Rapid Communications

Matrix formulation for non-Abelian families

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We generalize the K matrix formulation to nontrivial non-Abelian families of (2+1)-dimensional topological orders. Given a topological order $\mathcal{C}$, any topological order in the same non-Abelian family as $\mathcal{C}$ can be efficiently described by $\boldsymbol{a} = (a_I)$, where a_I are Abelian anyons in $\mathcal{C}$, together with a symmetric invertible matrix K, $K_{IJ} = k_{IJ} - t_{a_I,a_J}$, where k_{IJ} are integers, k_{II} are even, and t_{a_I,a_J} are the mutual statistics between a_I, a_J. In particular, when $\mathcal{C}$ is a root whose rank is the smallest in the family, K becomes an integer matrix. Our results make it possible to generate the data of large numbers of topological orders instantly, thus providing a large reservoir of potentially useful topological materials.

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Introduction. Topological phases of matter have drawn more and more research interest during recent years. A most remarkable feature of topological phases is that there can be several quantum states which are "topologically" degenerate. Such degeneracy is robust against any local perturbation, thus these states can be employed as qubits that are automatically immune to local noises. Given the possible application in quantum memory and quantum computation, it is then natural to ask how to produce the desired topological degeneracy.

One source of topological degeneracy is to put a topologically ordered system on a manifold with nontrivial topology [1–4]. This approach is not ideal: For one reason, it is not easy to shape a physical system into a nontrivial manifold such as a torus; for another, to manipulate the degenerate ground states one has to perform nonlocal operations.

Another source of topological degeneracy is to trap several anyonic quasiparticles. By braiding and fusion of these anyons, it is possible to realize universal topological quantum computation [5]. For an anyon i, we use the quantity d_i, called the quantum dimension, to measure the effective topological degeneracy carried by i. When there is a large number N of anyon i trapped, the topological degeneracy is of the order d_i^N.

Thus for anyons to produce a desired topological degeneracy, it is necessary that $d_i > 1$. An anyon with $d_i = 1$ is called Abelian while with $d_i > 1$ is called non-Abelian. If all the anyons in a topological order are Abelian, it is called an Abelian topological order. Abelian topological orders are useless in the braiding-fusion-based topological quantum computation, unless, e.g., one further employs more exotic boundaries and defects to enhance the computational capability [6–8], where the region bounded by boundaries or defects can be effectively viewed as a composite non-Abelian anyon.

In Ref. [9] we proposed the generalized hierarchy construction that can add or remove Abelian anyons to or from any topological order. Two topological orders which can be connected by such a construction are of the same "non-Abelian family," which is the equivalence class up to Abelian topological orders. The non-Abelian family captures the invariants of non-Abelian anyons, and we expect that topological orders in the same non-Abelian family behave similarly in topological quantum computation.

However, the construction in Ref. [9] is performed in a step-by-step manner. Given a topological order $\mathcal{C}$, it is not easy to calculate the property of another topological order in the same non-Abelian family that requires several steps of hierarchy constructions from $\mathcal{C}$. This Rapid Communication aims at resolving such difficulty. We showed that given a topological order $\mathcal{C}$, any topological order in the same non-Abelian family can be efficiently represented by a sequence of Abelian anyons in $\mathcal{C}$ together with a K matrix. When $\mathcal{C}$ is the trivial topological order, our result reduces to the original K matrix formulation for Abelian topological orders [10].

One-Step Generalized Hierarchy Construction. We first review and refine the construction proposed in Ref. [9]. The main idea is to let Abelian anyons form an effective Laughlin-like state [11]. This idea dates back to Haldane and Halperin, known as "hierarchy" construction [12,13]. But below we discuss it on a more general level.

We start with a topological phase $\mathcal{C}$. The anyons in $\mathcal{C}$ are labeled by $i, j, k, \ldots$. The most important data that characterize a topological order are the fusion rules N_k^{ij}, topological spin s_i, and topological T, S matrices. The fusion of anyons in $\mathcal{C}$ is given by

$$i \otimes j = \bigoplus_k N_k^{ij} k. \tag{1}$$

Quantum dimensions d_i are positive numbers satisfying

$$d_i d_j = \sum_k N_k^{ij} d_k. \tag{2}$$

By the Perron-Frobenius theorem, the positive solutions to the above is unique: d_i has to be the largest positive eigenvalue of matrix N_i with $(N_i)_{jk} = N_k^{ij}$.

The topological spin (or simply spin) s_i of anyon i is the fractional part of their angular momentum L^z: $s_i = \text{mod}(L_i^z, 1)$. s_i determines the self statistics of i: Exchanging two i anyons leads to the phase factor $e^{2\pi i s_i}$. The last piece of data to characterize topological orders is the chiral central charge c, which is the number of right-moving edge modes minus the number of left-moving edge modes.

T, S matrices can be calculated from (N_k^{ij}, s_i). In fact, (T, S) and (N_k^{ij}, s_i) determine each other,

$$T_{ij} = e^{2\pi i s_i}\delta_{ij}, \quad S_{ij} = \sum_k e^{2\pi i(s_i+s_j-s_k)}\frac{N_k^{ij}d_k}{D},$$
$$e^{2\pi i s_i} = T_{ii}, \quad N_k^{ij} = \sum_l \frac{S_{li}S_{lj}\overline{S_{lk}}}{S_{l\mathbf{1}}}. \tag{3}$$

where $D = \sqrt{\sum d_i^2}$ is the total quantum dimension. T, S matrices form a project representation of $SL(2, \mathbb{Z})$ where the projective phase factor is determined by the chiral central charge c,

$$S^4 = 1, \quad (TS)^3 = e^{2\pi i \frac{c}{8}} S^2. \tag{4}$$

Let a_c be an Abelian anyon in $\mathcal{C}$ with topological spin s_{a_c}. We try to make a_c form the Laughlin state on top of $\mathcal{C}$,

$$\langle\{z_a\}|\Psi\rangle = \prod_{a<b}(z_a - z_b)^{M_c} e^{-\frac{1}{4}\sum |z_a|^2}. \tag{5}$$

The resulting topological phase is determined by $\mathcal{C}$, a_c, and M_c, which will be denoted by $\mathcal{C}_{a_c,M_c}$. Here, z_a, z_b are the positions of a_c anyons. M_c must be consistent with anyon statistics. Considering exchanging two a_c anyons, we obtain a phase factor $e^{2\pi i \frac{M_c}{2}}$ from the wave function and a phase factor $e^{2\pi i s_{a_c}}$ from anyonic statistics. To be consistent, the total phase factor must be 1,

$$\frac{M_c}{2} + s_{a_c} \in \mathbb{Z}. \tag{6}$$

So we need to take $M_c = m_c - 2s_{a_c}$, where m_c is an even integer.

Anyon i in the phase $\mathcal{C}$ may be dressed with a flux M_i in the new phase $\mathcal{C}_{a_c,M_c}$,

$$\Psi(i, M_i) = \prod_b(\xi_i - z_b)^{M_i}\prod_{a<b}(z_a - z_b)^{M_c} e^{-\frac{1}{4}\sum |z_a|^2}. \tag{7}$$

Here, ξ_i is the position of anyon i. Thus an anyon in the new phase is represented by a pair (i, M_i). Again, M_i cannot be arbitrary. If a_c has trivial mutual statistics with i, M_i can be any integer. Otherwise, consider moving a_c around (i, M_i) and we obtain a phase factor $e^{2\pi i M_i}$ from the flux M_i and a phase factor $e^{2\pi i t_i}$ from the mutual statistics between a_c and i. The mutual statistics can be extracted from the S matrix, $e^{2\pi i t_i} = DS_{ia_c^*}/d_i$, $t_{a_c} = 2s_{a_c}$. To be consistent, the total phase factor must be 1,

$$M_i + t_i \in \mathbb{Z}. \tag{8}$$

Since the anyon a_c dressed with a flux M_c is a "trivial excitation" in the new phase,

$$\Psi(a_c, M_c) \sim \prod_b^n(\xi_{a_c} - z_b)^{M_c}\prod_{a<b}^n(z_a - z_b)^{M_c}$$
$$= \prod_{a<b}^{n+1}(z_a - z_b)^{M_c},$$
$$(a_c, M_c) \sim (\mathbf{1}, 0), \tag{9}$$

we have the equivalence relation

$$(i, M_i) \sim (i \otimes a_c, M_i + M_c). \tag{10}$$

It is then straightforward to derive the data of the resulting topological order $\mathcal{C}_{a_c,M_c}$:

(1) The spin of (i, M_i) is given by the spin of i plus the "spin" of the flux M_i,

$$s_{(i,M_i)} = s_i + \frac{M_i^2}{2M_c}. \tag{11}$$

(2) To fuse anyons $(i, M_i), (j, M_j)$ in the new phase, just fuse i, j as in the old phase, and add up the flux,

$$(i, M_i) \otimes (j, M_j) = \bigoplus_k N_k^{ij}(k, M_i + M_j), \tag{12}$$

and then apply the equivalence relation (10). In other words, the new fusion rules are

$$N_{(k,M_k)}^{(i,M_i),(j,M_j)} = N_k^{ij}\delta_{M_i+M_j,M_k}, \tag{13}$$

up to the equivalence relation (10).

(3) The quantum dimensions remain the same,

$$d_{(i,M_i)} = d_i, \tag{14}$$

since they are clearly the unique positive solution to

$$d_{(i,M_i)}d_{(j,M_j)} = \sum_{(k,M_k)} N_{(k,M_k)}^{(i,M_i),(j,M_j)} d_{(k,M_k)}$$
$$= \sum_k N_k^{ij} d_{(k,M_i+M_j)}. \tag{15}$$

(4) By a direct analysis [9] of the equivalence relation (10), we see that the rank (number of anyon types) of $\mathcal{C}_{a_c,M_c}$ is

$$N^{\mathcal{C}_{a_c,M_c}} = |M_c|N^{\mathcal{C}}. \tag{16}$$

(5) The S matrix, calculated via (3), is

$$S_{(i,M_i),(j,M_j)}^{\mathcal{C}_{a_c,M_c}} = \frac{1}{\sqrt{|M_c|}} S_{ij}^{\mathcal{C}} e^{-2\pi i \frac{M_iM_j}{M_c}}. \tag{17}$$

(6) The chiral central charge, calculated via (4), is

$$c^{\mathcal{C}_{a_c,M_c}} = c^{\mathcal{C}} + \operatorname{sgn} M_c, \tag{18}$$

where $\operatorname{sgn} M_c = \frac{|M_c|}{M_c}$ is the sign of M_c.

The one-step hierarchy construction is reversible. In $\mathcal{C}_{a_c,M_c}$, choosing $a_c' = (\mathbf{1}, 1)$, $s_{a_c'} = \frac{1}{2M_c}$, $m_c' = 0$, $M_c' = -1/M_c$, and repeating the construction, we will go back to $\mathcal{C}$. Therefore, hierarchy construction defines an equivalence relation between topological phases. We call the corresponding equivalence classes the "non-Abelian families." Each non-Abelian family has "root" phases with the smallest rank. Let $\mathcal{C}_{Ab}$ denote the full subcategory of all Abelian anyons in $\mathcal{C}$. $\mathcal{C}$ is a root if [14] and only if [9,14] $\mathcal{C}_{Ab}$ is a symmetric fusion category, namely, all the Abelian anyons are bosons or fermions with trivial mutual statistics with each other.

Multiple Steps of Construction and the Matrix Formulation. Now we consider multiple steps of hierarchy constructions and try to write down the final result at once. Note that in the flux label M_i we need to use the mutual statistics in the previous step, and things get involved when there are multiple steps. To separate out the mutual statistics and thus make things clearer, we use the "integer convention" (i, m), instead of the "flux convention" (i, M_i), where $m - t_i = M_i$.

Now consider starting from a topological order $\mathcal{C}$ and performing one-step construction κ times. For the first step we take $a_1 \in \mathcal{C}_{Ab}$ and even integer k_{11}. For the second step we take an Abelian anyon $(a_2 \in \mathcal{C}_{Ab}, k_{12})$ and even integer k_{22}, where k_{12} is an integer. For the third step we take an Abelian anyon $((a_3 \in \mathcal{C}_{Ab}, k_{13}), k_{23})$ and even integer k_{33}, where k_{13}, k_{23} are integers. We keep moving on and we see that the steps can be summarized by a_I and k_{IJ}. Define a corresponding integer symmetric κ by κ matrix by setting $k_{IJ} = k_{JI}$. Denote by $t_{i,a}$ the mutual statistics between anyon i and Abelian anyon a in $\mathcal{C}$ ($e^{2\pi i t_{i,a}}$ is the phase factor of braiding a around i), by s_i the spin of anyon i in $\mathcal{C}$, and set $t_{a,a} = 2s_a$. Let the K matrix be $K_{IJ} = k_{IJ} - t_{a_I,a_J}$.

Physically, we let the Abelian anyons a_I, $I = 1, 2, \dots, \kappa$ form a multilayer Laughlin-like state

$$\prod \left(z_a^{(I)} - z_b^{(J)}\right)^{K_{IJ}}, \tag{19}$$

where I labels the layer and $z_a^{(I)}$ is the position of the a_I anyon. By a similar argument as in the one-step case, we know that $K_{IJ} + t_{a_I,a_J}$ must be an integer and $K_{II} + t_{a_I,a_I}$ must be an even integer.

Though we are using the integer convention, note that similar to the one-step case, it is the combination $K_{IJ} = k_{IJ} - t_{a_I,a_J}$ or $M_c = m_c - t_{a_c}$ that determines the final topological order, not the integer k_{IJ} or m_c alone. The meaning of k_{IJ} or m_c depends on the choice of mutual statistics $t_{i,a}$.

The fusion rule and T, S matrices of the resulting topological order after κ steps can be calculated efficiently via the K matrix as stated in Theorem 1. This result generalizes the K matrix formulation for Abelian topological orders [10].

Theorem 1. The topological order constructed from $\mathcal{C}$ via κ steps can be summarized by a_I and K_{IJ}, where $I, J = 1, \dots, \kappa$, $a_I \in \mathcal{C}_{Ab}$, $\det K \neq 0$, $K_{IJ} = K_{JI}$, $K_{IJ} + t_{a_I,a_J}$ are integers, and $K_{II} + t_{a_I,a_I}$ are even. Let $\boldsymbol{a}$ formally denote the vector (a_I) and $\mathcal{C}_{\boldsymbol{a},K}$ denote the resulting topological order. $\mathcal{C}_{\boldsymbol{a},K}$ is as follows:

(1) Fix a choice of mutual statistics t_{i,a_I} in $\mathcal{C}$. Let $\boldsymbol{t}_i$ be the κ-dimensional vector (t_{i,a_I}). Anyons are labeled by $(i \in \mathcal{C}, \boldsymbol{l})$, where $\boldsymbol{l}$ is a κ-dimensional integer vector, subject to the following equivalence relations,

$$(i, \boldsymbol{l}) \sim (i \otimes a_I, \boldsymbol{l} + K_I - \boldsymbol{t}_i + \boldsymbol{t}_{i\otimes a_I}), \tag{20}$$

where K_I is the Ith column vector of K. For a different choice of mutual statistics, or representative $i' \sim i$, t_{i',a_I} differs from t_{i,a_I} by an integer, and $(i', \boldsymbol{l} + \boldsymbol{t}_{i'} - \boldsymbol{t}_i) \sim (i, \boldsymbol{l})$. $\mathcal{C}_{\boldsymbol{a},K}$ does not depend on the choice of mutual statistics or representative in $\mathcal{C}$.

(2) Fusion is given by

$$(i, \boldsymbol{l}) \otimes (j, \boldsymbol{k}) = \oplus_s N_s^{ij}(s, \boldsymbol{l} + \boldsymbol{k} - \boldsymbol{t}_i - \boldsymbol{t}_j + \boldsymbol{t}_s). \tag{21}$$

(3) The spin of $(i, \boldsymbol{l})$ is

$$s_{(i,\boldsymbol{l})} = s_i + \tfrac{1}{2}(\boldsymbol{l} - \boldsymbol{t}_i)^T K^{-1}(\boldsymbol{l} - \boldsymbol{t}_i). \tag{22}$$

(4) The S matrix is

$$S_{(i,\boldsymbol{l})(j,\boldsymbol{k})} = \frac{1}{\sqrt{|\det K|}} S_{ij} e^{-2\pi i(\boldsymbol{l}-\boldsymbol{t}_i)^T K^{-1}(\boldsymbol{k}-\boldsymbol{t}_j)}. \tag{23}$$

(5) The rank is $N^{\mathcal{C}_{\boldsymbol{a},K}} = |\det K| N^{\mathcal{C}}$. The chiral central charge is $c^{\mathcal{C}_{\boldsymbol{a},K}} = c^{\mathcal{C}} + \operatorname{sgn} K$. Here, $\operatorname{sgn} K$ denotes the index of the matrix K, namely, the number of positive eigenvalues minus the number of negative eigenvalues.

Proof. We postpone the lengthy proof to the Appendix. ■

When $\mathcal{C}$ is a root whose Abelian anyons $\mathcal{C}_{Ab}$ are a symmetric fusion category, a_I, a_J are mutually trivial, and t_{a_I,a_J} are all integers. In particular, we can choose $t_{a_I,a_I} = 1$ when a_I is fermionic, and other $t_{a_I,a_J} = 0$. In this case the K matrix is an integer matrix and K_{II} is even when a_I is a boson and odd when a_I is a fermion.

Equivalence Relation of the Constructed Topological Orders. Starting from the same topological order $\mathcal{C}$, different paths of construction may result in the same topological order. It is natural to ask what is the equivalence relation of $(\boldsymbol{a}, K)$. For now, we know three ways to generate equivalent $\mathcal{C}_{\boldsymbol{a},K}$:

(1) The equivalence between the starting point $F : \mathcal{C} \simeq \mathcal{D}$ naturally gives rise to equivalence $\mathcal{C}_{\boldsymbol{a},K} \simeq \mathcal{D}_{F(\boldsymbol{a}),K}$.

(2) "Integer linear recombination" of a_I, $W \in GL(\kappa, \mathbb{Z})$ (namely, W is an integer matrix with $\det W = \pm 1$), $\mathcal{C}_{\boldsymbol{a},K} \simeq \mathcal{C}_{W\boldsymbol{a},WKW^T}$. We call such a transformation the $GL(\mathbb{Z})$ transformation.

(3) The reversibility of one-step construction means that the topological order constructed from $\mathcal{C}$ with $\binom{a_1 = a_c}{a_2 = \boldsymbol{1}}$, $K = \begin{pmatrix} M_c & 1 \\ 1 & 0 \end{pmatrix}$ is equivalent to $\mathcal{C}$. Also (a_I, K_{IJ}) is equivalent to $\begin{pmatrix} a_I \\ a \\ \boldsymbol{1} \end{pmatrix}$, $\begin{pmatrix} K_{IJ} & \boldsymbol{l}_c - \boldsymbol{t}_a & 0 \\ \boldsymbol{l}_c^T - \boldsymbol{t}_a^T & m_c - 2s_a & 1 \\ 0 & 1 & 0 \end{pmatrix}$, where a can be any Abelian anyon in $\mathcal{C}_{Ab}$. Note that under $GL(\mathbb{Z})$ transformation, $\begin{pmatrix} K_{IJ} & \boldsymbol{l}_c - \boldsymbol{t}_a & 0 \\ \boldsymbol{l}_c^T - \boldsymbol{t}_a^T & m_c - 2s_a & 1 \\ 0 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} K_{IJ} & 0 & 0 \\ 0 & -2s_a & 1 \\ 0 & 1 & 0 \end{pmatrix} = K \oplus \begin{pmatrix} -t_{a,a} & 1 \\ 1 & 0 \end{pmatrix}$. Therefore, we have $(\boldsymbol{a}, K) \sim \left(\boldsymbol{a} \oplus \binom{a}{\boldsymbol{1}}, K \oplus \begin{pmatrix} -t_{a,a} & 1 \\ 1 & 0 \end{pmatrix}\right)$. We refer to $\left(\binom{a}{\boldsymbol{1}}, \begin{pmatrix} -t_{a,a} & 1 \\ 1 & 0 \end{pmatrix}\right)$ as the "trivial bilayer."

Conjecture 1. $\mathcal{C}_{\boldsymbol{a},K}$ and $\mathcal{C}_{\boldsymbol{a}',K'}$ (with exactly the same chiral central charge, not modulo 8) are equivalent if and only if, up to automorphisms of $\mathcal{C}$ and $GL(\mathbb{Z})$ transformations, $(\boldsymbol{a} \oplus \boldsymbol{b}, K \oplus X) \sim (\boldsymbol{a}' \oplus \boldsymbol{b}', K' \oplus X')$, where $(\boldsymbol{b}, X)$ and $(\boldsymbol{b}', X')$ are direct sums of trivial bilayers $\left(\binom{a}{\boldsymbol{1}}, \begin{pmatrix} -t_{a,a} & 1 \\ 1 & 0 \end{pmatrix}\right)$.

The Formal Categorical Formulation. We give the formal basis-independent formulation of the above constructions. Let $\mathcal{C}$ be a braided fusion category, $\alpha_{A,B,C}$, $c_{A,B}$ denote the associator and braiding in $\mathcal{C}$, $\underline{\mathcal{C}_{Ab}}$ denote the Abelian group corresponding to the pointed subcategory $\mathcal{C}_{Ab}$, and $t : \mathrm{Irr}(\mathcal{C}) \times \underline{\mathcal{C}_{Ab}} \to \mathbb{Q}$ denote the mutual statistics between simple objects and pointed ones, namely, $e^{2\pi i t(i,a)} = \frac{1}{d_i} \mathrm{Tr}\, c_{a,i} c_{i,a}$; in particular, the diagonal entries are related to exchange statistics $e^{i\pi t(a,a)} = \mathrm{Tr}\, c_{a,a}$.

Let $\mathbb{Z}^\kappa$ be a free Abelian group with κ generators. It can be naturally extended to a κ-dimensional vector space over $\mathbb{Q}$. Let $\overline{\mathbb{Z}}^\kappa := \mathrm{Hom}(\mathbb{Z}^\kappa, \mathbb{Q})$ denote the "dual space," the space of $\mathbb{Q}$-linear functions. Conventionally, we use $x, y, \dots$ to denote elements in $\mathbb{Z}^\kappa$ and $f(-), g(-), \dots$, or simply f, g when not confusing, to denote functions in $\overline{\mathbb{Z}}^\kappa$.

Let $K : \mathbb{Z}^\kappa \times \mathbb{Z}^\kappa \to \mathbb{Q}$ be a nondegenerate symmetric bilinear form. It defines an isomorphism from $\mathbb{Z}^\kappa$ to $\overline{\mathbb{Z}}^\kappa$, by $x \mapsto K(x, -) = K(-, x)$. Denote the inverse map by $\tilde{K}$, thus

$$\tilde{K}(K(x, -)) = x, \quad K(\tilde{K}(f), x) = f(x). \tag{24}$$

There is then a natural nondegenerate symmetric bilinear form $\overline{K}$ on $\overline{\mathbb{Z}}^\kappa$ induced from K, via

$$\overline{K}(f,g)=K(\tilde{K}(f),\tilde{K}(g))=f(\tilde{K}(g))=g(\tilde{K}(f)). \quad (25)$$

If one chooses a basis of $\mathbb{Z}^\kappa$ and the corresponding dual basis of $\overline{\mathbb{Z}}^\kappa$, the matrices of K and $\overline{K}$ are inverse to each other.

We also need to choose κ Abelian anyons for each step. This is concluded in a group homomorphism $\boldsymbol{a}:\mathbb{Z}^\kappa\to\mathcal{C}_{Ab}$. The bilinear form K needs to satisfy the even integral condition, namely, $\forall x,y,\ K(x,y)+t(\boldsymbol{a}(x),\boldsymbol{a}(y))\in\mathbb{Z}$, and $K(x,x)+t(\boldsymbol{a}(x),\boldsymbol{a}(x))\in 2\mathbb{Z}$.

For a κ step construction, we first define a semisimple category $\mathcal{C}^\uparrow_{\boldsymbol{a},K}$. $\mathcal{C}^\uparrow_{\boldsymbol{a},K}$ is graded by $\overline{\mathbb{Z}}^\kappa/K(2\ker\boldsymbol{a},-)$ (not faithful). Take a representative $f\in\overline{\mathbb{Z}}^\kappa$, the component $(\mathcal{C}^\uparrow_{\boldsymbol{a},K})_f$ is a full subcategory of $\mathcal{C}$ with simple objects i satisfying $f(-)+t(i,\boldsymbol{a}(-))\in\mathbb{Z}$ [note that $K(x,-)$ is an integer for $x\in\ker\boldsymbol{a}$, so this is well defined for $f+K(2\ker\boldsymbol{a},-)$]. Denote the simple objects in $\mathcal{C}^\uparrow_{\boldsymbol{a},K}$ by i_f. We then define the tensor product and braiding in $\mathcal{C}^\uparrow_{\boldsymbol{a},K}$,

$$i_f\otimes j_g=(i\otimes j)_{f+g}=\oplus_k N^{ij}_k k_{f+g}, \quad (26)$$

$$\alpha_{i_f,j_g,k_h}=\alpha_{i,j,k}, \quad (27)$$

$$c_{i_f,j_g}=c_{i,j}e^{i\pi\overline{K}(f,g)}. \quad (28)$$

Equation (28) is independent of the choice of representative: $\forall x\in\ker\boldsymbol{a}$,

$$\begin{aligned}c_{i_{f+K(2x,-)},j_g}&=c_{i_f,j_g}e^{i\pi\overline{K}(K(2x,-),g)}\\&=c_{i_f,j_g}e^{2\pi i g(x)}.\end{aligned} \quad (29)$$

Since $t(j,\boldsymbol{a}(x))=t(j,0)\in\mathbb{Z}$ and $g(x)+t(i,\boldsymbol{a}(x))\in\mathbb{Z}$, clearly $g(x)\in\mathbb{Z}$ as desired. Thus $\mathcal{C}^\uparrow_{\boldsymbol{a},K}$ is a braided fusion category graded by $\overline{\mathbb{Z}}^\kappa/K(2\ker\boldsymbol{a},-)$. It is obvious that $d_{i_f}=d_i$.

Observe that for any $x\in\mathbb{Z}^\kappa$, $\boldsymbol{a}(x)_{K(x,-)}$ is a self-boson and mutually trivial to any object i_f. $\boldsymbol{a}(x)_{K(x,-)}$ is a self-boson since

$$\begin{aligned}\mathrm{Tr}\,c_{\boldsymbol{a}(x)_{K(x,-)},\boldsymbol{a}(x)_{K(x,-)}}&=\mathrm{Tr}\,c_{\boldsymbol{a}(x),\boldsymbol{a}(x)}e^{i\pi\overline{K}(K(x,-),K(x,-))}\\&=e^{i\pi[t(\boldsymbol{a}(x),\boldsymbol{a}(x))+K(x,x)]}=1.\end{aligned} \quad (30)$$

$\boldsymbol{a}(x)_{K(x,-)}$ is in the Müger center [15] (mutually trivial to any object i_f) since

$$\begin{aligned}\frac{1}{d_i}\mathrm{Tr}\,c_{i_f,\boldsymbol{a}(x)_{K(x,-)}}c_{\boldsymbol{a}(x)_{K(x,-)},i_f}&=e^{2\pi i[t(i,\boldsymbol{a}(x))+\overline{K}(f,K(x,-))]}\\&=e^{2\pi i[t(i,\boldsymbol{a}(x))+f(x)]}=1.\end{aligned} \quad (31)$$

Therefore, $\{\boldsymbol{a}(x)_{K(x,-)},x\in\mathbb{Z}^\kappa\}$ generates a symmetric fusion subcategory in the Müger center of $\mathcal{C}^\uparrow_{\boldsymbol{a},K}$ which is equivalent to $\mathrm{Rep}(\langle\boldsymbol{a}(x)_{K(x,-)}\rangle\simeq\mathbb{Z}^\kappa/2\ker\boldsymbol{a})$. Condense it [16] [take the category of local modules over $\mathrm{Fun}(\mathbb{Z}^\kappa/2\ker\boldsymbol{a})$], and we obtain the final result $\mathcal{C}_{\boldsymbol{a},K}=(\mathcal{C}^\uparrow_{\boldsymbol{a},K})^{\mathrm{loc}}_{\mathrm{Fun}(\mathbb{Z}^\kappa/2\ker\boldsymbol{a})}$. In general, the associator (F matrix) will change and get complicated after such an anyon condensation process. However, since the condensed anyons are in the Müger center, the braiding and fusion rules are preserved [16,17]. Thus if we are only interested in the simple data such as fusion rules and T,S matrices, it is fine to work in the larger category $\mathcal{C}^\uparrow_{\boldsymbol{a},K}$.

Conclusion and Outlook. In this Rapid Communication we introduced the matrix formulation for non-Abelian families, which makes it possible to generate any topological order in the same non-Abelian family as a given one almost instantly. We have provided a powerful tool, which, on one hand, can help group known topological orders [18–22] (or modular tensor categories [23]) into non-Abelian families, and for simplicity, only the data of one root is necessary to be listed explicitly; on the other hand, one can efficiently generate the data of infinitely many possible unknown topological orders.

The results in Ref. [9] already reduces the classification problem of all (2+1)-dimensional topological orders to the classification of all root topological orders, namely, in which the Abelian anyons have trivial self- and mutual statistics. The results in this Rapid Communication further make this reduction an efficient and simple algorithm. In the end, we only need to maintain a list of root topological orders. It will be interesting to find the canonical (the simplest) form of $(\boldsymbol{a},K)$, and then we will have a simple name for each topological order: the root $\mathcal{C}$ plus the canonical form of $(\boldsymbol{a},K)$. Moreover, after fixing a root $\mathcal{C}$, we should be able to extract all possible *non-Abelian invariants* [14] of this family by studying $\mathcal{C}$ and the pair $(\boldsymbol{a},K)$. These non-Abelian invariants will surely deepen our understanding of topological phases of matter, as well as of the application of topological materials in quantum computation.

Although the topological orders in the same non-Abelian family have similar non-Abelian properties, rendering them similar in braiding-fusion-based topological quantum computation, they can be very different from other perspectives. For example, we already know that the chiral central charge changes by integers, thus the topological orders must have different edge states. Moreover, starting from a root topological order with integer chiral central charge, it is possible to construct a new topological order with nonchiral or even gapped edge states. Thus, our construction provides a large reservoir for topological defects and edge states which may have future applications. The other properties that have not been touched upon here, of the large numbers of topological orders accessible from our construction, may also have potential applications.

Our construction can also be viewed as a generalization of anyon condensation [16,24–30], where anyons are forced to form an effective trivial state, and the condensed anyons are necessarily bosons. We make anyons form effective Laughlin states, and our results imply that the multilayer Laughlin states are the most general types of states Abelian anyons can form. From this point of view, it is natural to ask what kind of nontrivial effective states non-Abelian anyons can form. Future research along this line may reveal more exotic relations between topological phases, by nontrivial condensations of non-Abelian anyons, and further simplify our understanding of topological orders.

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APPENDIX: PROOF OF THEOREM 1

We prove the theorem by induction. It is obviously true for $\kappa = 1$. Now assume that it is true for $\kappa - 1$ where $\kappa > 1$. Let K_0 be the corresponding $\kappa - 1$ by $\kappa - 1$ matrix. From $\kappa - 1$ to κ we choose $a_c = (a_\kappa, \boldsymbol{l}_c)$ and even integer m_c. The new K matrix is

$$K_1 = \begin{pmatrix} K_0 & \boldsymbol{l}_c - \boldsymbol{t}_{a_\kappa} \\ (\boldsymbol{l}_c - \boldsymbol{t}_{a_\kappa})^T & m_c - 2s_{a_\kappa} \end{pmatrix}. \tag{A1}$$

The spin of a_c is

$$s_{a_c} = s_{a_\kappa} + \tfrac{1}{2}(\boldsymbol{l}_c - \boldsymbol{t}_{a_\kappa})^T K_0^{-1}(\boldsymbol{l}_c - \boldsymbol{t}_{a_\kappa}), \tag{A2}$$

and the mutual statistics between $(i, \boldsymbol{l}_0)$ and a_c is

$$t_{(i,\boldsymbol{l}_0)} = t_{i,a_\kappa} + (\boldsymbol{l}_c - \boldsymbol{t}_{a_\kappa})^T K_0^{-1}(\boldsymbol{l}_0 - \boldsymbol{t}_i). \tag{A3}$$

First, as long as $m_c - 2s_{a_c} \neq 0$, K_1 is invertible with

$$K_1^{-1} = \begin{pmatrix} K_0^{-1} + \frac{K_0^{-1}(\boldsymbol{l}_c - \boldsymbol{t}_{a_\kappa})(\boldsymbol{l}_c - \boldsymbol{t}_{a_\kappa})^T K_0^{-1}}{m_c - 2s_{a_c}} & -\frac{K_0^{-1}(\boldsymbol{l}_c - \boldsymbol{t}_{a_\kappa})}{m_c - 2s_{a_c}} \\ -\frac{(\boldsymbol{l}_c - \boldsymbol{t}_{a_\kappa})^T K_0^{-1}}{m_c - 2s_{a_c}} & \frac{1}{m_c - 2s_{a_c}} \end{pmatrix}. \tag{A4}$$

Also,

$$\begin{aligned} \det(K_1) &= \det \begin{pmatrix} K_0 & \boldsymbol{l}_c - \boldsymbol{t}_{a_\kappa} \\ (\boldsymbol{l}_c - \boldsymbol{t}_{a_\kappa})^T & m_c - 2s_{a_\kappa} \end{pmatrix} \\ &= \det(K_0)[m_c - 2s_{a_\kappa} - (\boldsymbol{l}_c - \boldsymbol{t}_{a_\kappa})^T K_0^{-1}(\boldsymbol{l}_c - \boldsymbol{t}_{a_\kappa})] \\ &= (m_c - 2s_{a_c}) \det(K_0). \end{aligned} \tag{A5}$$

Thus $\det K$ accounts for the increment of rank, total quantum dimension, as well as the normalization of S matrix. Also $\operatorname{sgn} K = \operatorname{sgn} K_0 + \operatorname{sgn}(m_c - 2s_{a_c})$ accounts for the increment of chiral central charge.

The new anyons are labeled by $(i, \boldsymbol{l}_0, m)$, where m is an integer. Combine $\boldsymbol{l}$ and m into a κ-dimensional vector $\boldsymbol{l}^T = (\boldsymbol{l}_0^T, m)$. We only need to verify the spin, equivalence relations and fusion rule of $(i, \boldsymbol{l})$; S matrix follows directly.

The spin of $(i, \boldsymbol{l}_0, m) = (i, \boldsymbol{l})$ is

$$\begin{aligned} s_{(i,\boldsymbol{l})} &= s_{(i,\boldsymbol{l}_0)} + \frac{(m - t_{(i,\boldsymbol{l}_0)})^2}{2(m_c - 2s_{a_c})} \\ &= s_i + \frac{1}{2}(\boldsymbol{l}_0 - \boldsymbol{t}_i)^T K_0^{-1}(\boldsymbol{l}_0 - \boldsymbol{t}_i) + \frac{(m - t_{(i,\boldsymbol{l}_0)})^2}{2(m_c - 2s_{a_c})}, \end{aligned} \tag{A6}$$

while (using the same notation for $\kappa - 1$ and κ dimensional $\boldsymbol{t}_i$)

$$\begin{aligned} \frac{1}{2}(\boldsymbol{l} - \boldsymbol{t}_i)^T K_1^{-1}(\boldsymbol{l} - \boldsymbol{t}_i) &= \frac{1}{2}((\boldsymbol{l}_0 - \boldsymbol{t}_i)^T, m - t_{i,a_\kappa}) K_1^{-1} \begin{pmatrix} \boldsymbol{l}_0 - \boldsymbol{t}_i \\ m - t_{i,a_\kappa} \end{pmatrix} \\ &= \frac{1}{2}\left[(\boldsymbol{l}_0 - \boldsymbol{t}_i)^T K_0^{-1}(\boldsymbol{l}_0 - \boldsymbol{t}_i) + \frac{(m - t_{i,a_\kappa})^2 + (t_{(i,\boldsymbol{l}_0)} - t_{i,a_\kappa})^2 - 2(m - t_{i,a_\kappa})(t_{(i,\boldsymbol{l}_0)} - t_{i,a_\kappa})}{m_c - 2s_{a_c}}\right]. \end{aligned} \tag{A7}$$

Indeed, we have

$$s_{(i,\boldsymbol{l})} = s_i + \tfrac{1}{2}(\boldsymbol{l} - \boldsymbol{t}_i)^T K_1^{-1}(\boldsymbol{l} - \boldsymbol{t}_i). \tag{A8}$$

For $\kappa - 1$ we have equivalence relations

$$(i, \boldsymbol{l}_0) \sim (i \otimes a_I, \boldsymbol{l}_0 + (K_0)_I - \boldsymbol{t}_i + \boldsymbol{t}_{i\otimes a_I}). \tag{A9}$$

For $(i, \boldsymbol{l}_0, m)$, one equivalence relation comes from condensing $a_c = (a_\kappa, \boldsymbol{l}_c)$ with even integer m_c,

$$(i, \boldsymbol{l}_0, m) \sim (i \otimes a_\kappa, \boldsymbol{l}_0 + \boldsymbol{l}_c - \boldsymbol{t}_i - \boldsymbol{t}_{a_\kappa} + \boldsymbol{t}_{i\otimes a_\kappa}, m + m_c - t_{(i,\boldsymbol{l}_0)} - t_{(a_\kappa,\boldsymbol{l}_c)} + t_{(i\otimes a_\kappa, \boldsymbol{l}_0 + \boldsymbol{l}_c - \boldsymbol{t}_i - \boldsymbol{t}_{a_\kappa} + \boldsymbol{t}_{i\otimes a_\kappa})}), \tag{A10}$$

where

$$\begin{aligned} &-t_{(i,\boldsymbol{l}_0)} - t_{(a_\kappa,\boldsymbol{l}_c)} + t_{(i\otimes a_\kappa, \boldsymbol{l}_0 + \boldsymbol{l}_c - \boldsymbol{t}_i - \boldsymbol{t}_{a_\kappa} + \boldsymbol{t}_{i\otimes a_\kappa})} \\ &\quad = -t_{i,a_\kappa} - t_{a_\kappa,a_\kappa} + t_{i\otimes a_\kappa,a_\kappa} + (\boldsymbol{l}_c - \boldsymbol{t}_{a_\kappa})^T K_0^{-1}(\boldsymbol{t}_i - \boldsymbol{l}_0 + \boldsymbol{t}_{a_\kappa} - \boldsymbol{l}_c - \boldsymbol{t}_{i\otimes a_\kappa} + \boldsymbol{l}_0 + \boldsymbol{l}_c - \boldsymbol{t}_i - \boldsymbol{t}_{a_\kappa} + \boldsymbol{t}_{i\otimes a_\kappa}) \\ &\quad = -t_{i,a_\kappa} - t_{a_\kappa,a_\kappa} + t_{i\otimes a_\kappa,a_\kappa}. \end{aligned} \tag{A11}$$

Thus

$$(i, \boldsymbol{l}) \sim (i \otimes a_\kappa, \boldsymbol{l} + (K_1)_\kappa - \boldsymbol{t}_i + \boldsymbol{t}_{i\otimes a_\kappa}), \tag{A12}$$

where $(K_1)_\kappa^T = (\boldsymbol{l}_c^T - \boldsymbol{t}_{a_\kappa}^T, m_c - 2s_{a_\kappa})$. The other equivalence relations come from choosing a different representative of $(i, \boldsymbol{l}_0)$; for $I = 1, \ldots, \kappa - 1$,

$$(i, \boldsymbol{l}_0, m) \sim (i \otimes a_I, \boldsymbol{l}_0 + (K_0)_I - \boldsymbol{t}_i + \boldsymbol{t}_{i\otimes a_I}, m - t_{(i,\boldsymbol{l}_0)} + t_{(i\otimes a_I, \boldsymbol{l}_0 + K_I - \boldsymbol{t}_i + \boldsymbol{t}_{i\otimes a_I})}), \tag{A13}$$

where

$$\begin{aligned} &- t_{(i,\boldsymbol{l}_0)} + t_{(i\otimes a_I, \boldsymbol{l}_0 + (K_0)_I - \boldsymbol{t}_i + \boldsymbol{t}_{i\otimes a_I})} \\ &\quad = -t_{i,a_\kappa} + t_{i\otimes a_I,a_\kappa} + (\boldsymbol{l}_c - \boldsymbol{t}_{a_\kappa})^T K_0^{-1}(\boldsymbol{t}_i - \boldsymbol{l}_0 - \boldsymbol{t}_{i\otimes a_I} + \boldsymbol{l}_0 + (K_0)_I - \boldsymbol{t}_i + \boldsymbol{t}_{i\otimes a_I}) \\ &\quad = -t_{i,a_\kappa} + t_{i\otimes a_\kappa,a_\kappa} + (\boldsymbol{l}_c - \boldsymbol{t}_{a_\kappa})_I^T. \end{aligned} \tag{A14}$$

Thus

$$(i,\boldsymbol{l}) \sim (i\otimes a_I, \boldsymbol{l} + (K_1)_I - \boldsymbol{t}_i + \boldsymbol{t}_{i\otimes a_I}), \tag{A15}$$

where $(K_1)_I^T = ((K_0)_I^T, (\boldsymbol{l}_c - \boldsymbol{t}_{a_\kappa})_I)$, $I = 1,\dots,\kappa-1$.

The fusion of $(i,\boldsymbol{l}_0,m)$ and $(j,\boldsymbol{k}_0,n)$ is

$$(i,\boldsymbol{l}_0,m)\otimes(j,\boldsymbol{k}_0,n) = \oplus N_s^{ij}(s,\boldsymbol{l}_0+\boldsymbol{k}_0-\boldsymbol{t}_i-\boldsymbol{t}_j+\boldsymbol{t}_s, m+n-t_{(i,\boldsymbol{l}_0)}-t_{(j,\boldsymbol{k}_0)}+t_{(s,\boldsymbol{l}_0+\boldsymbol{k}_0-\boldsymbol{t}_i-\boldsymbol{t}_j+\boldsymbol{t}_s)}), \tag{A16}$$

where

$$\begin{aligned} &-t_{(i,\boldsymbol{l}_0)}-t_{(j,\boldsymbol{k}_0)}+t_{(s,\boldsymbol{l}_0+\boldsymbol{k}_0-\boldsymbol{t}_i-\boldsymbol{t}_j+\boldsymbol{t}_s)}\\ &= -t_{i,a_\kappa}-t_{j,a_\kappa}+t_{s,a_\kappa}+(\boldsymbol{l}_c-\boldsymbol{t}_{a_\kappa})^TK_0^{-1}(\boldsymbol{t}_i-\boldsymbol{l}_0+\boldsymbol{t}_j-\boldsymbol{k}_0-\boldsymbol{t}_s+\boldsymbol{l}_0+\boldsymbol{k}_0-\boldsymbol{t}_i-\boldsymbol{t}_j+\boldsymbol{t}_s)\\ &= -t_{i,a_\kappa}-t_{j,a_\kappa}+t_{s,a_\kappa}. \end{aligned} \tag{A17}$$

Thus we do have

$$(i,\boldsymbol{l})\otimes(j,\boldsymbol{k}) = \oplus_s N_s^{ij}(s,\boldsymbol{l}+\boldsymbol{k}-\boldsymbol{t}_i-\boldsymbol{t}_j+\boldsymbol{t}_s). \tag{A18}$$

In the above proof, we need to assume that $\det K_0 \neq 0$. As we prove by induction, this in fact means that we need to assume that $\det(K_{IJ}, I,J=1,2,\dots,n)\neq 0$ for any $n<\kappa-1$. However, such an assumption is inessential and can be dropped, given the following transformation on $(\boldsymbol{a},K)$: For an integer matrix W with $\det W=\pm1$, (a_I,K_{IJ}) is equivalent to $(a_I' = \otimes_J a_J^{\otimes W_{IJ}}, K' = WKW^T)$. The fact that $t_{a_I',a_J'} = \sum_{PQ} W_{IP} t_{a_P,a_Q} W_{JQ}$ implies the transformation for the K matrix. As a_I are in an Abelian group, it is convenient to write in the additive convention $a_I' = \sum_J W_{IJ}a_I$, or simply $\boldsymbol{a}' = W\boldsymbol{a}$. Thus $\mathcal{C}_{\boldsymbol{a},K}\simeq \mathcal{C}_{W\boldsymbol{a},WKW^T}$ for integer matrix W with $\det W=\pm1$. More precisely, the equivalence is given by $(i,\boldsymbol{l})\mapsto(i,W\boldsymbol{l})$. Note that $\boldsymbol{t}_i' = (t_{i,a_I'}) = W\boldsymbol{t}_i$. It is straightforward to check that this map is compatible with the equivalence relation (20), and preserves fusion (21), spin (22), and S matrix (23).

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