


Self-dual $\nu = 1$ bosonic quantum Hall state in mixed-dimensional QEDWei-Han Hsiao  and Dam Thanh Son*Kadanoff Center for Theoretical Physics, University of Chicago, Chicago, Illinois 60637, USA*

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We consider a $(2 + 1)$ -dimensional Wilson-Fisher boson coupled to a $(3 + 1)$ -dimensional $U(1)$ gauge field. This theory possesses a strong-weak duality in terms of the coupling constant e and is self-dual at a particular value of e . We derive exact relations between transport coefficients for a $\nu = 1$ quantum Hall state at the self-dual point. Using boson-fermion duality, we map the $\nu = 1$ bosonic quantum Hall state to a Fermi sea of the dual fermion and observe that the exact relationships between transport coefficients at the bosonic self-dual point are reproduced by a simple random-phase approximation (RPA), coupled with a Drude formula, in the fermionic theory. We explain this success of the RPA by pointing out a cancellation of a parity-breaking term in the fermion theory which occurs only at the self-dual point, resulting in the fermion self-dual theory explored previously. In addition, we argue that the equivalence of the self-dual structure can be understood in terms of electromagnetic duality or modular invariance, and these features are not inherited by the nonrelativistic cousins of the present model.

DOI: [10.1103/PhysRevB.100.235150](https://doi.org/10.1103/PhysRevB.100.235150)**I. INTRODUCTION**

Dualities provide powerful tools for the study of phases of matter and emergent properties of strongly correlated systems, often allowing one to derive results not accessible by perturbative methods. Particularly useful are strong-weak dualities and self-dualities, which, in many cases, provide quantitative predictions in addition to qualitative insights. Well-known examples include the two-dimensional classical Ising model and the one-dimensional quantum Ising model [1]. In both systems, the location of the phase transition can be found from the condition of self-duality.

Recently, field-theoretical infrared dualities in $2 + 1$ dimensions have attracted the attention of the condensed-matter and high-energy communities. These dualities are the relativistic version of nonrelativistic flux attachment and can also be regarded as a higher-dimensional generalization of $(1 + 1)$ -dimensional bosonization [2,3]. A large number of equivalent pairs of field theories can be derived from a single “seed” boson-fermion duality [4,5]. Models in this web of dualities have found applications in the $\nu = \frac{1}{2}$ quantum Hall problem [6], the strongly interacting surface states of topological insulators [7,8], and the deconfined quantum critical points [9,10]. Progress has been made in trying to come up with a microscopic derivation of the seed duality [11–13] and in extracting quantitative predictions from the dualities [14,15]. In particular, in our previous work [15], we made use of the self-duality property of a certain theory to derive constraints on the physical response at the self-dual point.

The goal of this paper is to extend the method of Ref. [15] to bosonic quantum Hall states. Our model consists of a $(2 + 1)$ -dimensional $[(2 + 1)D]$ Wilson-Fisher boson coupled to a $U(1)$ gauge field propagating in $(3 + 1)$ dimensions and is self-dual at a certain value of the coupling constant. The self-duality allows one, in analogy with the fermionic case

considered in Ref. [15], to derive nontrivial relations between transport coefficients at quantum Hall filling factor $\nu = 1$. These nontrivial relations, Eqs. (21), (22), and (23) below, include a semicircle law for the conductivities, a relationship between the thermal Hall angle and the Hall angle, and a relationship between the bulk thermal Hall conductivity and the thermoelectric coefficients.

The $\nu = 1$ bosonic quantum Hall state allows an interpretation in terms of a Fermi liquid [16] of composite fermions. Therefore, one may pose the question. What are the properties of the composite fermions that guarantee the self-dual properties of the bosonic theory? In this paper we will show that it is a discrete symmetry, which we call T_f , which acts on the composite fermion like a time reversal. The simplest random-phase approximation (RPA) preserves T_f at any value of the gauge coupling [17], but beyond RPA the symmetry is realized only at the self-dual value of the coupling.

This paper is structured as follows. In Sec. II, we first introduce our notations for the building blocks that will be used later for constructing the action. In Sec. III, we review the self-dualities studied in Ref. [15] in a manner that incorporates both bosonic and fermionic particle-vortex dualities. In Sec. IV, we discuss how fractional bosonic quantum Hall states and the self-dual structures can be understood in terms of the composite fermions. In particular, we focus on the $\nu = 1$ state and examine how transport properties at the self-dual point can be understood using a simple fermionic picture. It is found that there the self-dual point on the bosonic side corresponds to a self-dual, time-reversal symmetric point on the fermionic side, and we again elaborate its relation with electromagnetic duality. Such a simultaneously self-dual and time-reversal-symmetric structure can be understood in terms of modular coupling constant and its transformation under $PSL(2, \mathbb{Z})$. At the end, we give a short summary and discuss some open directions.

II. BUILDING BLOCKS

In the field theories that we will consider, the matter field will be a $(2+1)$ -dimensional $O(2)$ Wilson-Fisher (WF) boson, denoted by ϕ or $\tilde{\phi}$, or a two-component Dirac fermion, denoted by ψ or χ . When minimally coupled to gauge fields A , their actions are abbreviated as follows:

$$I_m[\phi, A] = |D_A \phi|^2 - |\phi|^4, \quad (1)$$

$$I_m[\psi, A] = i\bar{\psi}\not{D}_A\psi, \quad (2)$$

where $D_A = \partial - iA$. To write formulas that can be applied equally to the bosonic and fermionic cases, we will denote the matter field by Ψ_p , where $p = 1$ corresponds to the WF boson and $p = 2$ corresponds to the Dirac fermion: $\Psi_1 = \phi$, $\Psi_2 = \psi$.

For gauge fields, in this work we mainly consider three kinds of actions: a $(3+1)$ D Maxwell term I_M and two types of $(2+1)$ D Chern-Simons (CS) terms, I_{BF} and I_{CS} . They are given as follows:

$$I_M[A; e] = -\frac{1}{4e^2} \int d^4x F_{\mu\nu} F^{\mu\nu}, \quad (3)$$

$$I_{BF}[a, A] = \frac{1}{2\pi} \int d^3x \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu A_\lambda, \quad (4)$$

$$I_{CS}[A] = \frac{1}{4\pi} \int d^3x \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda. \quad (5)$$

Note that the Maxwell term involves integrating over a $(3+1)$ -dimensional space-time. On some occasions, it is convenient to consider its reduction to a nonlocal term in $2+1$ dimensions [18]:

$$I_P[A; e] = -\frac{i}{2e^2} \int d^3x d^3x' F_{\mu\nu}(x) \frac{1}{\sqrt{-\partial^2}} F^{\mu\nu}(x'). \quad (6)$$

Finally, the action $I_\theta[A, \Delta\theta]$ refers to an axion action with a θ -angle difference $\Delta\theta$ across a given domain wall:

$$I_\theta[A, \Delta\theta] = \frac{1}{32\pi^2} \int d^4x \theta(z) \epsilon^{\mu\nu\lambda\rho} F_{\mu\nu} F_{\lambda\rho}, \quad (7)$$

where $\theta(z)$ has a jump at $z = 0$, $\theta(+\epsilon) - \theta(-\epsilon) = \Delta\theta$, and is constant elsewhere.

III. REVIEW OF SELF-DUAL MIXED-DIMENSIONAL QED

A. Dualities

In this section we review self-dual theories involving a $(2+1)$ -dimensional field theory on a $z = 0$ brane coupled to a gauge field (3) propagating in $(3+1)$ -dimensional Minkowski space-time. First, let us recall the bosonic [19–21] and fermionic particle-vortex dualities [6–8]. In our notation, these dualities can be written in a uniform way,

$$I_m[\Psi_p, A] \leftrightarrow I_m[\tilde{\Psi}_p, a] + \frac{1}{p} I_{BF}[a, A]. \quad (8)$$

In Eq. (8), A_μ is considered a background (probe) field. We add to both sides of Eq. (8) a $(3+1)$ D Maxwell term (3) and integrate over A_μ . The duality now becomes

$$I_m[\Psi_p, A] + I_M[A; e] \leftrightarrow I_m[\tilde{\Psi}_p, a] + \frac{1}{p} I_{BF}[a, A] + I_M[A; e]. \quad (9)$$

This procedure introduces an exactly marginal coupling e into the theory. The theories on the two sides of Eq. (8) are now of the “brane-world” type, with matter living on a $(2+1)$ D “brane” interacting with a $(3+1)$ D “bulk” field. This type of theory can be regarded as a Lorentz-invariant version of condensed-matter systems, such as graphene or a superconducting thin film, with noninstantaneous Coulomb interaction. In the rest of the paper, we often call the tilde variables $\tilde{\Psi}_p$ the “composite particles” or “vortices” of the ordinary matter fields Ψ_p .

The operator mapping between the two sides of the duality can be found by equating the variations of the two sides of Eq. (8) with respect to A and from the equation of motion obtained by varying the action on the right-hand side of Eq. (8) with respect to a ,

$$J_p^\mu = \frac{1}{2\pi p} \epsilon^{\mu\nu\lambda} \partial_\nu a_\lambda, \quad (10)$$

$$\tilde{J}_p^\mu = -\frac{1}{2\pi p} \epsilon^{\mu\nu\lambda} \partial_\nu A_\lambda. \quad (11)$$

The $\mu = 0$ component reads [22]

$$\rho_p = -\frac{b}{2\pi p}, \quad \tilde{\rho}_p = \frac{B}{2\pi p}. \quad (12)$$

The implication of Eq. (12) is that a particle on one side of the duality maps to a flux on the other side. Moreover, the flux is double in the fermionic case ($p = 2$). Next, we can also look at the relationship between the filling fraction $\nu = 2\pi\rho/B$ on the two sides. From Eq. (12) we find

$$\nu\tilde{\nu} = \frac{2\pi\rho_p}{B} \frac{2\pi\tilde{\rho}_p}{b} = -\frac{1}{p^2}. \quad (13)$$

Similarly, the spatial components of Eqs. (10) and (11) read

$$J_p^i = -\frac{1}{2\pi p} \epsilon^{ij} e^j, \quad \tilde{J}_p^i = \frac{1}{2\pi p} \epsilon^{ij} E^j, \quad (14)$$

where ϵ^{ij} is the antisymmetric tensor, $\epsilon^{12} = -\epsilon^{21} = 1$.

One can take the Gaussian integral over A on the right-hand side (or the vortex side) of (8). The result is

$$I_m[\Psi_p, A] + I_M[A; e] \leftrightarrow I_m[\tilde{\Psi}_p, a] + I_M[a, \tilde{e}], \quad (15)$$

where

$$\tilde{e} = \frac{4\pi p}{e}. \quad (16)$$

The duality is now a strong-weak duality and is reminiscent of electromagnetic duality. In particular, there exists a self-dual point $e^2 = 4\pi p$ at which both sides of the theories become exactly the same. The latter should be of no surprise since the coupling adA is an \mathbf{S} operation on three-dimensional a conformal field theory [23,24]. It also helps explain the existence of the strong-weak duality as a legacy of the modular symmetry.

One can use duality to constrain the physics at finite chemical potential and magnetic field. According to Eq. (13), electromagnetic duality maps a state with filling factor ν to a state with filling factor $\tilde{\nu} = -(p^2\nu)^{-1}$. In particular, in the fermionic case $p = 2$, the duality maps $\nu = 1/2$ to $\tilde{\nu} = -1/2$; that is, it maps a filled zeroth Landau level to an empty one [25]. Combined with time reversal, the duality mapping can relate the physics at filling factors ν and $-\tilde{\nu} = (p^2\nu)^{-1}$.

In particular, at $\nu = \pm p^{-1}$ and when the coupling constant is tuned to the self-dual value, duality combined with time reversal maps the theory to itself, a fact that we will explore in the next section.

B. Transport coefficients

We now use duality to put constraints on transport properties. They include the electronic conductivity σ , thermoelectric tensor α , thermal conductivity in the absence of electric field $\bar{\kappa}$, and thermal conductivity in the absence of electric current κ . They are defined by the following identities (with the p subscript suppressed):

$$J^i = \sigma^{ij} E^j + \alpha^{ij} \partial_j T, \quad (17a)$$

$$q^i = -T \alpha^{ij} E^j - \bar{\kappa}^{ij} \partial_j T, \quad (17b)$$

$$\kappa = \bar{\kappa} - T \alpha \sigma^{-1} \alpha, \quad (17c)$$

where q^i is the heat current. Similar equations apply to the dual theory, where all quantities are replaced by their version with a tilde,

$$\tilde{J}^i = \tilde{\sigma}^{ij} \tilde{E}^j + \tilde{\alpha}^{ij} \partial_j T, \quad (18a)$$

$$q^i = -T \tilde{\alpha}^{ij} \tilde{E}^j - \tilde{\kappa}^{ij} \partial_j T, \quad (18b)$$

$$\tilde{\kappa} = \tilde{\kappa} - T \tilde{\alpha} \tilde{\sigma}^{-1} \tilde{\alpha}. \quad (18c)$$

Note that the temperature T and the heat current q^i are invariant under electromagnetic duality. First, let us look at the J equation. Using the duality mappings (14), it can be shown that

$$\sigma = -\frac{1}{(2\pi p)^2} \epsilon \tilde{\sigma}^{-1} \epsilon, \quad (19a)$$

$$\alpha = (2\pi p) \sigma \epsilon \tilde{\alpha} = \frac{1}{(2\pi p)} \epsilon \tilde{\sigma}^{-1} \tilde{\alpha}, \quad (19b)$$

$$\bar{\kappa} = \tilde{\kappa} - T \tilde{\alpha} \tilde{\sigma}^{-1} \tilde{\alpha} = \tilde{\kappa}. \quad (19c)$$

Identities similar to Eqs. (19) are explicitly found in Ref. [26], where hydrodynamics equipped with bulk electromagnetic duality is studied. In the context of holographic duality, similar results, but with different numerical factors, are found in Ref. [27].

Equations (19) become much more restrictive when a self-dual structure is present. Let us now consider the self-dual value $e^2 = 4\pi p$ and self-dual configuration $\nu = \pm p^{-1}$, and for simplicity we consider the response at infinite wavelength and finite frequency. The conductivity tensor σ , due to rotational invariance, must have the form

$$\sigma^{ij} = \sigma_{xx} \delta^{ij} + \sigma_{xy} \epsilon^{ij}. \quad (20)$$

Under duality, since ν flips sign, σ changes to its transpose. For instance, $\tilde{\sigma} = \sigma^T$. Consequently, we can further simplify Eqs. (19) and conclude that

$$\sigma_{xx}^2 + \sigma_{xy}^2 = \frac{1}{(2\pi p)^2}, \quad (21)$$

$$\frac{\alpha_{xy}}{\alpha_{xx}} = \tan\left(\frac{\pi}{4} + \frac{\theta_H}{2}\right), \quad \theta_H = \arctan \frac{\sigma_{xy}}{\sigma_{xx}}, \quad (22)$$

$$\bar{\kappa}_{xy} = -\kappa_{xy} = \frac{T}{2} \frac{\alpha_{xx}^2 + \alpha_{xy}^2}{1/(2\pi p)}. \quad (23)$$

The Hall angle θ_H is a function of ω/T , where T is the temperature. Although we cannot compute θ_H at arbitrary T , there are two limit cases to be illustrated. In the clean ballistic limit $\omega/T \rightarrow \infty$, $\sigma_{xx} = 0$ and $\sigma_{xy} = \frac{1}{2\pi p} \Rightarrow \theta_H = \frac{\pi}{2}$. In the opposite hydrodynamic limit $T/\omega \rightarrow \infty$, $\sigma_{xy} = 0$, and thus, $\theta_H = 0$. These are the main results obtained in the previous work [15], yet in the present paper we perform the derivation by putting bosonic and fermionic dualities on equal footing. In the rest of the paper, we are going to focus on the $p = 1$ case.

IV. BOSONIC QUANTUM HALL STATE

A. Dual descriptions of the bosonic quantum Hall state

In this section we present the main result, discussing various descriptions of our bosonic theory near filling factor $\nu = 1$. A potential way to realize the bosonic quantum Hall states is by using the rapidly rotating Bose-Einstein condensate (BEC) [28]. It is anticipated that some quantum Hall states will emerge as the Abrikosov lattice melts [29]. We will use the flux attachment picture [30] to guide our intuition.

Particularly interesting is the limiting compressible state at $\nu = 1$. In analogy with the fermionic Halperin-Lee-Read (HLR) $\nu = \frac{1}{2}$ state, this state is expected to be a Fermi liquid of the composite fermions [31,32]. On the other hand, numerical studies on the rotating Bose-Einstein condensate suggested a gapped ground state [29,33], perhaps a Pfaffian state. The theory considered here differs from the one describing the rotating BEC by relativistic invariance and the gauge interactions between the bosons. To simplify further discussion we will assume no pairing instability of the composite fermions, or if there is such an instability, we are at a temperature above the critical temperature.

The minimal ingredient to model the bosonic quantum Hall physics is the Wilson-Fisher boson defined by (1). Making use of the duality web [4,5,34], the theory on the particle side ϕ is dual to a fermionic theory,

$$I_m[\phi, A] + I_M[A; e] \quad (24a)$$

\Downarrow

$$I_m[\psi, a] - \frac{1}{2} I_{CS}[a] - I_{BF}[a, A] - I_{CS}[A] + I_M[A; e], \quad (24b)$$

while on the vortex side $\tilde{\phi}$ one has the duality

$$I_m[\tilde{\phi}, \tilde{a}] + I_{BF}[\tilde{a}, A] + I_M[A; e] \quad (24c)$$

\Downarrow

$$I_m[\chi, c] + \frac{1}{2} I_{CS}[c] - I_{BF}[c, A] + I_{CS}[A] + I_M[A; e]. \quad (24d)$$

We define the physical time reversal as the symmetry under which the Wilson-Fisher boson (24a) is invariant. The naive time reversal and charge conjugation on the fermionic sides of the duality will be denoted as T_f and C_f . They act on the gauge fields as

$$T_f : \alpha_0 \rightarrow \alpha_0, \alpha_i \rightarrow -\alpha_i \quad (\alpha = a, c, A), \quad (25)$$

$$C_f : a_\mu \rightarrow -a_\mu, c_\mu \rightarrow -c_\mu, A_\mu \rightarrow A_\mu, \quad (26)$$

and on the fermionic fields ψ and χ in such a way that the fermionic kinetic terms are invariant.

There are two apparent puzzles: (i) The Lagrangians (24b) and (24d) map to each other under $\mathbf{C}_T\mathbf{T}_f$. On the other hand, the mapping between (24a) and (24c) is nonlocal. (ii) Only one of the four equivalent theories written in Eqs. (24) [the $(2+1)$ -dimensional Wilson-Fisher boson] is manifestly time reversal invariant, whereas the other theories seem not to be, at least classically. These puzzles are explained in Ref. [4]: the invariance under \mathbf{T} , manifesting as a classical symmetry of the Wilson-Fisher boson (24a), emerges on the fermionic sides as a quantum symmetry, which maps a theory to its dual. It has also been pointed out in Ref. [34] that particle-vortex dualities, under bosonization or fermionization, become local symmetry operations.

We will see in the following that the naive time-reversal operation on the fermion side of the duality is analogous to the particle-vortex duality in the Dirac composite fermion theory [6]. On top of that, after introducing I_M there exists a value of e^2 at which the fermionic theories are \mathbf{T}_f invariant.

The four theories defined in Eqs. (24) are equivalent, so a given state can be described in all four theories. A state with a certain filling factor ν_ϕ in the original theory of ϕ maps to states with different filling factors in the other three theories. We can look at the duality mappings given by these theories by varying Lagrangians with respect to A , a , \tilde{a} , and c :

$$J_{\text{phys}}^\mu = J_\phi^\mu = -\frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu a_\lambda - \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu A_\lambda \quad (27)$$

$$= \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu \tilde{a}_\lambda = \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu A_\lambda - \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu c_\lambda, \quad (28)$$

$$J_\psi^\mu = \frac{1}{4\pi}\epsilon^{\mu\nu\lambda}\partial_\nu a_\lambda + \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu A_\lambda, \quad (29)$$

$$J_{\tilde{\phi}}^\mu = -\frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu A_\lambda, \quad (30)$$

$$J_\chi^\mu = -\frac{1}{4\pi}\epsilon^{\mu\nu\lambda}\partial_\nu c_\lambda + \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu A_\lambda. \quad (31)$$

First, we look at the zeroth components of (27) and (29):

$$\rho_\phi = \frac{b}{2\pi} + \frac{B}{2\pi} \Rightarrow \nu_\phi = \frac{\langle b \rangle}{\langle B \rangle} + 1, \quad (32)$$

$$\rho_\psi = -\frac{b}{4\pi} - \frac{B}{2\pi}. \quad (33)$$

Therefore, if the original boson has $\nu_\phi = 1$, then $\langle b \rangle = 0$. The ψ fermions have a finite density $\rho_\psi = -\langle B \rangle / (2\pi)$ and live in an average zero magnetic field and therefore can form a Fermi liquid.

At more general filling fractions, we find

$$(\nu_\phi - 1)\left(\nu_\psi + \frac{1}{2}\right) = -1. \quad (34)$$

In particular, if ψ forms an integer quantum Hall state with $\nu_\psi = n + \frac{1}{2}$, then

$$\nu_\phi = 1 - \frac{2}{2\nu_\psi + 1} = \frac{n}{n+1}, \quad (35)$$

which are the Jain sequences for bosons [33]. Equation (35) is reminiscent of the conventional flux attachment since duality is in essence the relativistic counterpart. Note that if

TABLE I. Some examples of filling fractions under duality mapping.

Field	ν	$\nu(n = \infty)$	$\nu(n = 0)$
ϕ	$\frac{n}{n+1}$	1	0
$\tilde{\phi}$	$-\frac{n+1}{n}$	-1	∞
ψ	$n + \frac{1}{2}$	∞	$\frac{1}{2}$
χ	$\frac{-1}{2(2n+1)}$	0	$-\frac{1}{2}$

we choose $\nu_\psi = -(n + \frac{1}{2})$, then $\nu_\phi = \frac{n+2}{n+1} = 2 - \frac{n}{n+1}$. The transition from filling factor ν to $2 - \nu$ is the ‘‘particle-hole’’ transformation for bosons, considered in Ref. [35]. Here we see that a symmetry operation corresponds to the time reversal of composite fermions, which flips the sign of ν_ψ . However, since such time reversal is not the symmetry of the fermionic theory, the physics of the bosonic states with filling factors ν and $2 - \nu$ are not equivalent.

Straightforwardly, one can make use of Eq. (30) to eliminate A in Eqs. (28) and (31) and derive the mapping between the filling fractions in the $\tilde{\phi}$ and the χ theories. It turns out the relation is the same as Eq. (34),

$$(\nu_{\tilde{\phi}} - 1)\left(\nu_\chi + \frac{1}{2}\right) = -1. \quad (36)$$

Since the theories involving ϕ and $\tilde{\phi}$ are particle-vortex duals of each other, the connection between the filling factors in the two theories is given by the standard relation

$$\nu_\phi \nu_{\tilde{\phi}} = -1, \quad (37)$$

and from Eqs. (34) and (36), we obtain

$$\nu_\psi \nu_\chi = -\frac{1}{4}. \quad (38)$$

In particular, when the original boson is in the Jain state with $\nu_\phi = \frac{n}{n+1}$, the χ fermion has filling factor $\nu_\chi = -\frac{1}{2(2n+1)}$, which is a fractional quantum Hall state (a Jain state). The $\nu_\phi = 1$ state corresponds to $\nu_\chi = 0$, i.e., the half-filled Landau level of χ . We list some examples in Table I.

B. Fermionic representations of bosonic observables

In Sec. III B we reviewed relations between transport coefficients, Eqs. (19a), (19b), and (19c), in theories that map to each other under particle-vortex duality. We can also derive the connection between the transport coefficients between the bosonic and fermionic sides of each duality (24). We write down the spatial components of Eqs. (27) and (29) and introduce the relevant transport coefficients,

$$\mathbf{J}_\phi = \sigma_\phi \mathbf{E} + \alpha_\phi \nabla T = \frac{1}{2\pi} \epsilon \mathbf{e} + \frac{1}{2\pi} \epsilon \mathbf{E}, \quad (39)$$

$$\mathbf{J}_\psi = \sigma_\psi \mathbf{e} + \alpha_\psi \nabla T = -\frac{1}{4\pi} \epsilon \mathbf{e} - \frac{1}{2\pi} \epsilon \mathbf{E}. \quad (40)$$

In the same manner as in Ref. [17], the consistency between the two equations requires

$$\epsilon \left(\sigma_\phi - \frac{\epsilon}{2\pi} \right) \epsilon^{-1} \left(\sigma_\psi + \frac{\epsilon}{4\pi} \right) = \frac{1}{(2\pi)^2}, \quad (41a)$$

$$\alpha_\phi = -\frac{\epsilon}{2\pi} \left(\sigma_\psi + \frac{\epsilon}{4\pi} \right)^{-1} \alpha_\psi. \quad (41b)$$

In addition to electrical and thermoelectric responses, we further look at thermal transport. As the heat current should have the same form in either of the dual theories, $\mathbf{q} = -\bar{\kappa}_\phi \nabla T - \alpha_\phi T \mathbf{E} = -\bar{\kappa}_\psi \nabla T - \alpha_\psi T \mathbf{e}$, we can then obtain

$$\bar{\kappa}_\phi = \bar{\kappa}_\psi - \alpha_\psi \left(\sigma_\psi + \frac{1}{4\pi} \right)^{-1} \alpha_\psi T. \quad (41c)$$

It is straightforward to show these relationships hold with the replacement $\phi \rightarrow \tilde{\phi}$ and $\psi \rightarrow \chi$.

One can also verify that if the transport coefficients in the two bosonic (ϕ and $\tilde{\phi}$) theories are related by Eqs. (19) with $p = 1$, then Eqs. (41) (and similar equations with the replacement $\phi \rightarrow \tilde{\phi}$ and $\psi \rightarrow \chi$) imply that the transport coefficients in the two fermionic theories satisfy the duality relations with $p = 2$. This is consistent with ψ and χ being related by particle-vortex duality.

C. Transport in self-dual boson: A fermionic view

We now look at the self-dual $\nu_\phi = 1$ state and show that the constraints that follow from duality can be understood using the fermionic picture, in which ψ forms a Fermi surface. At self-duality $(\sigma_\phi^{xx})^2 + (\sigma_\phi^{xy})^2 = \frac{1}{(2\pi)^2}$, and one can parametrize $\sigma_\phi^{xx} = \frac{1}{2\pi} \cos \theta_H$ and $\sigma_\phi^{xy} = \frac{1}{2\pi} \sin \theta_H$, where θ_H is the Hall angle. Using Eq. (41a), we have

$$\sigma_\psi^{xx} = \frac{1}{4\pi} \tan \left(\frac{\theta_H}{2} + \frac{\pi}{4} \right), \quad \sigma_\psi^{xy} = 0. \quad (42)$$

In reverse, if the Hall conductivity of the composite fermion σ_ψ^{xy} vanishes, then the bosonic conductivity tensor satisfies the self-duality constraint.

Note that the average magnetic field acting on the composite fermion is zero, and the vanishing of σ_ψ^{xy} takes place in the simple Drude model of transport. It is instructive to derive all three self-duality constraints for bosonic transport from the Drude model of the composite fermion. This model gives only diagonal transport tensors

$$\sigma_\psi^{ij} = \sigma_\psi \delta^{ij}, \quad \alpha_\psi^{ij} = \alpha_\psi \delta^{ij}, \quad \kappa_\psi^{ij} = \kappa_\psi \delta^{ij}. \quad (43)$$

Plugging σ_ψ^{ij} into Eq. (41a), we find

$$\begin{aligned} \sigma_\phi^{xx} &= \frac{1}{4\pi^2} \frac{\sigma_\psi^{xx}}{(\sigma_\psi^{xx})^2 + \frac{1}{(4\pi)^2}}, \\ \sigma_\phi^{xy} &= \frac{1}{2\pi} - \frac{1}{16\pi^3} \frac{1}{(\sigma_\psi^{xx})^2 + \frac{1}{(4\pi)^2}} \\ &\Rightarrow (\sigma_\phi^{xx})^2 + (\sigma_\phi^{xy})^2 = \frac{1}{(2\pi)^2}. \end{aligned} \quad (44)$$

Given the above, we can parametrize σ_ϕ using the Hall angle. Then by the same token, using Eqs. (41b) and (42), we find

$$\frac{\alpha_\phi^{xy}}{\alpha_\phi^{xx}} = 4\pi \sigma_\psi^{xx} = \tan \left(\frac{\pi}{4} + \frac{\theta_H}{2} \right). \quad (45)$$

Finally, we look at the difference $\bar{\kappa}_\phi^{xy} - \bar{\kappa}_\phi^{yx}$. Using Eqs. (41c) and (42), it is

$$\bar{\kappa}_\phi^{xy} - \bar{\kappa}_\phi^{yx} = \frac{8\pi T \alpha_\psi^2}{(16\pi^2 \sigma_\psi^2 + 1)} = 8\pi T \alpha_\psi^2 \cos^2 \left(\frac{\pi}{4} + \frac{\theta}{2} \right).$$

To find α_ψ^2 in terms of α_ϕ , we use (41b) and (45) to get

$$\alpha_\psi^2 = \frac{1}{4} \sec^2 \left(\frac{\pi}{4} + \frac{\theta}{2} \right) \left[(\alpha_\phi^{xx})^2 + (\alpha_\phi^{xy})^2 \right],$$

and as a consequence,

$$\bar{\kappa}_\phi^{xy} - \bar{\kappa}_\phi^{yx} = 2\pi T \left[(\alpha_\phi^{xx})^2 + (\alpha_\phi^{xy})^2 \right]. \quad (46)$$

To summarize, the exact relationships (21), (22), and (23), which are the consequences of self-duality in the bosonic theory, can be derived assuming (43) on the fermion side, which would appear naturally in the simplest Drude model of transport.

The above ‘‘derivation’’ of the self-duality constraints on bosonic transport gives rise to a puzzle. If we recall the derivations in the self-dual QEDs, these identities hold under two assumptions: (i) $e^2 = 4\pi p$, and (ii) the matter field is tuned at self-dual filling $\nu = 1/p$. On the other hand, the derivation through the fermionic Drude model does not seem to require one to tune $e^2 = 4\pi$. In the literature, a similar argument was used [17] to derive the semicircle law (21) in the absence of any self-duality. In the following section, we explain why the fine-tuning of e^2 is required for the derivation to work.

D. Self-duality and manifest T_r symmetry

In the previous section, we showed that all self-dual properties of boson field ϕ can be understood in terms of those of ψ at the level of Drude approximation. This argument does not make use of the value of e and thereby is independent of self-dual structure, implying that the exact relationships between transport coefficients that we have derived using self-duality are valid even when the coupling constant is away from the self-dual point. In this section we show that the naive argument is not correct and, indeed, the fine tuning of the coupling constant is required.

We note that those transport properties hold as long as $(\sigma_\psi, \alpha_\psi, \bar{\kappa}_\psi, \kappa_\psi)$ have no off-diagonal and parity-breaking components and are isotropic, being proportional to δ^{ij} . We have also argued that, since the average magnetic field is zero, Drude approximation indeed gives zero off-diagonal transport coefficients. However, the approximation neglects gauge field fluctuations. In a theory where the dynamics of the gauge field violates parity and time-reversal symmetries, a contribution of the type depicted in the two-loop self-energy diagram in Fig. 1 necessarily introduces nonzero Hall transport.

It remains for us to explain why at a particular value of the gauge coupling, namely, the self-dual value $e^2 = 4\pi$, gauge field fluctuations do not lead to nondiagonal transport coefficients.

To do that, we integrate over A in Eq. (24b) and obtain an effective action with a nonlocal term for a (a pseudo quantum electrodynamics in the terminology of Ref. [18]):

$$I_{\text{eff}} = I_m[\psi, a] + \frac{e^2 - \tilde{e}^2}{2(e^2 + \tilde{e}^2)} I_{\text{CS}}[a] + I_P[a; g], \quad (47)$$

$$\frac{1}{g^2} = \frac{1}{e^2} \frac{1}{[1 + (4\pi/e^2)^2]} = \frac{1}{e^2 + \tilde{e}^2}. \quad (48)$$

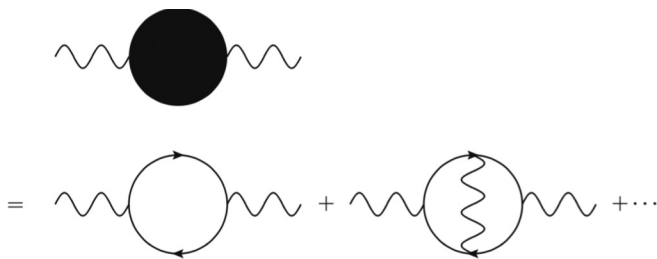


FIG. 1. In the present work, transport coefficients are defined as the 1PI diagram in the first line. In the second line, the leading diagram on the left is given by the RPA, which can be isotropic. As higher-order diagrams such as the right one in the second line are included, $\langle aa \rangle$ will contribute to parity-violating off-diagonal terms at general e^2 and invalidate the assumption of Drude values.

We see that at $e^2 = 4\pi$, the integration procedure completely cancels out $-\frac{1}{8\pi}ada$ in the original Lagrangian. This provides an explanation for the puzzle: only at this value of e^2 would the off-diagonal transport coefficient in the fermionic theory remain zero once one goes beyond the mean-field approximation and integrates over gauge field fluctuations.

Note that the coincidence between self-dual and time-reversal restoration is also found and discussed in the wire-construction version of the model [34].

E. Electromagnetic duality revisited

In Sec. IV D we saw that, as $e^2 = 4\pi$, the fermionic theories are also tuned to their self-dual point. It is instructive to examine this fact from the point of view of electromagnetic duality in the bulk. In this section we show that at $e^2 = 4\pi$, the gauge field seen by ψ is the electromagnetic dual of one seen by χ . To avoid confusion, we introduce an index to distinguish the gauge field that couples to the matter field in each of the four theories, so

$$\begin{aligned} \phi \text{ couples to } A, \quad \psi \text{ couples to } a_\psi, \\ \tilde{\phi} \text{ couples to } a_{\tilde{\phi}}, \quad \chi \text{ couples to } a_\chi. \end{aligned} \quad (49)$$

For a gauge field whose action contains a $(3+1)D$ Maxwell term I_M , one can impose certain orbifold conditions [15]. For example, in the theory of ϕ , one can require

$$\begin{aligned} A_\mu(z) &= A_\mu(-z), \quad \mu = t, x, y, \\ A_z(z) &= -A_z(-z). \end{aligned} \quad (50)$$

Such orbifold conditions can be imposed without changing the theory because the parts of A with opposite parities (odd part of A_μ and even part of A_z) decouple [15].

We first relate fields A and a_ψ . Using the boundary condition for the bulk Maxwell equation and the relations between J_ϕ^μ , a_ψ , and A , Eqs. (27) and (29), we have

$$\frac{2\pi}{e^2} \Delta E_z = \frac{4\pi}{e^2} E_z(0^+) = b_\psi + B_z, \quad (51a)$$

$$\frac{2\pi}{e^2} \Delta \mathbf{B}_\parallel = \frac{4\pi}{e^2} \mathbf{B}_\parallel(0^+) = -\mathbf{e}_\psi - \mathbf{E}_\parallel. \quad (51b)$$

In the formulas that follow, quantities that are discontinuous across the plane $z = 0$ will be assumed to be evaluated at $z =$

$+0$ unless we explicitly specify otherwise. Next, we perform an electromagnetic duality transformation in the bulk:

$$\mathbf{E} = -\text{sgn}(z)\tilde{\mathbf{B}}, \quad (52)$$

$$\mathbf{B} = \text{sgn}(z)\tilde{\mathbf{E}}. \quad (53)$$

Note that we have chosen opposite sign conventions in the definition of the dual gauge fields \mathbf{E} and \mathbf{B} on the two sides of the brane. Equations (51) now can be rewritten as

$$-\frac{4\pi}{e^2} \tilde{B}_z = b_\psi + \tilde{E}_z, \quad (54)$$

$$\frac{4\pi}{e^2} \tilde{\mathbf{E}}_\parallel = -\mathbf{e}_\psi + \tilde{\mathbf{B}}_\parallel. \quad (55)$$

As suggested by this relation, in the dual ψ theory, we extend gauge fields a_ψ into the bulk by

$$\begin{pmatrix} \mathbf{e}_\psi \\ \mathbf{b}_\psi \end{pmatrix} = \frac{1}{e} \begin{pmatrix} -\tilde{e} & e \\ -e & -\tilde{e} \end{pmatrix} \begin{pmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{B}} \end{pmatrix}, \quad \tilde{e} = \frac{4\pi}{e}. \quad (56)$$

The relation between A and $a_{\tilde{\phi}}$ was shown previously in Ref. [15]. Because of particle vortex duality,

$$\rho_{\tilde{\phi}} = \frac{1}{2\pi} B_z = \frac{1}{2\pi} \tilde{E}_z(0^+), \quad (57)$$

$$\mathbf{J}_{\tilde{\phi}} = \frac{1}{2\pi} \epsilon \mathbf{E}_\parallel = -\frac{1}{2\pi} \epsilon \tilde{\mathbf{B}}_\parallel(0^+). \quad (58)$$

$a_{\tilde{\phi}}$ is identified on the brane and extended to the bulk with A (or \tilde{A}) via

$$a_{\tilde{\phi}} = \frac{4\pi}{e^2} \tilde{A}. \quad (59)$$

Finally, we relate a_χ and A using (28), (30), (31), (57), and (58). After eliminating J_ϕ^μ , in terms of \tilde{A} ,

$$\tilde{E}_z = b_\chi + \frac{4\pi}{e^2} \tilde{B}_z, \quad (60)$$

$$\tilde{\mathbf{B}}_\parallel = -\mathbf{e}_\chi - \frac{4\pi}{e^2} \tilde{\mathbf{E}}_\parallel. \quad (61)$$

Again, as suggested by this relation, in χ theory we extend a_χ into the bulk by

$$\begin{pmatrix} \mathbf{e}_\chi \\ \mathbf{b}_\chi \end{pmatrix} = \frac{1}{e} \begin{pmatrix} -\tilde{e} & -e \\ e & -\tilde{e} \end{pmatrix} \begin{pmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{B}} \end{pmatrix}. \quad (62)$$

Now we can relate a_ψ and a_χ by these relations:

$$\begin{pmatrix} \mathbf{e}_\chi \\ \mathbf{b}_\chi \end{pmatrix} = \frac{1}{e^2 + \tilde{e}^2} \begin{pmatrix} \tilde{e}^2 - e^2 & 8\pi \\ -8\pi & \tilde{e}^2 - e^2 \end{pmatrix} \begin{pmatrix} \mathbf{e}_\psi \\ \mathbf{b}_\psi \end{pmatrix}. \quad (63)$$

Therefore, at $e^2 = 4\pi$,

$$\mathbf{e}_\chi = \mathbf{b}_\psi, \quad (64)$$

$$\mathbf{b}_\chi = -\mathbf{e}_\psi. \quad (65)$$

To summarize, at any e^2 , a_ϕ and A are related by electromagnetic duality and a_ψ and a_χ can be related by (63). However, at $e^2 = 4\pi$, where ϕ and $\tilde{\phi}$ become self-dual, a_χ and a_ψ also become the electromagnetic duals of each other.

Continuing this procedure, we may look at the effective action led by this integration process. Using duality mapping (29),

$$\rho_\psi = -\frac{b_{\psi z}}{4\pi} - \frac{\tilde{E}_z}{2\pi}, \quad (66)$$

$$\mathbf{J}_\psi = -\frac{1}{4\pi} \mathbf{e}_{\psi\parallel} \times \hat{\mathbf{z}} + \frac{1}{2\pi} \tilde{\mathbf{B}}_\parallel \times \hat{\mathbf{z}}. \quad (67)$$

Inverting (56), the above equations become

$$e_{\psi z} = \frac{e^2 + \tilde{e}^2}{2} \left(\rho_\psi - \frac{1}{4\pi^2} \frac{(e^2 - \tilde{e}^2)\pi}{e^2 + \tilde{e}^2} b_{\psi z} \right), \quad (68)$$

$$\mathbf{b}_{\psi\parallel} = \frac{e^2 + \tilde{e}^2}{2} \left(\mathbf{J}_\psi \times \hat{\mathbf{z}} + \frac{1}{4\pi^2} \frac{(e^2 - \tilde{e}^2)\pi}{e^2 + \tilde{e}^2} \mathbf{e}_{\psi\parallel} \right). \quad (69)$$

Similarly, inverting (62) gives us

$$e_{\chi z} = \frac{e^2 + \tilde{e}^2}{2} \left(\rho_\chi + \frac{1}{4\pi^2} \frac{(e^2 - \tilde{e}^2)\pi}{e^2 + \tilde{e}^2} b_{\chi z} \right), \quad (70)$$

$$\mathbf{b}_{\chi\parallel} = \frac{e^2 + \tilde{e}^2}{2} \left(\mathbf{J}_\chi \times \hat{\mathbf{z}} - \frac{1}{4\pi^2} \frac{(e^2 - \tilde{e}^2)\pi}{e^2 + \tilde{e}^2} \mathbf{e}_{\chi\parallel} \right). \quad (71)$$

Referring to the general theory of axion electrodynamics [36], these boundary conditions are given by the actions

$$I_m[\psi, a_\psi] + I_M[a_\psi, g] + \frac{1}{4\pi^2} \int d^4x \theta(z) \mathbf{e}_\psi \cdot \mathbf{b}_\psi, \quad (72)$$

$$I_m[\chi, a_\chi] + I_M[a_\chi, g] - \frac{1}{4\pi^2} \int d^4x \theta(z) \mathbf{e}_\chi \cdot \mathbf{b}_\chi, \quad (73)$$

$$g^2 = e^2 + \tilde{e}^2, \quad \theta(z) = \frac{\text{sgn}(z)}{2} \left(\frac{e^2 - \tilde{e}^2}{e^2 + \tilde{e}^2} \right) \pi. \quad (74)$$

The choice of the θ term accommodates the orbifold boundary conditions. From this perspective, we learn that in the presence of the bulk fluctuation, bosonization dualities have the following structure:

$$I_m[\phi, A] + I_M[A, e] \leftrightarrow I_m[\psi, a] + I_M[a, g] + I_\theta[a, \Delta\theta]$$

$$\updownarrow \qquad \qquad \qquad \updownarrow$$

$$I_m[\tilde{\phi}, \tilde{A}] + I_M[\tilde{A}, \tilde{e}] \leftrightarrow I_m[\chi, a] + I_M[a, g] + I_\theta[a, -\Delta\theta].$$

In terms of the bulk field actions, the manifest \mathbb{T} -symmetric point occurs when we flow e^2 to \tilde{e}^2 , tuning $\Delta\theta = 0$.

V. MODULAR INVARIANCE

We have defined the mixed-dimensional QED so that the gauge field propagates on both sides of the $(2+1)$ D brane. In Ref. [15] we compared this picture with the alternative picture often used in the literature (for example, in Ref. [4]), where the brane is the boundary of space and the bulk fields propagate only on one side of it. There we see that the self-dual structure is also evident in the latter model, in which fermion and boson self-dual couplings are $g_f^2 = 4\pi$ and $g_b^2 = 2\pi$, respectively.

Let us argue that the fermionic self-dual point $g_f^2 = 4\pi$ is equivalent to $g_b^2 = 2\pi$ from this perspective. More concretely, we consider a bulk electromagnetic action with complex coupling constant τ with a boundary matter minimally coupled to the bulk field. The mixed Chern-Simons term adA and the Chern-Simons term AdA are regarded as the results of \mathbf{S} and

\mathbb{T} . The action of deforming action, via a Gaussian integral, is rephrased in terms of mapping coupling constants via [23,24]

$$\mathbf{S}[\tau] = -\frac{1}{\tau}, \quad (75)$$

$$\mathbb{T}[\tau] = \tau + 1. \quad (76)$$

We denote the bulk action as $I_M[\tau]$ and the boundary minimally coupled matter action as $I_{b/f}[\dots]$. As a warm-up, we may recall the fermionic self-dual theories can be stated concisely as

$$I_f[\psi] + I_M[\tau_f] \leftrightarrow I_f[\tilde{\psi}] + I_M[\tilde{\tau}_f], \quad (77)$$

with

$$\tilde{\tau}_f = \mathbf{ST}^{-2}\mathbf{ST}^{-1}[\tau_f] = \frac{\tau_f - 1}{2\tau_f - 1}. \quad (78)$$

Similarly, the bosonic self-dual theories are stated as

$$I_b[\phi] + I_M[\tau_b] \leftrightarrow I_b[\tilde{\phi}] + I_M[\tilde{\tau}_b], \quad (79)$$

with

$$\tilde{\tau}_b = \mathbf{S}[\tau_b] = -\frac{1}{\tau_b}. \quad (80)$$

The boson-fermion duality states that

$$I_b[\phi] + I_M[\tau_b] \leftrightarrow I_f[\psi] + I_M[\tau_f], \quad (81)$$

where [37]

$$\tau_f = \frac{1}{1 - \tau_b}, \quad \mathbb{T}\mathbf{S}[\tau_f] = \tau_b. \quad (82)$$

At the bosonic self-dual point $\tau_b = \tilde{\tau}_b = i$, we see fermion theories are also pinned at the self-dual point $\tau_f = \tilde{\tau}_f = \frac{1}{2}(1 + i)$. To avoid possible confusion, we note that up to a \mathbb{T} operation the self-dual coupling of fermions can also be

$$\tau_f = -\frac{1}{2} + \frac{i}{2}. \quad (83)$$

This picture can provide additional understanding of the fact that the parity-violating term in the effective action vanishes at the self-dual point. The relation between τ_ϕ and τ_ψ is given by

$$\tau_\phi = -\frac{1}{\tau_\psi} - 1 = \mathbb{T}^{-1}\mathbf{S}[\tau_\psi]. \quad (84)$$

Taking $\tau_\phi = i\tau''_\phi$,

$$\tau_\psi = -\frac{1}{1 + i\tau''_\phi} = -\frac{1}{1 + (\tau''_\phi)^2} (1 - i\tau''_\phi). \quad (85)$$

The parity-invariant condition

$$\text{Re}[\tau_\psi] = -\frac{1}{2} \Rightarrow \tau''_\phi = 1, \quad \text{Im}[\tau_\psi] = \frac{1}{2}. \quad (86)$$

The same argument applies in the reverse, where

$$\tau_\psi = -\frac{1}{\tau_\phi + 1} = \mathbf{ST}[\tau_\phi]. \quad (87)$$

Taking $\tau_\psi = -\frac{1}{2} + i\frac{1}{2}\tau''_\psi$,

$$\tau_\phi = \frac{1 - (\tau''_\psi)^2}{1 + (\tau''_\psi)^2} + \frac{2i}{1 + (\tau''_\psi)^2}. \quad (88)$$

The parity-invariant condition

$$\text{Re}[\tau_\phi] = 0 \Rightarrow \tau_\psi'' = 1, \quad \tau_\phi = i. \quad (89)$$

As a simple exercise, we can also redefine the coupling constants such that τ_ϕ and τ_ψ share the same self-dual point as found in Refs. [34,38]. Explicitly, we define $-1/z_f = 2\tau_f - 1$ and $z_b = \tau_b$. Equation (82) becomes

$$z_f = \frac{z_b - 1}{z_b + 1}. \quad (90)$$

The self-dual point is moved to $z_f = z_b = i$.

VI. CONCLUSION

To summarize, In the first half of the paper we reviewed the self-dualities led by combining $(2+1)$ -dimensional particle-vortex dualities and $3+1$ U(1) gauge fields and conjectured a relation with \mathbf{S} duality. Those dualities also imply mappings between transport properties. In particular, at self-dual points and self-dual states, these mappings become more concrete

constraints on the entries of transport tensors. In the second part, we studied the $\nu = 1$ bosonic quantum Hall state using a description in terms of a Dirac composite fermion. We showed that the self-duality constraints on transport in the bosonic theory can be understood very easily as a consequence of the discrete symmetry of the composite fermion. We hope that the model studied in this paper will provide additional insights into the composite fermions in quantum Hall states.

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- [1] J. B. Kogut, *Rev. Mod. Phys.* **51**, 659 (1979).
 [2] S. Coleman, *Phys. Rev. D* **11**, 2088 (1975).
 [3] S. Mandelstam, *Phys. Rev. D* **11**, 3026 (1975).
 [4] N. Seiberg, T. Senthil, C. Wang, and E. Witten, *Ann. Phys. (NY)* **374**, 395 (2016).
 [5] A. Karch and D. Tong, *Phys. Rev. X* **6**, 031043 (2016).
 [6] D. T. Son, *Phys. Rev. X* **5**, 031027 (2015).
 [7] C. Wang and T. Senthil, *Phys. Rev. X* **5**, 041031 (2015).
 [8] M. A. Metlitski and A. Vishwanath, *Phys. Rev. B* **93**, 245151 (2016).
 [9] C. Wang, A. Nahum, M. A. Metlitski, C. Xu, and T. Senthil, *Phys. Rev. X* **7**, 031051 (2017).
 [10] Y. Q. Qin, Y.-Y. He, Y.-Z. You, Z.-Y. Lu, A. Sen, A. W. Sandvik, C. Xu, and Z. Y. Meng, *Phys. Rev. X* **7**, 031052 (2017).
 [11] S. Kachru, M. Mulligan, G. Torroba, and H. Wang, *Phys. Rev. D* **94**, 085009 (2016).
 [12] S. Kachru, M. Mulligan, G. Torroba, and H. Wang, *Phys. Rev. Lett.* **118**, 011602 (2017).
 [13] J.-Y. Chen, J. H. Son, C. Wang, and S. Raghu, *Phys. Rev. Lett.* **120**, 016602 (2018).
 [14] M. Cheng and C. Xu, *Phys. Rev. B* **94**, 214415 (2016).
 [15] W.-H. Hsiao and D. T. Son, *Phys. Rev. B* **96**, 075127 (2017).
 [16] Strictly speaking, it is a marginal Fermi liquid.
 [17] M. Mulligan, *Phys. Rev. B* **95**, 045118 (2017).
 [18] E. C. Marino, *Nucl. Phys. B* **408**, 551 (1993).
 [19] M. E. Peskin, *Ann. Phys. (NY)* **113**, 122 (1978).
 [20] C. Dasgupta and B. I. Halperin, *Phys. Rev. Lett.* **47**, 1556 (1981).
 [21] M. P. A. Fisher and D. H. Lee, *Phys. Rev. B* **39**, 2756 (1989).
 [22] We use the $(+ - -)$ signature for the metric and the convention of Ref. [39]; in particular, $F_{12} = -B$ and $F_{0i} = E^i$.
 [23] A. Kapustin and M. J. Strassler, *J. High Energy Phys.* **04** (1999) 021.
 [24] E. Witten, [arXiv:hep-th/0307041](https://arxiv.org/abs/hep-th/0307041).
 [25] The spectrum of the Dirac fermion in a magnetic field contains a Landau level at zero energy. The $\nu = 1/2$ state here refers to the state filling this zeroth Landau level. The nonrelativistic filling factor $\nu_{NR} = \frac{1}{2}$ corresponds to filling half of the zeroth Landau level $\nu = 0$ in this context.
 [26] A. Donos, J. P. Gauntlett, T. Griffin, N. Lohitsiri, and L. Melgar, *J. High Energy Phys.* **07** (2017) 006.
 [27] S. A. Hartnoll and C. P. Herzog, *Phys. Rev. D* **76**, 106012 (2007).
 [28] A. L. Fetter, *Rev. Mod. Phys.* **81**, 647 (2009).
 [29] N. R. Cooper, N. K. Wilkin, and J. M. F. Gunn, *Phys. Rev. Lett.* **87**, 120405 (2001).
 [30] J. K. Jain, *Phys. Rev. Lett.* **63**, 199 (1989).
 [31] V. Pasquier and F. Haldane, *Nucl. Phys. B* **516**, 719 (1998).
 [32] N. Read, *Phys. Rev. B* **58**, 16262 (1998).
 [33] N. Regnault and T. Jolicoeur, *Phys. Rev. Lett.* **91**, 030402 (2003).
 [34] D. F. Mross, J. Alicea, and O. I. Motrunich, *Phys. Rev. X* **7**, 041016 (2017).
 [35] C. Wang and T. Senthil, *Phys. Rev. B* **94**, 245107 (2016).
 [36] F. Wilczek, *Phys. Rev. Lett.* **58**, 1799 (1987).
 [37] It may differ by a T operation depending on how we choose the real part of τ_f . More precisely, this is the relation between the ϕ and χ coupling constants.
 [38] H. Goldman and E. Fradkin, *Phys. Rev. B* **97**, 195112 (2018).
 [39] L. D. Landau and E. M. Lifshitz, *Classical Theory of Fields*, 4th ed. (Pergamon, Oxford, 1975).