitted in the source terms since they contribute with terms that are $\mathcal{O}(1/Z^4)$ to the Boltzmann dynamics whereas we shall see that the leading order collision terms are $\mathcal{O}(1/Z^3)$.

Therefore we remain with

$$S_{\alpha\mu\nu,1/Z^2} = \frac{1}{Z}\sum_\gamma J_{\gamma\alpha}[\langle\hat{c}^\dagger_\alpha\hat{c}_\nu\rangle^{\rm corr}\langle\hat{c}^\dagger_\mu\hat{c}_\gamma\rangle^{\rm corr} - \langle\hat{c}^\dagger_\gamma\hat{c}_\nu\rangle^{\rm corr}\langle\hat{c}^\dagger_\mu\hat{c}_\alpha\rangle^{\rm corr}] - \left(\frac{V_{\mu\alpha}}{Z} - \frac{V_{\nu\alpha}}{Z}\right)\langle\hat{n}_\alpha\rangle(1-\langle\hat{n}_\alpha\rangle)\langle\hat{c}^\dagger_\mu\hat{c}_\nu\rangle^{\rm corr}$$
$$+ \frac{J_{\alpha\mu}}{Z}[(\langle\hat{n}_\mu\rangle - \langle\hat{n}_\alpha\rangle)\langle\hat{c}^\dagger_\alpha\hat{c}_\nu\rangle^{\rm corr}] - \frac{J_{\alpha\nu}}{Z}[(\langle\hat{n}_\nu\rangle - \langle\hat{n}_\alpha\rangle)\langle\hat{c}^\dagger_\mu\hat{c}_\alpha\rangle^{\rm corr}] + \ldots \tag{C20}$$

Within the source term $S_{\alpha\mu\nu,1/Z^3}$, only the four-point correlators are of interest,

$$S_{\alpha\mu\nu,1/Z^3} = \frac{1}{Z}\sum_\gamma J_{\gamma\alpha}[\langle\hat{c}^\dagger_\gamma\hat{c}_\alpha\hat{c}^\dagger_\mu\hat{c}_\nu\rangle^{\rm corr} - \langle\hat{c}^\dagger_\alpha\hat{c}_\gamma\hat{c}^\dagger_\mu\hat{c}_\nu\rangle^{\rm corr}] + \ldots \tag{C21}$$

After Fourier transformation and rotation in sublattice space, we find from (C19)–(C21) the generalization of the evolution equation (B6), i.e.,

$$i\partial_t g^{Xab}_{\mathbf{q},\mathbf{k}} = \big(-E^a_{\mathbf{q}} + E^b_{\mathbf{k}}\big)g^{Xab}_{\mathbf{q},\mathbf{k}} + S^{Xab}_{\mathbf{q},\mathbf{k},1/Z^2} + S^{Xab}_{\mathbf{q},\mathbf{k},1/Z^3} \tag{C22}$$

with

$$S^{Xab}_{\mathbf{q},\mathbf{k},1/Z^2} = \big(E^a_{\mathbf{q}} - E^b_{\mathbf{k}}\big)O^a_X(\mathbf{q})O^b_X(\mathbf{k})\big[-(n^X)^2 + n^X\big(f^a_{\mathbf{q}} + f^b_{\mathbf{k}}\big) - f^a_{\mathbf{q}}f^b_{\mathbf{k}}\big] - V_{\mathbf{q}+\mathbf{k}}O^a_{\bar{X}}(\mathbf{q})O^b_{\bar{X}}(\mathbf{k})(n^X-1)n^X\big(f^a_{\mathbf{q}} - f^b_{\mathbf{k}}\big) \tag{C23}$$

and

$$S^{Xab}_{\mathbf{q},\mathbf{k},1/Z^3} = \int_{\mathbf{p}}\sum_{c,d}\big(E^d_{\mathbf{k}+\mathbf{q}+\mathbf{p}} - E^c_{\mathbf{p}}\big)O^c_X(\mathbf{p})O^d_X(\mathbf{k}+\mathbf{q}+\mathbf{p})h^{cdab}_{\mathbf{p},-\mathbf{k}-\mathbf{q}-\mathbf{p},\mathbf{q}} \tag{C24}$$

where we rotated the four-point correlators according to

$$h^{abcd}_{\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3} = \sum_{XYVW} O^a_X(\mathbf{p}_1)O^b_Y(\mathbf{p}_2)O^c_V(\mathbf{p}_3)O^d_W(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)h^{XYVW}_{\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3}\,. \tag{C25}$$

4. Four-point correlators

The dynamics of the Fourier components $h^{XYVW}_{\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3}$ can be deduced from a generalization of (B10), i.e.,

$$i\partial_t\langle\hat{c}^\dagger_\alpha\hat{c}_\beta\hat{c}^\dagger_\mu\hat{c}_\nu\rangle^{\rm corr} = \frac{1}{Z}\sum_\gamma J_{\gamma\alpha}\langle\hat{c}^\dagger_\gamma\hat{c}_\beta\hat{c}^\dagger_\mu\hat{c}_\nu\rangle^{\rm corr} - \frac{1}{Z}\sum_\gamma J_{\gamma\beta}\langle\hat{c}^\dagger_\alpha\hat{c}_\gamma\hat{c}^\dagger_\mu\hat{c}_\nu\rangle^{\rm corr}$$
$$+ \frac{1}{Z}\sum_\gamma J_{\gamma\mu}\langle\hat{c}^\dagger_\alpha\hat{c}_\beta\hat{c}^\dagger_\gamma\hat{c}_\nu\rangle^{\rm corr} - \frac{1}{Z}\sum_\gamma J_{\gamma\nu}\langle\hat{c}^\dagger_\alpha\hat{c}_\beta\hat{c}^\dagger_\mu\hat{c}_\gamma\rangle^{\rm corr}$$
$$- \frac{1}{Z}\sum_\gamma(V_{\alpha\gamma} - V_{\beta\gamma} + V_{\mu\gamma} - V_{\nu\gamma})\langle\hat{n}_\gamma\rangle\langle\hat{c}^\dagger_\alpha\hat{c}_\beta\hat{c}^\dagger_\mu\hat{c}_\nu\rangle^{\rm corr} + S_{\alpha\beta\mu\nu,1/Z^3} + \mathcal{O}(1/Z^4)\,. \tag{C26}$$

The inhomogeneity

$$S_{\alpha\beta\mu\nu,1/Z^3} = \frac{J_{\alpha\beta}}{Z}[\langle\hat{n}_\beta\hat{c}^\dagger_\mu\hat{c}_\nu\rangle^{\rm corr} - \langle\hat{n}_\alpha\hat{c}^\dagger_\mu\hat{c}_\nu\rangle^{\rm corr} + \langle\hat{c}^\dagger_\mu\hat{c}_\beta\rangle^{\rm corr}\langle\hat{c}^\dagger_\beta\hat{c}_\nu\rangle^{\rm corr} - \langle\hat{c}^\dagger_\mu\hat{c}_\alpha\rangle^{\rm corr}\langle\hat{c}^\dagger_\alpha\hat{c}_\nu\rangle^{\rm corr}]$$
$$+ \frac{J_{\alpha\nu}}{Z}[\langle\hat{n}_\alpha\hat{c}^\dagger_\mu\hat{c}_\beta\rangle^{\rm corr} - \langle\hat{n}_\nu\hat{c}^\dagger_\mu\hat{c}_\beta\rangle^{\rm corr} + \langle\hat{c}^\dagger_\mu\hat{c}_\alpha\rangle^{\rm corr}\langle\hat{c}^\dagger_\alpha\hat{c}_\beta\rangle^{\rm corr} - \langle\hat{c}^\dagger_\mu\hat{c}_\nu\rangle^{\rm corr}\langle\hat{c}^\dagger_\nu\hat{c}_\beta\rangle^{\rm corr}]$$
$$+ \frac{J_{\beta\mu}}{Z}[\langle n_\mu\hat{c}^\dagger_\alpha\hat{c}_\nu\rangle^{\rm corr} - \langle\hat{n}_\beta\hat{c}^\dagger_\alpha\hat{c}_\nu\rangle^{\rm corr} + \langle\hat{c}^\dagger_\alpha\hat{c}_\mu\rangle^{\rm corr}\langle\hat{c}^\dagger_\mu\hat{c}_\nu\rangle^{\rm corr} - \langle\hat{c}^\dagger_\alpha\hat{c}_\beta\rangle^{\rm corr}\langle\hat{c}^\dagger_\beta\hat{c}_\nu\rangle^{\rm corr}]$$
$$+ \frac{J_{\mu\nu}}{Z}[\langle\hat{n}_\nu\hat{c}^\dagger_\alpha\hat{c}_\beta\rangle^{\rm corr} - \langle\hat{n}_\mu\hat{c}^\dagger_\alpha\hat{c}_\beta\rangle^{\rm corr} + \langle\hat{c}^\dagger_\alpha\hat{c}_\nu\rangle^{\rm corr}\langle\hat{c}^\dagger_\nu\hat{c}_\beta\rangle^{\rm corr} - \langle\hat{c}^\dagger_\alpha\hat{c}_\mu\rangle^{\rm corr}\langle\hat{c}^\dagger_\mu\hat{c}_\beta\rangle^{\rm corr}]$$
$$- \frac{1}{Z}\sum_\gamma(V_{\alpha\gamma} - V_{\beta\gamma})\langle\hat{n}_\gamma\hat{c}^\dagger_\mu\hat{c}_\nu\rangle^{\rm corr}\langle\hat{c}^\dagger_\alpha\hat{c}_\beta\rangle^{\rm corr} - \frac{1}{Z}\sum_\gamma(V_{\nu\gamma} - V_{\alpha\gamma})\langle\hat{n}_\gamma\hat{c}^\dagger_\mu\hat{c}_\beta\rangle^{\rm corr}\langle\hat{c}^\dagger_\alpha\hat{c}_\nu\rangle^{\rm corr}$$
$$- \frac{1}{Z}\sum_\gamma(V_{\beta\gamma} - V_{\mu\gamma})\langle\hat{n}_\gamma\hat{c}^\dagger_\alpha\hat{c}_\nu\rangle^{\rm corr}\langle\hat{c}^\dagger_\mu\hat{c}_\beta\rangle^{\rm corr} - \frac{1}{Z}\sum_\gamma(V_{\mu\gamma} - V_{\nu\gamma})\langle\hat{n}_\gamma\hat{c}^\dagger_\alpha\hat{c}_\beta\rangle^{\rm corr}\langle\hat{c}^\dagger_\mu\hat{c}_\nu\rangle^{\rm corr}$$
$$- \frac{V_{\alpha\beta}}{Z}(\langle\hat{n}_\beta\rangle - \langle\hat{n}_\alpha\rangle)\langle\hat{c}^\dagger_\alpha\hat{c}_\nu\rangle^{\rm corr}\langle\hat{c}^\dagger_\mu\hat{c}_\beta\rangle^{\rm corr} - \frac{V_{\mu\nu}}{Z}(\langle\hat{n}_\nu\rangle - \langle\hat{n}_\mu\rangle)\langle\hat{c}^\dagger_\alpha\hat{c}_\nu\rangle^{\rm corr}\langle\hat{c}^\dagger_\mu\hat{c}_\beta\rangle^{\rm corr}$$
$$- \frac{V_{\alpha\nu}}{Z}(\langle\hat{n}_\alpha\rangle - \langle\hat{n}_\nu\rangle)\langle\hat{c}^\dagger_\alpha\hat{c}_\beta\rangle^{\rm corr}\langle\hat{c}^\dagger_\mu\hat{c}_\nu\rangle^{\rm corr} - \frac{V_{\beta\mu}}{Z}(\langle\hat{n}_\mu\rangle - \langle\hat{n}_\beta\rangle)\langle\hat{c}^\dagger_\alpha\hat{c}_\beta\rangle^{\rm corr}\langle\hat{c}^\dagger_\mu\hat{c}_\nu\rangle^{\rm corr} \tag{C27}$$

contains additional terms compared to (B11) due to the presence of the charge density wave. A transformation of (C26) and (C27) to Fourier space and a subsequent rotation in sublattice space leads to

$$i\partial_t h^{abcd}_{\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3} = \big(-E^a_{\mathbf{p}_1} + E^b_{\mathbf{p}_2} - E^c_{\mathbf{p}_3} + E^d_{\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3}\big)h^{abcd}_{\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3} + S^{abcd}_{\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,1/Z^3} + \mathcal{O}(1/Z^4) \tag{C28}$$

with

$$\begin{aligned} S^{abcd}_{\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,1/Z^3} = {} & \sum_X O^a_X(\mathbf{p}_1)O^b_{\bar{X}}(\mathbf{p}_2)O^c_{\bar{X}}(\mathbf{p}_3)O^d_X(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)V_{\mathbf{p}_2+\mathbf{p}_3}(n^{\bar{X}}-n^X)\big[f^c_{\mathbf{p}_3}f^d_{\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3} - n^X f^c_{\mathbf{p}_3} - n^{\bar{X}} f^d_{\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3}\big] \\ & + \sum_X O^a_X(\mathbf{p}_1)O^b_X(\mathbf{p}_2)O^c_X(\mathbf{p}_3)O^d_X(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)\big[\big(E^b_{\mathbf{p}_2}-E^a_{\mathbf{p}_1}\big)f^c_{\mathbf{p}_3}f^d_{\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3} - n^X E^c_{\mathbf{p}_3}f^d_{\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3} + n^X E^d_{\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3}f^c_{\mathbf{p}_3}\big] \\ & + \sum_X g^{Xcb}_{\mathbf{p}_3,\mathbf{p}_2}\big[\big(E^a_{\mathbf{p}_1}-E^d_{\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3}\big)O^a_X(\mathbf{p}_1)O^d_X(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3) - V_{\mathbf{p}_2+\mathbf{p}_3}O^a_X(\mathbf{p}_1)O^d_X(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)\big(f^a_{\mathbf{p}_1}-n^X\big) \\ & + V_{\mathbf{p}_2+\mathbf{p}_3}O^a_{\bar{X}}(\mathbf{p}_1)O^d_{\bar{X}}(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)\big(f^d_{\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3}-n^{\bar{X}}\big)\big] \\ & - (\{a,\mathbf{p}_1\}\leftrightarrow\{c,\mathbf{p}_3\}) + (\{a,\mathbf{p}_1\}\leftrightarrow\{c,\mathbf{p}_3\},\{b,\mathbf{p}_2\}\leftrightarrow\{d,-\mathbf{p}_1-\mathbf{p}_2-\mathbf{p}_3\}) - (\{b,\mathbf{p}_2\}\leftrightarrow\{d,-\mathbf{p}_1-\mathbf{p}_2-\mathbf{p}_3\})\,. \end{aligned} \tag{C29}$$

5. Boltzmann dynamics

As in the previous section, the differential equations for the three-point correlators (C22) and the four-point correlators (C28) are solved within Markov approximation. When the resulting expressions are inserted into the evolution equation for the particle and hole distribution functions (C18), we find

$$i\partial_t f^a_{\mathbf{k}} = -\int_{\mathbf{q}}\sum_{X,b} V_{\mathbf{k}+\mathbf{q}}\frac{iO^b_X(\mathbf{q})O^a_X(\mathbf{k})}{i\big(E^b_{\mathbf{q}}-E^a_{\mathbf{k}}\big)-\epsilon}\left[S^{\bar{X},ba}_{\mathbf{q},\mathbf{k},1/Z^2} + \int_{\mathbf{p}}\sum_{c,d}\big(E^d_{\mathbf{k}+\mathbf{q}+\mathbf{p}}-E^c_{\mathbf{p}}\big)\frac{iO^c_X(\mathbf{p})O^d_X(\mathbf{k}+\mathbf{q}+\mathbf{p})S^{cdba}_{\mathbf{p},-\mathbf{k}-\mathbf{q}-\mathbf{p},\mathbf{q},1/Z^3}}{i\big(E^c_{\mathbf{p}}-E^d_{\mathbf{k}+\mathbf{q}+\mathbf{p}}+E^b_{\mathbf{q}}-E^a_{\mathbf{k}}\big)-\epsilon}\right] - \text{c.c.} \tag{C30}$$

After some algebra and taking the continuum limit, one can show that the Boltzmann equations take the form

$$\partial_t f^d_{\mathbf{k}} = -2\pi\int_{\mathbf{q},\mathbf{p}}\sum_{a,b,c} M^{abcd}_{\mathbf{p}+\mathbf{q},\mathbf{p},\mathbf{k}-\mathbf{q},\mathbf{k}}\delta\big(E^a_{\mathbf{p}+\mathbf{q}}-E^b_{\mathbf{p}}+E^c_{\mathbf{k}-\mathbf{q}}-E^d_{\mathbf{k}}\big)\big[f^d_{\mathbf{k}}f^b_{\mathbf{p}}\big(1-f^c_{\mathbf{k}-\mathbf{q}}\big)\big(1-f^a_{\mathbf{p}+\mathbf{q}}\big) - f^a_{\mathbf{p}+\mathbf{q}}f^c_{\mathbf{p}-\mathbf{q}}\big(1-f^d_{\mathbf{k}}\big)\big(1-f^b_{\mathbf{p}}\big)\big] \tag{C31}$$

with the transition matrix elements given by

$$\begin{aligned} M^{abcd}_{\mathbf{p}+\mathbf{q},\mathbf{p},\mathbf{k}-\mathbf{q},\mathbf{k}} = {} & \sum_{X,Y} V_{\mathbf{q}}O^a_X(\mathbf{p}+\mathbf{q})O^b_X(\mathbf{p})O^c_{\bar{X}}(\mathbf{k}-\mathbf{q})O^d_{\bar{X}}(\mathbf{k}) \\ & \times\big[V_{\mathbf{q}}O^a_Y(\mathbf{p}+\mathbf{q})O^b_Y(\mathbf{p})O^c_{\bar{Y}}(\mathbf{k}-\mathbf{q})O^d_{\bar{Y}}(\mathbf{k}) - V_{\mathbf{k}-\mathbf{p}-\mathbf{q}}O^a_Y(\mathbf{p}+\mathbf{q})O^b_{\bar{Y}}(\mathbf{p})O^c_{\bar{Y}}(\mathbf{k}-\mathbf{q})O^d_Y(\mathbf{k})\big]\,. \end{aligned} \tag{C32}$$

6. Charge density background

The collision dynamics has also an impact on the charge density background. We know from (C2) that the change of the local charge density is determined by the off-diagonal correlation functions $f^{\mathrm{corr},a\bar{a}}_{\mathbf{k}}$. From the relation (C11), we find that their dynamics is determined by the Boltzmann collisions of the particle and hole distribution functions $f^a_{\mathbf{k}}$. After some algebra one arrives at the result

$$\begin{aligned} \partial_t n^{\mathcal{A}} = {} & -\partial_t n^{\mathcal{B}} = -2\pi\int_{\mathbf{k},\mathbf{q},\mathbf{p}}\sum_{a,b,c,d} N^{abcd}_{\mathbf{p}+\mathbf{q},\mathbf{p},\mathbf{k}-\mathbf{q},\mathbf{k}}\delta\big(E^a_{\mathbf{p}+\mathbf{q}}-E^b_{\mathbf{p}}+E^c_{\mathbf{k}-\mathbf{q}}-E^d_{\mathbf{k}}\big) \\ & \times\big[f^d_{\mathbf{k}}f^b_{\mathbf{p}}\big(1-f^c_{\mathbf{k}-\mathbf{q}}\big)\big(1-f^a_{\mathbf{p}+\mathbf{q}}\big) - f^a_{\mathbf{p}+\mathbf{q}}f^c_{\mathbf{p}-\mathbf{q}}\big(1-f^d_{\mathbf{k}}\big)\big(1-f^b_{\mathbf{p}}\big)\big] \end{aligned} \tag{C33}$$

with

$$\begin{aligned} N^{abcd}_{\mathbf{p}+\mathbf{q},\mathbf{p},\mathbf{k}-\mathbf{q},\mathbf{k}} = {} & \frac{J_{\mathbf{k}}}{\omega_{\mathbf{k}}}\sum_{X,Y} V_{\mathbf{q}}O^a_X(\mathbf{p}+\mathbf{q})O^b_X(\mathbf{p})O^c_{\bar{X}}(\mathbf{k}-\mathbf{q})O^{\bar{d}}_{\bar{X}}(\mathbf{k}) \\ & \times\big[V_{\mathbf{q}}O^a_Y(\mathbf{p}+\mathbf{q})O^b_Y(\mathbf{p})O^c_{\bar{Y}}(\mathbf{k}-\mathbf{q})O^d_{\bar{Y}}(\mathbf{k}) - V_{\mathbf{k}-\mathbf{p}-\mathbf{q}}O^a_Y(\mathbf{p}+\mathbf{q})O^b_{\bar{Y}}(\mathbf{p})O^c_{\bar{Y}}(\mathbf{k}-\mathbf{q})O^d_Y(\mathbf{k})\big]\,. \end{aligned} \tag{C34}$$

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