Quantum Monte Carlo study of the superfluid density in quasi-one-dimensional systems of hard-core bosons: Effect of the suppression of phase slippage

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We study the superfluid density of hard-core bosons on quasi-one-dimensional lattices using the quantum Monte Carlo method. Because of phase slippage, the superfluid density drops quickly to zero at finite temperatures with increasing the system length ℓ and the superfluid transition temperature is zero in one spatial dimension and also in quasi-one dimension in the limit of $\ell \to \infty$. We calculate the superfluid density of a model where no phase slippage is allowed and show that the superfluid density remains finite at finite temperatures even in the one-dimensional limit. We also discuss how finite superfluid density can be observed in a quasi-one-dimensional system using a torsional oscillator.

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I. INTRODUCTION

Superfluidity is one of the most fascinating phenomena in condensed matter systems. It is well known that the spatial dimensionality plays a remarkable role in superfluidity. In two dimensions, the Berezinskii-Kosterlitz-Thouless (BKT) transition [1,2], a unique topological phase transition, occurs at a finite temperature despite the absence of true long-range order [3].

Recently, ⁴He atoms confined in straight nanopores have attracted much attention [4–17]. For example, Wada and coworkers studied He atoms adsorbed on the inner walls of one-dimensional pores of porous material, FSM (folded sheet mesoporous materials)-16 [4–7]; typical pore length is 200–300 nm and its diameter *R* can be systematically changed from R = 1.5 nm to 4.7 nm. Superfluid density was measured using a torsional oscillator and a frequency shift was found to set in at a temperature close to the BKT transition temperature, T_{BKT} [1,2], determined by the areal density of adsorbed He atoms. They carefully analyzed the results and concluded that the frequency shift was caused by finite superfluid density in the one-dimensional part of the system, i.e., the one-dimensional He tube [6,7].

On the other hand, Taniguchi and Suzuki studied superfluidity of liquid ⁴He filling nanopores [8–10,12,14]. Superfluid density was then measured with a torsional oscillator. They found a two-step increase, that is, they found an additional increase in the resonance frequency at a temperature lower than the bulk λ transition temperature. They also ascribed the second increase to the onset of superfluidity of liquid He filling one-dimensional pores [8–10].

In these experiments, three-dimensionality plays only a minor role in contrast to the previous experiments using interconnected porous materials [3,18–22], because the pores are connected only via their ends. Moreover, as the pores are regularly arranged, randomness caused by irregular connection of pores, which has significant effects in Vycor glass, for example, is also considered to be irrelevant.

It is a well-established fact that no Bose-Einstein condensation (BEC) occurs at a finite temperature in one or two dimensions [23,24]. However, the relation of existence or absence of BEC to superfluidity is not necessarily a well-understood problem [25]. In particular, we should note that superfluidity is detected dynamically in torsional oscillator experiments. Therefore, in discussing superfluidity in one dimension or quasi-one dimension, a dynamical aspect of the phenomenon has to be considered. For example, Shevchenko showed that the characteristic temperature for one-dimensional superfluidity is given by

$$T_{\rm c} \sim \frac{\hbar^2 n_1}{k_{\rm B} M \ell_z},\tag{1}$$

where n_1 is the one-dimensional number density of boson atoms, M the atomic mass, and ℓ_z the one-dimensional length of the system [26]. Although it simply vanishes as $\ell_z \rightarrow \infty$, he argued that superfluidity would be observed at a much higher temperature than T_c , if $\omega \tau \gg 1$ is fulfilled, where ω is the frequency at which superfluidity is measured (e.g., the frequency of a torsional oscillator) and τ is the relaxation time of the supercurrent [26].

A more explicit argument was independently given by Machta and Guyer [27,28]. They proposed two different definitions of superfluid density. One is denoted by ρ_s , which is the coefficient of the increase in the free energy in the presence of a slow supercurrent, and the other by ρ_p , which is the coefficient of the increase in the free energy caused by an infinitesimal phase twist between both ends of the system. They found the relation

$$\rho_{\rm p}(T) \simeq 2L_{\rm eff} \frac{k_{\rm B}T}{J} \exp\left[-\frac{L_{\rm eff}}{2\rho_{\rm s}(T)} \frac{k_{\rm B}T}{J}\right], \qquad (2)$$

at temperature $T \gg J\rho_s/(k_B L_{eff})$, where $L_{eff} = \ell_z/(n_1a^2)$, $J = \hbar^2/(Ma^2)$, and *a* is the average interparticle distance [28]. The superfluid densities are normalized so that $\rho_{s,p}(T=0)=1$. Out of the two superfluid densities, $\rho_p(T)$ is affected by phase slippage and readily vanishes at $T \gtrsim J\rho_s/(k_B L_{eff}) \simeq \hbar^2 n_1/(k_B M \ell_z) \sim T_c$. In other words, it vanishes at any finite temperatures in the limit of $\ell_z \to \infty$. On the other hand, $\rho_s(T)$ does not suffer from phase slippage and can be finite at finite temperatures. They argued that it is $\rho_s(T)$ that is observed in torsional oscillator experiments, but did not discuss the explicit temperature dependence of $\rho_s(T)$. A similar relation between $\rho_s(T)$ and $\rho_p(T)$ was also derived by Prokof'ev and Svistunov [29]. It should also be noted that it is $\rho_p(T)$ that is obtained with calculations under the thermal equilibrium condition.

It is not trivial which one, $\rho_s(T)$ or $\rho_p(T)$, will be observed in a torsional oscillator experiment. If $\omega \tau \gg 1$ at low temperatures, it must be $\rho_s(T)$ that will be observed in an experiment as was suggested by Machta and Guyer [28]. However, τ is temperature dependent and should decrease as *T* increases. Therefore, $\rho_p(T)$ will be observed once $\omega \tau$ becomes small enough. In actual dynamical experiments, this kind of crossover will be observed.

Superfluid density in one-dimensional systems was also analyzed using the Tomonaga-Luttinger theory [30-32]. In particular, the dynamical aspect of superfluid behavior was discussed by Eggel *et al.* [32] and experimental results [14] were analyzed based on this theory. However, the theory is limited to low temperatures and the temperature range or the range of pore radius where the theory can be justified is not clear.

Quantum Monte Carlo simulations were also performed for liquid ⁴He confined in nanopores [33,34]. The results, in particular, those in narrower pores, R < 0.4 nm with Rbeing the radius of a pore, are successfully analyzed using the Tomonaga-Luttinger theory. However, for wider pores, R > 0.9 nm, the system length used in the simulations may not be long enough to study the quasi-one-dimensional cases.

Superfluid density in quasi-one-dimensional systems was also analyzed on the basis of classical spin models (*XY* models) [35,36]. Superfluid density $\rho_s(T)$ that is not affected by phase slippage in quasi-one dimension was calculated using a special boundary condition or a restricted sampling method [36]. It was then found that, without the effect of phase slippage, superfluid density can survive up to the transition temperature of the extended film or the bulk system even in the one-dimensional limit [36]. Although the main conclusion in Ref. [36] is expected to be also valid in quantum systems, it is highly desirable to demonstrate it explicitly in a quantum system. This is precisely the purpose of this paper.

In this study, we examine superfluid density of hard-core bosons on quasi-one-dimensional lattices using the quantum Monte Carlo method. As was done in Ref. [36], we calculate superfluid density $\rho_s(T)$ by modifying the model used in the calculation. In this study, we suppress phase slippage by introducing special transfer integrals. We then show that superfluid density can remain finite up to the BKT transition temperature T_{BKT} or the bulk transition temperature T_{λ} even in the one-dimensional limit when the effect of phase slippage is completely suppressed. The rest of this paper is organized as follows. Section II introduces a hard-core Bose-Hubbard model and modify it so that the phase slippage is prevented. In addition, we define the superfluid density for this modified model. Section III presents the results of the simulations, which clearly show that the superfluid density can be finite at high temperatures when the phase slippage is not allowed. Section IV summarizes this paper.

II. MODEL AND METHOD

A. Model without phase slippage

In order to study a quasi-one-dimensional system such as ⁴He atoms in nanopores, we consider hard-core bosons on an anisotropic square or cubic lattice described by the following Hamiltonian:

$$\mathcal{H} = -\sum_{\langle i,j \rangle} (t_{ij} b_i^{\dagger} b_j + \text{H.c.}), \qquad (3)$$

where b_i (b_i^{\dagger}) is the annihilation (creation) operator of a boson at site *i* and no multiple occupancy at the same site is allowed because of the strong repulsion between bosons, i.e., $b_i^{\dagger}b_i = 0$ or 1. For simplicity, we consider only the transfer integral t_{ij} between the nearest neighboring sites, and accordingly the sum in Eq. (3) runs over all the nearest neighboring sites $\langle i, j \rangle$. Furthermore, we do not consider the interaction between bosons at different sites. In this study, we set the boson density at half-filling, i.e., $N = 0.5N_L$, where N (N_L) is the total number of bosons (lattice sites), and thus the chemical potential μ is always zero. Note that this model can be mapped onto the spin S = 1/2 XY model with only the nearest-neighbor exchange interaction with no external magnetic field [37].

To simulate ⁴He atoms adsorbed on the inner walls of nanopores [4,6,7,11,13,17], we consider an anisotropic two-dimensional square lattice, i.e., a film, composed of $L_x \times L_z$ sites with $L_z \gg L_x$ (see Fig. 1). The periodic condition is imposed in both directions. We thus consider



FIG. 1. (a) Quasi-one-dimensional lattice system with the periodic boundary condition along the z-direction, forming a torus-like geometry. Phase slippage is suppressed in a spin model if the central hole of the torus is closed [36]. This is equivalent to setting the transfer integrals along a single row (indicated by the red line) to be infinite in the hard-core boson model studied here. The inner part is empty in the film case (b) and it is filled with lattice sites in the bar case (c).

hard-core bosons on a long tube, as schematically shown in Fig. 1(b). For a film, the effective length $L_{\text{eff}} = \ell_z/(n_1a^2) \sim L_z/L_x \sim \ell_z/\ell_x$, i.e., the aspect ratio of the anisotropic lattice, because $n_1 = N/\ell_z = N_L/(2\ell_z)$ and $L_\alpha \sim \ell_\alpha/a$ ($\alpha = x$ and z). Experimentally, the aspect ratio L_z/L_x can be estimated to be 15–50 [4,6,7,11,13,17]. On the other hand, to simulate ⁴He atoms filling nanopores, we consider an anisotropic three-dimensional cubic lattice, i.e., a bar, composed of $N_L = L_x L_y L_z$ sites with $L_z \gg L_x$, L_y [see Figs. 1(a) and 1(c)]. For a bar, the effective length L_{eff} can be estimated as $L_{\text{eff}} \sim L_z/(L_x L_y) \sim \ell_z a/(\ell_x \ell_y)$ and typically $L_z/(L_x L_y) =$ 5–30 [5,8–10,15]. In our simulations, the periodic boundary condition is imposed in the z-direction and the open boundary condition is applied in the remaining two directions.

The superfluid density $\rho_{\rm p}(T)$ in these systems vanishes at finite temperatures in the limit of $L_z \rightarrow \infty$ because of phase slippage. To calculate the superfluid density $\rho_s(T)$ that is not affected by phase slippage, a slight modification of the model is required. In a spin model, phase slippage is suppressed when all the L_z spins in a single row of the lattice (out of L_x or $L_x \times L_y$ rows) are replaced with a single spin [36]. By doing so, one can close the central hole of the torus as is shown in Fig. 1(a). This is equivalent to setting the exchange interaction in this row to be infinity. The exchange interaction in the XYmodel is mapped to the transfer integral in the hard-core boson system studied here. Thus, by setting the transfer integral in a single row of the lattice (out of L_x or $L_x \times L_y$ rows) along the z direction to be infinity, we can prohibit phase slippage. With this modification, the translational invariance along the transverse directions (x- or/and y-direction) is violated, but the system remains translationally invariant in the z-direction.

In the numerical calculations, we set the transfer integral t_{ij} along this single row to be t^* , which is much larger than the other transfer integral $t_{ij} = t$. Typically, we set $t^* \sim L_z t$ but we also investigate the dependence of the results on the choice of t^* . As is shown in the following, by calculating the superfluid density of the model with this transfer integral t^* , we can obtain the superfluid density $\rho_s(T)$ that is not affected by phase slippage.

B. Method: Superfluid density

To calculate superfluid density, we apply the world-line Monte Carlo method employing the directed-loop implementation [38–40] of the worm algorithm [41]. The well-known definition of the superfluid density [42] in a spatially homogeneous system is given by

$$\rho_s = \frac{\sum_{\alpha} \left\langle L_{\alpha}^2 W_{\alpha}^2 \right\rangle}{2t \,\beta N_{\rm L}},\tag{4}$$

where $L = (L_x, L_y, L_z)$ stands for the linear system size in a three-dimensional cubic lattice system, $W = (W_x, W_y, W_z)$ is the winding number in each spatial direction, $\langle \cdots \rangle =$ $\text{Tr}(e^{-\beta \mathcal{H}} \cdots)/\text{Tr}e^{-\beta \mathcal{H}}, \beta = 1/T$, and the Boltzmann constant k_B is set to be 1. The total number of lattice sites is $N_L =$ $L_x L_y L_z$. The hopping integral t in Eq. (4) is assumed to be uniform. When we calculate the winding number W_α ($\alpha =$ x, y, and z), we count the number of kinks of world lines that correspond to the hopping operator $b_i^{\dagger} b_j$. For example, the winding number in the z direction can be explicitly written as

$$L_z^2 W_z^2 = \left[\sum_{b_z} \left(n_{b_z}^+ - n_{b_z}^- \right) \right]^2,$$
(5)

where the summation of b_z runs over all bonds along the *z*-direction. The number of kinks of world lines on the b_z -th bond in the positive (negative) *z*-direction, $n_{b_z}^+(n_{b_z}^-)$, is given by $n_{b_z}^+ = b_{i+e_z}^{\dagger}b_i$ $(n_{b_z}^- = b_i^{\dagger}b_{i+e_z})$ with site $i + e_z$ being the nearest-neighbor site of site *i* in the positive *z* direction.

In the present system, the definition of superfluid density has to be generalized to take account of the non-uniform transfer integral [43]. Allowing for the bond-dependent transfer integral, the superfluid density in the z direction is given as

$$\rho_s^z = \frac{\langle L_z^2 \tilde{W}_z^2 \rangle}{2t\beta L_z (L_x^{d-1} - 1)},\tag{6}$$

where the normalized winding number is

$$L_{z}^{2}\tilde{W}_{z}^{2} = \left[t\sum_{b_{z}}\frac{\left(n_{b_{z}}^{+}-n_{b_{z}}^{-}\right)}{t_{b_{z}}}\right]^{2}$$
(7)

with $t_{b_z} = t^*$ along the bonds in the special row of the lattice (denoted by the red line in Fig. 1) and $t_{b_z} = t$ along the other bonds, and d = 2 (3) in the system of a film (bar) geometry. Here, we assume that $L_x = L_y$ in the system of the bar geometry.

III. RESULTS

A. Film: Anisotropic two-dimensional lattices

Figure 2 shows temperature dependence of the superfluid density in the *z* direction of hard-core bosons on an anisotropic two-dimensional lattice (i.e., a film) of different sizes. When the system is isotropic and large, that is, $L_x = L_z \gg 1$ and $t^* = t$, the superfluid density is found to vanish at $T \simeq 0.7t$ [see Fig. 2(c)], which is close to the known results $T_{\text{BKT}}/t = 0.68606$ in two dimensions [44]. In the one-dimensional limit, i.e., $L_z \gg L_x$, with $t^* = t$, the superfluid density vanishes at a much lower temperature than T_{BKT} , in agreement with the theoretical prediction [26,28,29] and the previous result for a classical model [35] [see the results for $N_{\text{L}} = 480 \times 8$ and 480×1 in Fig. 2(c)].

We then suppress phase slippage by setting $t^* = L_z t$ to find that the superfluid density remains finite at finite temperatures. In Fig. 2(a), L_x is fixed at $L_x = 8$ and the length L_z of the system is changed. It is observed that the temperature dependence hardly depends on L_z although the aspect ratio L_z/L_x significantly changes. This result clearly shows that the superfluid density remains finite up to $T \simeq t$ even in the onedimensional limit of $L_z \rightarrow \infty$ as long as the phase slippage is suppressed.

In Fig. 2(b), on the other hand, L_z is kept constant at $L_z = 480$ and L_x is varied. As L_x increases, the superfluid density is found to drop more sharply as a function of T. As L_x approaches to L_z , the result almost converges to that in the two dimensional case for $N_L = 480 \times 480$ with $t^* = t$ shown in Fig. 2(c). Figure 2(c) shows the superfluid density for a



FIG. 2. Superfluid density ρ_s^z along the z-direction in the film geometry of different lattice sizes ($N_L = L_z \times L_x$) with $t^* = tL_z$. (a) L_z is varied with keeping $L_x = 8$, (b) L_x is varied with keeping $L_z = 480$, and (c) L_x and L_z are varied with keeping the aspect ratio $L_z/L_x = 30$. For comparison, the results for $N_L = 8 \times 8$, 480×480 , 480×8 , and 480×1 with $t = t^*$ are also shown in (c). The dashed line in (c) represents $\rho_s^z = T/(\pi t)$. The BKT transition temperature T_{BKT} for the two-dimensional system determined previously by the quantum Monte Carlo method is $T_{\text{BKT}}/t = 0.68606$ [44].

fixed aspect ratio $L_z/L_x = 30$. We find that the temperature dependence of the superfluid density is very similar to that shown in Fig. 2(b).

These results clearly demonstrate that the temperature dependence of the superfluid density is similar for all cases on the two-dimensional, quasi-one-dimensional, or one-dimensional lattice, that is, the superfluid density remains finite at finite temperatures, provided that the phase slippage is prohibited. It is also noticed that the temperature dependence of the superfluid density is primarily determined by L_x .

In Fig. 2(c), the dashed line represents the universal jump of the superfluid density for the BKT transition [45]. The result for $L_z/L_x = 30$ appears to merge at $T \simeq 0.7t$ to the universal jump line, as the data for $L_x = L_y$ (i.e., $N_L = 8 \times 8$ and 480×480) do. This strongly suggests that the system undergoes a transition that belongs to the BKT universality class. In Sec. III D, we shall perform the scaling analysis to show that the transition is indeed the BKT transition and estimate the transition temperature.

B. Bar: Anisotropic three-dimensional lattices

Now, we study the superfluid density of hard-core bosons on an anisotropic cubic lattice composed of $N_{\rm L} = L_x \times L_y \times L_z$ sites with $L_z \gg L_x$, L_y . As in the case with the film geometry, the superfluid density rapidly diminishes at temperatures much smaller than the bulk transition temperature $T_\lambda \simeq 2t$



FIG. 3. Superfluid density ρ_s^z along the z-direction in the bar geometry of different lattice sizes $(N_L = L_z \times L_x \times L_y)$ with $t^* = L_z t$. (a) L_z is varied with keeping $L_x = L_y = 4$ and (b) $L_x = L_y$ is varied with keeping $L_z = 480$.

[46–50] when $L_z \gg L_x = L_y$ and $t^* = t$, although those results are not presented here.

Figure 3(a) shows the results of the superfluid density for different values of L_z with keeping $L_x = L_y = 4$ and $t^* =$ $L_z t$ to suppress the phase slippage. It is observed that the superfluid density is now survived up to the bulk transition temperature $T_{\lambda} \simeq 2t$ [46–50]. Interestingly, the results hardly depend on the value of L_z and remain intact even in the one-dimensional limit of $L_z \rightarrow \infty$. This is very similar to the results for the film case [see Fig. 2(a)]. Figure 3(b) shows the results for different values of $L_x = L_y$ with keeping $L_z =$ 480 and $t^* = L_z t$. As L_x increases, the superfluid density vanishes more steeply with T. However, it quickly converges in increasing L_x . These results shown in Figs. 3(a) and 3(b) imply, as in the case of the film geometry, that the superfluid density remains finite up to the bulk transition temperature even in the one-dimensional limit of $L_z \rightarrow \infty$, as long as the phase slippage is prohibited.

C. t*-dependence

Thus far, we have set $t^* = L_z t$ to suppress the effect of phase slippage. However, this value is chosen rather arbitrarily. Here, we examine the dependence of the superfluid density on the value of t^* and show that the results do not depend on the precise value of t^* as long as it is large enough (i.e., $t^* \gtrsim L_z t/8$ for $L_z = 480$).

Figure 4 shows the superfluid density for different values of t^* in the film geometry of $N_L = 480 \times 8$. As t^* increases, the superfluid density at low temperatures increases, because of the suppression of the phase slippage, and the results are essentially converged for $t^* \gtrsim L_z t/8$. This implies that the results obtained above are not the results for a particular value of t^* , but represent the characteristic behavior of the superfluid density in the systems where phase slippage is suppressed.

D. Finite size scaling for ρ_s^z

1. Film geometry

When the system size increases, the system ultimately reaches the thermodynamic limit irrespective of the shape of the system. For the film geometry with a fixed aspect ratio



FIG. 4. Superfluid density ρ_s^z for different values of t^* in the film geometry of $N_{\rm L} = 480 \times 8$ sites with $L_z = 480$ and $L_x = 8$.

of $R_{\text{film}} = L_z/L_x$, we can reach the thermodynamic limit of the two dimensional system as $L_z \to \infty$ even when $L_x \ll L_z$. However, it is known that ρ_s^z depends on the aspect ratio R_{film} , and the temperature where superfluidity sets in decreases from T_{BKT} for $R_{\text{film}} = 1$ with increasing R_{film} , when the phase slippage is not prohibited [44].

As shown in the previous section, if the phase slippage is suppressed, ρ_s^z can be clearly finite up to the temperature close to T_{BKT} for the isotropic two-dimensional lattice with $R_{\text{film}}=1$, even when $R_{\text{film}} \gg 1$ [see Fig. 2(c)]. Here, using the finite size scaling, we show that the transition in the absence of the phase slippage is indeed the BKT transition and estimate the transition temperature.

Assuming that the transition is the BKT transition, we can expect that the quantity

$$x = \frac{\pi}{2} \frac{2t\rho_s^2}{T} - 2,$$
 (8)

i.e., the deviation of the superfluid density from the universal value at $T = T_{BKT}$, satisfies the following finite size scaling equation:

$$x(T, L_z) = l^{-1} F((K - K_{\text{BKT}})l^2),$$
(9)

where K = t/T and $l = \ln(L_z/L_0)$ with L_0 being a phenomenological constant [51,52]. Figure 5 shows the scaling plot of $x(T, L_z)$ for the systems in the film geometry with a fixed value of $R_{\text{film}} = 30$ and $t^* = L_z t$. We employ the Baysian analysis [53,54] to find the best scaling function. It is clearly observed in Fig. 5 that the numerical data for different sizes collapse excellently onto a universal curve. The estimated values are $K_{\text{BKT}} = 1.49$ (2) and $\ln L_0 = -3$ (1). This confirms that the transition is indeed the BKT transition for the systems in the film geometry with no phase slippage allowed. The estimated transition temperature $T_{\text{BKT}}/t = 0.671$ (9) is to be compared with the value for the isotropic two-dimensional system, $T_{\text{BKT}}/t = 0.68606$ (16) [44], where the phase slippage is not prohibited.

2. Bar geometry

For anisotropic three-dimensional (3D) lattices (i.e., bars), the system also reaches the bulk limit when we increase L_x , L_y , and L_z with keeping its relative magnitude constant, $L_\alpha = c_\alpha L$ ($\alpha = x, y$, and z), where L and c_α are constant,



FIG. 5. Finite size scaling of superfluid density ρ_s^z in the film geometry with a fixed value of the aspect ratio $R_{\text{film}} = L_z/L_x = 30$ and $t^* = L_z t$ for $L_z = 240, 480, 720$, and 960. The black solid curve is the scaling function obtained by the Baysian scaling analysis [53,54].

even if $L_z \gg L_x$ and L_y . However, the effective length given by $L_{\text{eff}} \simeq L_z/(L_x L_y) = c_z/(c_x c_y L)$ becomes zero as $L \to \infty$. Therefore, this limit is rather trivial. In contrast, the limit of $L_z \rightarrow \infty$ with a constant $R_{\text{bar}} = L_z/(L_x L_y)$ is expected to be nontrivial and here we discuss the finite-size scaling for the bar systems in this limit. Figure 6 shows the results for the finite-size scaling of ρ_s^z with $R_{\text{bar}} = 10$. We obtained the transition temperature $T_c/t = 2.022(1)$ in this system when we use the known critical exponent v = 0.6717 of 3D-XY universality class yielded by classical Monte Carlo simulations[55,56]. The transition temperature T_c estimated here is shifted from $T_c/t = 2.0169$ (5) obtained for the half-filled three-dimensional hard-core boson model [50] but compared well. Therefore, our system in the bar geometry with fixed $R_{\rm bar}$, provided that phase slippage is prohibited, is not completely equivalent to the three-dimensional isotropic hardcore boson model but belongs to the three-dimensional XY universality class.



FIG. 6. Finite size scaling of superfluid density ρ_s^z in the bar geometry with a fixed value of the ratio $R_{\text{bar}} = L_z/(L_x L_y) = 10$ and $t^* = L_z t$ for $L_z = 360, 640, 1000$, and 1440. Using v = 0.6717, the transition temperature is estimated to be $T_c/t = 2.022(1)$.

IV. CONCLUSIONS

We have studied the superfluid density of the quasi-onedimensional hard-core boson system focusing on the effect of phase slippage. We have demonstrated that the phase slippage is suppressed by setting the large transfer integral between the neighboring sites along a single row of the system. We have successfully shown that superfluid density remains finite at high temperatures ($T_{\rm BKT}$ in the film geometry and T_{λ} in the bar geometry) even in the one-dimensional limit, $L_z \rightarrow \infty$, as long as the phase slippage is prohibited. In particular, we have found that the transition in the film geometry is the BKT transition if the phase slippage is suppressed.

Although we have shown that the superfluid density can be finite up to $T_{\rm BKT}$ or T_{λ} in a quasi-one-dimensional system, it does not necessarily mean that one always observes a finite superfluid density up to those temperatures in experiments. At very low temperatures, i.e., $T \ll T_{BKT}$ or T_{λ} , the relaxation time τ should be long enough compared with the inverse of the frequency ω at which the superfluidity is measured. Therefore $\omega \tau \gg 1$, and thus ρ_s is observed in a torsional oscillator experiment. As T increases, τ rapidly decreases and eventually $\omega \tau$ becomes much smaller than unity ($\omega \tau \ll 1$). In this case, it is ρ_p that is observed with a torsional oscillator. However, ρ_p readily vanishes at very low temperatures and thus $\rho_p \simeq 0$ at temperatures where $\omega \tau \ll 1$. Thus, in dynamical experiments, one observes a crossover from finite $\rho_{\rm s}$ to vanishing $\rho_{\rm p}$ (or vice versa) at a temperature where $\omega \tau \simeq 1$. The crossover temperature (or the onset temperature) $T_{\rm o}$ increases as ω increases. What we have found in this study is that the onset temperature would be $T_{\rm BKT}$ or T_{λ} in the limit of $\omega \tau \to \infty$. In other words, the upper limit of the onset temperature is T_{BKT} or T_{λ} in quasi-one dimensional systems. A crucial point to be emphasized is that the limiting value of the onset temperature remains to be $T_{\rm BKT}$ or T_{λ} even in the one-dimensional limit.

If the onset temperature T_0 is distant from T_{BKT} or T_{λ} , we expect to observe a two-step increase in the superfluid density in a torsional oscillator experiment. However, if the onset temperature is close the T_{BKT} or T_{λ} , it might be difficult to separate T_0 from T_{BKT} or T_{λ} . In a previous publication [36], it was argued that this difference might be the cause for a difference in observation in the film and bar geometries; $T_0 \ll T_{\lambda}$ in the bar geometry, but $T_0 \simeq T_{BKT}$ in the film geometry. A more detailed analysis of frequency dependence of the superfluid onset is required to clarify this point.

Now, two comments are in order on the direction of future study. First, it is desirable to extend the present calculation to a continuous system. It is not clear at all how we can suppress the phase slippage in a continuous system. A position-dependent mass might be a possible way to suppress the phase slippage. Next, it is desirable to calculate the superfluid density observed with a torsional oscillator in a given system directly under a non-equilibrium condition. For this, we further have to specify the microscopic mechanism of the dissipation of supercurrent, which determines τ , such as periodic or random potential caused by the substrate. However, currently, sufficient information is not available about such microscopic details of the system.

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