Ferromagnetism in the SU(n) Hubbard model with a nearly flat band

Kensuke Tamura [1,*] and Hosho Katsura [1,2]
[1] Department of Physics, Graduate School of Science, The University of Tokyo, 7-3-1 Hongo, Tokyo 113-0033, Japan
[2] Institute for Physics of Intelligence, The University of Tokyo, 7-3-1 Hongo, Tokyo 113-0033, Japan

(Received 12 September 2019; revised manuscript received 6 December 2019; published 20 December 2019)

We present rigorous results for the SU(n) Fermi-Hubbard model on the railroad-trestle lattice. We first study the model with a flat band at the bottom of the single-particle spectrum and prove that the ground states exhibit SU(n) ferromagnetism when the total fermion number is the same as the number of unit cells. We then perturb the model by adding extra hopping terms and make the flat band dispersive. Under the same filling condition, it is proved that the ground states of the perturbed model remain SU(n) ferromagnetic when the bottom band is nearly flat. This is the first rigorous example of the ferromagnetism in nonsingular SU(n) Hubbard models in which both the single-particle density of states and the on-site repulsive interaction are finite.

DOI: 10.1103/PhysRevB.100.214423

I. INTRODUCTION

Strongly correlated electron systems, in which the Coulomb interaction between electrons plays an essential role, can exhibit a variety of phenomena such as ferromagnetism, antiferromagnetism, and superconductivity. The Hubbard model has been introduced as a minimum model to describe such systems [1–3]. Despite its apparent simplicity, the intricate competition between the kinetic and the on-site Coulomb terms in the model is hard to deal with analytically. So far, exact and rigorous results have been mostly limited to one dimension [4] or systems with special hopping and filling [5–10].

Recently, it has become possible to simulate the Hubbard model using ultracold atoms in optical lattices [11–13]. Furthermore, it was proposed theoretically [14] and demonstrated experimentally [15] that multicomponent fermionic systems with SU(n) symmetry can be realized in cold-atom setups. These systems are well described by the SU(n) Fermi-Hubbard model, in which each atom carries n internal degrees of freedom. When $n = 2$, the model reduces to the original Hubbard model with spin-independent interaction. Although the SU(n) ($n > 2$) symmetry has been less explored in the condensed matter literature, there is a growing interest in recent years in studying the SU(n) Hubbard model theoretically. For example, it is argued that the SU(n) Hubbard model can exhibit exotic phases that do not appear in the SU(2) counterpart [14,16,17]. Besides, enlarged symmetry other than SU(n), such as SO(5), in higher spin systems has also been discussed [18].

The SU(n) Hubbard model is, in general, harder to theoretically study than the SU(2) Hubbard model. It has been reported that the Nagaoka ferromagnetism [19,20], which is the first rigorous result for the SU(2) Hubbard model, can be generalized to the case of SU(n) [21,22]. Flat-band ferromagnetism is another example of rigorous results for the SU(2) Hubbard model. Here, a flat band refers to a structure of single-particle energy spectrum which has a macroscopic degeneracy. A tight-binding model with a flat band can be constructed using standard methods such as the line graph [23] and the cell constructions [5]. In the SU(2) case, it is known that if the system has a flat band at the bottom of the single-particle spectrum and the particle number is the same as the number of unit cells, the ground state of the model exhibits ferromagnetism [5,6,8,24]. An SU(n) counterpart of the flat-band ferromagnetism has also been discussed recently [25].

In this paper, we consider the SU(n) Hubbard model on a one-dimensional (1D) lattice called the railroad-trestle lattice and derive rigorous results. We first treat the model with a flat band at the bottom and prove that the model exhibits SU(n) ferromagnetism in its ground states provided that the on-site interaction is repulsive and the total fermion number is the same as the number of unit cells. This is a slight generalization of the result obtained by Liu *et al.* in Ref. [25], in the sense that our hopping Hamiltonian has one more parameter. We then discuss SU(n) ferromagnetism in a perturbed model obtained by adding extra hopping terms that make the flat band dispersive. We prove that this particular perturbation leaves the SU(n) ferromagnetic ground states unchanged when the band width of the bottom band is sufficiently narrow. This is our main result and can be thought of as an SU(n) extension of the previous theorem for the SU(2) Hubbard model with nearly flat bands [26].

The rest of this paper is organized as follows. In Sec. II, we introduce the SU(n) Hubbard model with completely flat band and prove that its ground states exhibit SU(n) ferromagnetism. In Sec. III, we study a model with nearly flat band and prove that the ground states remain SU(n) ferromagnetic when the repulsive interaction and the band gap are sufficiently large. We present our conclusions in Sec. IV.

II. MODEL WITH COMPLETELY FLAT BAND

Let M be an arbitrary positive integer and $\Lambda = \{1, 2, \dots, 2M\}$ be a set of $2M$ sites on the railroad-trestle

*tamura-kensuke265@g.ecc.u-tokyo.ac.jp

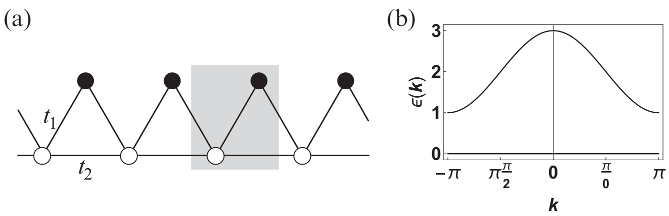

FIG. 1. (a) The railroad-trestle lattice with hopping amplitudes $t_1 = \nu t$ and $t_2 = \nu^2 t$. Odd (black) and even (white) sites have on-site potentials t and $2\nu^2 t$, respectively. The shaded region indicates the unit cell. (b) The energy bands for $t = 1$, $\nu = 1/\sqrt{2}$. The lowest band at zero energy is completely flat.

lattice [Fig. 1(a)]. We impose periodic boundary condition, so that site j and $j + 2M$ are identified. We denote by $\mathcal{E}$ and $\mathcal{O}$ subsets of Λ consisting of even sites and odd sites, respectively. We define creation and annihilation operators $c^\dagger_{x,\alpha}$ and $c_{x,\alpha}$ for a fermion at site $x \in \Lambda$ with color $\alpha = 1, \dots, n$. They satisfy $\{c_{x,\alpha}, c^\dagger_{y,\beta}\} = \delta_{\alpha,\beta}\delta_{x,y}$. The number operator of fermion at site x with color α is denoted by $n_{x,\alpha} = c^\dagger_{x,\alpha}c_{x,\alpha}$. We consider the SU($n$) Hubbard Hamiltonian

$$H_1 = H_{\rm hop} + H_{\rm int}, \tag{1}$$

$$H_{\rm hop} = \sum_{\alpha=1}^{n}\sum_{x,y\in\Lambda} t_{x,y} c^\dagger_{x,\alpha} c_{y,\alpha}, \tag{2}$$

$$H_{\rm int} = U \sum_{1\leqslant\alpha<\beta\leqslant n}\sum_{x\in\Lambda} n_{x,\alpha}n_{x,\beta}, \tag{3}$$

where $t_{2x-1,2x-1} = t$, $t_{2x,2x} = 2\nu^2 t$, $t_{2x-1,2x} = t_{2x,2x-1} = \nu t$, $t_{2x-2,2x} = t_{2x,2x-2} = \nu^2 t$, and the remaining elements of $t_{x,y}$ are zero [see Fig. 1(a)]. The parameters t, ν, and U are positive.

When $U = 0$, the model reduces a tight-binding model and we see that it has two bands with $\epsilon_1(k) = 0$, $\epsilon_2(k) = t(2\nu^2 + 1) + 2\nu^2 t \cos k$. Clearly, the lowest band is dispersionless, as shown in Fig. 1(b).

We define total number operators of fermion with color α and color raising and lowering operators as $F^{\alpha,\beta} = \sum_{x\in\Lambda} c^\dagger_{x,\alpha}c_{x,\beta}$. Since the Hamiltonian H_1 has SU(n) symmetry, they commute with H_1. We denote the eigenvalue of $F^{\alpha,\alpha}$ by N_α. Since $F^{\alpha,\alpha}$ commute with the Hamiltonian H_1, the eigenstates of H_1 are separated into different sectors labeled by $(N_1, \dots, N_n)$. If the total fermion number $N_{\rm f} = \sum_{x\in\Lambda}\sum_{\alpha=1}^{n} n_{x,\alpha}$ is fixed, N_α must satisfy $\sum_{\alpha=1}^{n} N_\alpha = N_{\rm f}$.

Now we define a new set of operators

$$a_{x,\alpha} := -\nu c_{x-1,\alpha} + c_{x,\alpha} - \nu c_{x+1,\alpha} \quad \text{for } x \in \mathcal{E}, \tag{4}$$

$$b_{x,\alpha} := \nu c_{x-1,\alpha} + c_{x,\alpha} + \nu c_{x+1,\alpha} \quad \text{for } x \in \mathcal{O}, \tag{5}$$

which satisfy

$$\{a_{x,\alpha}, b^\dagger_{y,\beta}\} = 0, \tag{6}$$

$$\{a_{x,\alpha}, a^\dagger_{y,\beta}\} = \begin{cases} \delta_{\alpha,\beta}(\nu^2+2) & \text{if } x = y, \\ \delta_{\alpha,\beta}\ \nu^2 & \text{if } x = y \pm 2, \\ 0 & \text{otherwise,} \end{cases} \tag{7}$$

$$\{b_{x,\alpha}, b^\dagger_{y,\beta}\} = \begin{cases} \delta_{\alpha,\beta}(\nu^2+2) & \text{if } x = y, \\ \delta_{\alpha,\beta}\ \nu^2 & \text{if } x = y \pm 2, \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

The hopping Hamiltonian $H_{\rm hop}$ is rewritten in terms of $b_{x,\alpha}$ and $b^\dagger_{x,\alpha}$ as

$$H_{\rm hop} = t\sum_{\alpha=1}^{n}\sum_{x\in\mathcal{O}} b^\dagger_{x,\alpha}b_{x,\alpha} \tag{9}$$

and hence positive semidefinite. The interaction term $H_{\rm int}$ is also positive semidefinite because $n_{x,\alpha}n_{x,\beta} = (c_{x,\alpha}c_{x,\beta})^\dagger c_{x,\alpha}c_{x,\beta}$. Therefore, the total Hamiltonian $H_1 = H_{\rm hop} + H_{\rm int}$ is positive semidefinite as well. From now on, we fix the total fermion number as $N_{\rm f} = |\mathcal{E}| = M$ and define a fully polarized state as $|\Phi_{{\rm all},\alpha}\rangle := \prod_{x\in\mathcal{E}} a^\dagger_{x,\alpha}|\Phi_{\rm vac}\rangle$, where $|\Phi_{\rm vac}\rangle$ is a vacuum state of $c_{x,\alpha}$. From the anticommutation relation (6), we find that $|\Phi_{{\rm all},\alpha}\rangle$ is an eigenstate of H_1 with eigenvalue zero. Since $H_1 \geqslant 0$, the fully polarized states are ground states of H_1. Because of the SU(n) symmetry, one obtains a general form of degenerate ground states as

$$\left|\Phi_{N_1,\dots,N_n}\right\rangle = (F^{n,1})^{N_n}\dots(F^{2,1})^{N_2}|\Phi_{{\rm all},1}\rangle, \tag{10}$$

where $N_1 = M - \sum_{\alpha=2}^{n} N_\alpha$. We also refer to states of the form Eq. (10) as fully polarized states [27].

The first result of this paper is the following.

Theorem 1. Consider the Hubbard Hamiltonian (1) with the total fermion number $N_{\rm f} = M$. For arbitrary $t > 0$ and $U > 0$, the ground states of the Hamiltonian (1) are the fully polarized states and unique apart from trivial degeneracy due to the SU(n) symmetry.

Proof of Theorem 1. Let $|\Phi_{\rm GS}\rangle$ be an arbitrary ground state of H_1 with $N_{\rm f} = M$. Since the ground-state energy is zero, we have $H_1|\Phi_{\rm GS}\rangle = 0$. The inequalities $H_{\rm hop} \geqslant 0$ and $H_{\rm int} \geqslant 0$ imply that $H_{\rm hop}|\Phi_{\rm GS}\rangle = 0$ and $H_{\rm int}|\Phi_{\rm GS}\rangle = 0$, which means that

$$b_{x,\alpha}|\Phi_{\rm GS}\rangle = 0 \ \text{ for any } x \in \mathcal{O} \text{ and } \alpha = 1, \dots, n, \tag{11}$$

$$c_{x,\alpha}c_{x,\beta}|\Phi_{\rm GS}\rangle = 0 \ \text{ for any } x \in \Lambda \text{ and } \alpha \neq \beta. \tag{12}$$

Since $a_{x,\alpha}$ and $b_{x,\alpha}$ obey the anticommutation relation (6), the condition (11) implies that $|\Phi_{\rm GS}\rangle$ does not contain any $b^\dagger_{x,\alpha}$ operator when it is constructed by acting with creation operators on the vacuum state. Therefore, it is written as

$$|\Phi_{\rm GS}\rangle = \sum_{\substack{A_1,A_2,\dots A_n\subset\mathcal{E} \\ \sum_{\alpha=1}^{n}|A_\alpha|=M}} f(\{A_\alpha\})\left(\prod_{x\in A_1} a^\dagger_{x,1}\right)\dots\left(\prod_{x\in A_n} a^\dagger_{x,n}\right)\Phi_{\rm vac}\rangle, \tag{13}$$

where A_α is a subset of $\mathcal{E}$ and $f(\{A_\alpha\})$ is a certain coefficient.

Next, we make use of the condition (12). We take an even site $x \in \mathcal{E}$. Using the anticommutation relation $\{c_{x,\alpha}, a^\dagger_{y,\beta}\} = \delta_{\alpha,\beta}\delta_{x,y}$ and Eq. (12), we see that $f(\{A_\alpha\}) = 0$ if there exist A_α and A_β such that $A_\alpha \cap A_\beta \neq \emptyset$. Since $\sum_{\alpha=1}^{n}|A_\alpha| = M$ and $A_\alpha \cap A_\beta = \emptyset$ for $\alpha \neq \beta$, we find that $\cup_{\alpha=1}^{n} A_\alpha = \mathcal{E}$. This means that the ground state is rewritten as

$$|\Phi_{\rm GS}\rangle = \sum_{\boldsymbol{\alpha}} C(\boldsymbol{\alpha})\left(\prod_{x\in\mathcal{E}} a^\dagger_{x,\alpha_x}\right)|\Phi_{\rm vac}\rangle, \tag{14}$$

where the sum is over all possible color configurations $\boldsymbol{\alpha} = (\alpha_x)_{x\in\mathcal{E}}$ with $\alpha_x = 1,\dots,n$. Then, we consider the condition (12) for $x\in\mathcal{O}$. By using

$$\{c_{x,\alpha}, a^\dagger_{y,\beta}\} = \begin{cases} -\nu\delta_{\alpha,\beta} & \text{if } y = x\pm 1, \\ 0 & \text{otherwise,}\end{cases} \tag{15}$$

we get

$$c_{x,\alpha}c_{x,\beta}|\Phi_{\rm GS}\rangle = \sum_{\substack{\boldsymbol{\alpha}\\ \text{s.t.}\alpha_p=\beta,\\ \alpha_q=\alpha}} \nu^2[C(\boldsymbol{\alpha}) - C(\boldsymbol{\alpha}_{p\leftrightarrow q})]\left(\prod_{y\in\mathcal{E}\setminus\{x\pm1\}} a^\dagger_{y,\alpha_y}\right)|\Phi_{\rm vac}\rangle, \tag{16}$$

where $p = x-1$ and $q = x+1$. The color configuration $\boldsymbol{\alpha}_{p\leftrightarrow q}$ is obtained from $\boldsymbol{\alpha}$ by swapping α_p and α_q. Since all the states in the sum are linearly independent, we find from the condition (12) that $C(\boldsymbol{\alpha}) = C(\boldsymbol{\alpha}_{p\leftrightarrow q})$ for all $\boldsymbol{\alpha}$ and all $x\in\mathcal{O}$. As the two localized states on neighboring even sites share an odd site between them, we see that

$$C(\boldsymbol{\alpha}) = C(\boldsymbol{\alpha}_{x\leftrightarrow y}), \tag{17}$$

where x, y are arbitrary different sites in $\mathcal{E}$.

To show that states satisfying Eq. (17) are the fully polarized states, i.e., SU(n) ferromagnetic, we introduce a concept of a word [28]. A *word* $w = (w_1,\dots,w_M)$ is a sequence of integers where $w_i\in\{1,\dots,n\}$ for all i. We denote by $|w|_\alpha$ the number of occurences of α in w. We define the set of words for which $|w|_\alpha = N_\alpha$ holds as follows: $W(N_1,\dots,N_n) = \{w|\ |w|_\alpha = N_\alpha,\ \alpha = 1,\dots,n\}$. For example, $W(2,0,1)$ consists of (1, 1, 3), (1, 3, 1), and (3, 1, 1). It follows from Eq. (17) that the ground state of H_1 in the sector labeled by $(N_1,\dots,N_n)$ can be written as

$$|\widetilde{\Phi}_{N_1,\dots,N_n}\rangle = \sum_{w\in W(N_1,\dots,N_n)} a^\dagger_{2,w_1}a^\dagger_{4,w_2}\dots a^\dagger_{2M,w_M}|\Phi_{\rm vac}\rangle. \tag{18}$$

Now using commutation relations $[F^{\beta,\alpha}, a^\dagger_{x,\gamma}] = \delta_{\alpha,\gamma}a^\dagger_{x,\beta}$ for all $x\in\mathcal{E}$, we see that

$$(F^{2,1})^{N_2}|\Phi_{{\rm all},1}\rangle = \sum_{w\in W(M-N_2,N_2)} a^\dagger_{2,w_1}a^\dagger_{4,w_2}\dots a^\dagger_{2M,w_M}|\Phi_{\rm vac}\rangle. \tag{19}$$

By repeating the procedure, we have the desired result $|\widetilde{\Phi}_{N_1,\dots,N_n}\rangle = |\Phi_{N_1,\dots,N_n}\rangle$. This proves that the ground states of H_1 are fully polarized states. ■

III. MODEL WITH NEARLY FLAT BAND

So far, we have considered the flat-band model, but this is an idealized case in which the lowest energy band becomes completely dispersionless. As a more realistic model, we consider a model with nearly flat band by adding a perturbation to the model in the previous section [29]. Here we define another Hubbard model on the same lattice as Theorem 1:

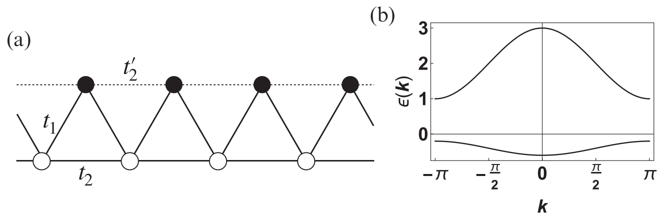

FIG. 2. (a) The lattice geometry of $H'_{\rm hop}$. The hopping amplitudes are given by $t_1 = \nu(t+s)$, $t_2 = \nu^2 t$, and $t'_2 = -\nu^2 s$. Odd (black) and even (white) sites have on-site potentials $t - 2\nu^2 s$ and $-s + 2\nu^2 t$, respectively. The corresponding energy bands are shown in panel (b) for $t = 1$, $\nu = 1/\sqrt{2}$, $s = 1/10$.

$$H_2 = H'_{\rm hop} + H_{\rm int}, \tag{20}$$

where $H'_{\rm hop}$ is defined as

$$H'_{\rm hop} = -s\sum_{\alpha=1}^n\sum_{x\in\mathcal{E}} a^\dagger_{x,\alpha}a_{x,\alpha} + t\sum_{\alpha=1}^n\sum_{x\in\mathcal{O}} b^\dagger_{x,\alpha}b_{x,\alpha}, \tag{21}$$

and $H_{\rm int}$ is defined in Eq. (3) with parameters $s, t, U > 0$. When the hopping Hamiltonian $H'_{\rm hop}$ is written in terms of original fermion operator $c_{x,\alpha}$, it takes the form $H'_{\rm hop} = \sum_\alpha\sum_{x,y\in\Lambda} t'_{xy}c^\dagger_{x,\alpha}c_{y,\alpha}$, where $t'_{2x-1,2x-1} = t - 2\nu^2 s$, $t'_{2x,2x} = -s + 2\nu^2 t$, $t'_{2x-1,2x} = t'_{2x,2x-1} = \nu(t+s)$, $t'_{2x-2,2x} = t'_{2x,2x-2} = \nu^2 t$, $t'_{2x-1,2x+1} = t'_{2x+1,2x-1} = -\nu^2 s$, and the remaining elements of $t'_{x,y}$ are zero. When we consider the single-particle problem, we obtain two bands with $\epsilon_1(k) = -s(2\nu^2+1) - 2\nu^2 s\cos k$, $\epsilon_2(k) = t(2\nu^2+1) + 2\nu^2 t\cos k$ [see Fig. 2(b)]. We see that the lowest band is no longer flat; however, it can be regarded as a nearly flat band when s is small enough. We focus on this model in the following and prove a theorem on the ferromagnetism.

Theorem 2. Consider the Hamiltonian (20) with the total fermion number $N_{\rm f} = M$. For sufficiently large $t/s > 0$ and $U/s > 0$, the ground states are the fully polarized states and unique apart from the trivial degeneracy due to the SU(n) symmetry.

Proof of Theorem 2. First, we decompose the Hamiltonian (20) into the sum of local Hamiltonians as

$$H_2 = -sM(2\nu^2+1) + \lambda H_{\rm flat} + \sum_{x\in\mathcal{E}} h_x, \tag{22}$$

where

$$H_{\rm flat} = \sum_{\alpha=1}^n\sum_{x\in\mathcal{O}} b^\dagger_{x,\alpha}b_{x,\alpha} + \sum_{x\in\Lambda}\sum_{\alpha<\beta} n_{x,\alpha}n_{x,\beta} \tag{23}$$

and

$$\begin{aligned} h_x = &\sum_{\alpha=1}^n\left[-sa^\dagger_{x,\alpha}a_{x,\alpha} + \frac{t-\lambda}{2}(b^\dagger_{x-1,\alpha}b_{x-1,\alpha} + b^\dagger_{x+1,\alpha}b_{x+1,\alpha})\right] \\ &+ \frac{\kappa(U-\lambda)}{4}n_{x-2}(n_{x-2}-1) + \frac{U-\lambda}{4}n_{x-1}(n_{x-1}-1) \\ &+ \frac{(1-\kappa)(U-\lambda)}{2}n_x(n_x-1) + \frac{U-\lambda}{4}n_{x+1}(n_{x+1}-1) \\ &+ \frac{\kappa(U-\lambda)}{4}n_{x+2}(n_{x+2}-1) + s(2\nu^2+1), \end{aligned} \tag{24}$$

where n_x is defined as $n_x = \sum_\alpha n_{x,\alpha}$. The two parameters λ and κ satisfy $0 < \lambda < \min\{t, U\}$ and $0 \leqslant \kappa < 1$. To prove Theorem 2, we use the following lemmas.

Lemma 1. Suppose the local Hamiltonian h_x is positive semidefinite for any $x \in \mathcal{E}$. Then the ground states of the Hamiltonian (22), and hence Eq. (20), are fully polarized states and unique apart from the trivial degeneracy due to the SU(n) symmetry.

Lemma 2. Suppose that t, U are infinitely large and $0 < \kappa < 1$. Then the local Hamiltonian (24) is positive semidefinite. (We take λ and κ to be proportional to s.)

We note that h_x can be regarded as a finite-dimensional matrix independent of the system size since the local Hamiltonian h_x acts nontrivially only on a finite number of sites. This means that the energy levels of h_x depend continuously on the parameters. Therefore, Lemma 2 guarantees that h_x is positive semidefinite when t, U are finite but sufficiently large. Then Lemma 1 implies that the ground states of the Hamiltonian (20) are fully polarized states, which proves Theorem 2. ■

Below, we prove Lemmas 1 and 2.

Proof of Lemma 1. First, it is noted that a fully polarized state $|\Phi_{\text{all},1}\rangle = (\prod_{x\in\mathcal{E}} a^\dagger_{x,1})|\Phi_{\text{vac}}\rangle$ satisfies $h_x|\Phi_{\text{all},1}\rangle = 0$ for each h_x. Since h_x is SU(n) invariant, all fully polarized states have zero energy. We assume that $h_x \geqslant 0$ for all $x \in \mathcal{E}$. Let $|\Phi^{\text{flat}}_{\text{GS}}\rangle$ be an arbitrary ground state of H_{flat}. Since $H_{\text{flat}}|\Phi^{\text{flat}}_{\text{GS}}\rangle = 0$ and $h_x|\Phi^{\text{flat}}_{\text{GS}}\rangle = 0$, we see that $H_2|\Phi^{\text{flat}}_{\text{GS}}\rangle = -sM(2\nu^2+1)|\Phi^{\text{flat}}_{\text{GS}}\rangle$. From $h_x \geqslant 0$, the ground energy of H_2 is $-sM(2\nu^2+1)$. If $|\Phi_{\text{GS}}\rangle$ is an arbitrary ground state of H_2, it satisfies $H_2|\Phi_{\text{GS}}\rangle = -sM(2\nu^2+1)$. From $H_{\text{flat}} \geqslant 0$ and $h_x \geqslant 0$, we find $H_{\text{flat}}|\Phi_{\text{GS}}\rangle = 0$ and $h_x|\Phi_{\text{GS}}\rangle = 0$. This shows that any ground state of H_2 must be a ground state of H_{flat}. The Hamiltonian H_{flat} is nothing but the Hamiltonian H_1 with $t = U = 1$. Thus, the ground states of H_2 are fully polarized and unique. ■

We remark that one can check whether h_x is positive semidefinite by numerically diagonalizing a finite-dimensional matrix. The result for the SU(4) case is shown in Fig. 3.

Proof of Lemma 2. Because of the translational invariance, it suffices to show the case for h_0. The local Hamiltonian h_0 is regarded as an operator defined on five sites $\{-2, -1, 0, 1, 2\}$, where $x = -2, -1$, and 0 are identified with $x = 2M - 2, 2M - 1$, and $2M$, respectively. On these sites, we define operators

$$\tilde{a}_{-2,\alpha} := \frac{1}{\sqrt{\nu^2+1}}(c_{-2,\alpha} - \nu c_{-1,\alpha}), \tag{25}$$

$$\tilde{b}_{-1,\alpha} := \nu c_{-2,\alpha} + c_{-1,\alpha} + \nu c_{0,\alpha}, \tag{26}$$

$$\tilde{a}_{0,\alpha} := -\nu c_{-1,\alpha} + c_{0,\alpha} + -\nu c_{1,\alpha}, \tag{27}$$

$$\tilde{b}_{1,\alpha} := \nu c_{0,\alpha} + c_{1,\alpha} + \nu c_{2,\alpha}, \tag{28}$$

$$\tilde{a}_{2,\alpha} := \frac{1}{\sqrt{\nu^2+1}}(-\nu c_{1,\alpha} + c_{2,\alpha}). \tag{29}$$

These operators satisfy

$$\{\tilde{a}_{y,\alpha}, \tilde{a}^\dagger_{y',\beta}\} = \begin{cases} \delta_{\alpha,\beta}(2\nu^2+1) & \text{for } y = y' = 0, \\ \delta_{\alpha,\beta}\frac{\nu^2}{\sqrt{\nu^2+1}} & \text{for } |y-y'| = 2, \\ \delta_{\alpha,\beta}\delta_{y,y'} & \text{for } y, y' = \pm 2, \end{cases} \tag{30}$$

$$\{\tilde{a}_{y,\alpha}, \tilde{b}^\dagger_{y',\alpha}\} = 0. \tag{31}$$

Single-fermion states corresponding to these operators are linearly independent. To show the lemma, we only need to consider states $|\Phi\rangle$ which have finite energy in this limit, i.e., $\lim_{t,U\to\infty}\langle\Phi|\,h_0\,|\Phi\rangle < \infty$. The condition that $|\Phi\rangle$ has finite energy is equivalent to the following:

$$\tilde{b}_{y,\alpha}|\Phi\rangle = 0 \ \text{ for } y = \pm 1, \tag{32}$$

$$c_{y,\alpha}c_{y,\beta}|\Phi\rangle = 0 \ \text{ for } y = 0, \pm 1, \pm 2. \tag{33}$$

Let $|\Phi\rangle$ be a state which has finite energy. From Eqs. (32) and (33) with $y = -2, 0, 2$, $|\Phi\rangle$ is written as

$$|\Phi\rangle = \sum_{\substack{A_1,\ldots,A_n\subset\tilde{\mathcal{E}} \\ A_\alpha\cap A_\beta=\emptyset}} f(\{A_\alpha\})\left(\prod_{y\in A_1}\tilde{a}^\dagger_{y,1}\right)\cdots\left(\prod_{y\in A_n}\tilde{a}^\dagger_{y,n}\right)|\Phi_{\text{vac}}\rangle, \tag{34}$$

where $\tilde{\mathcal{E}} = \{-2, 0, 2\}$ and A_α is an arbitrary subset of $\tilde{\mathcal{E}}$. Since $\tilde{\mathcal{E}}$ contains three sites, the particle number of finite-energy states must be less than or equal to three. Using the condition Eq. (33) with $y = \pm 1$, we see that all the finite-energy states $|\Phi\rangle$ must be a fully polarized state over the five sites and have zero energy when the particle number is three. For one-particle states, all the eigenvalues of h_0 are nonnegative. Thus, we only need to verify the positive semidefiniteness for two-particle sectors labeled by $(N_\alpha, N_\beta) = (2, 0)$ and $(N_\alpha, N_\beta) = (1, 1)$. To this end, we solve the eigenvalue problem for Ph_0P, where P denotes the projection operator onto the space of finite-energy states. In the sector $(2, 0)$, we find that there are three eigenstates

$$|\Phi_1\rangle = \tilde{a}^\dagger_{-2,\alpha}\tilde{a}^\dagger_{0,\alpha}|\Phi_{\text{vac}}\rangle, \tag{35}$$

$$|\Phi_2\rangle = \tilde{a}^\dagger_{0,\alpha}\tilde{a}^\dagger_{2,\alpha}|\Phi_{\text{vac}}\rangle, \tag{36}$$

$$|\Phi_3\rangle = \left[\frac{\nu^2}{\nu^2+1}(\tilde{a}^\dagger_{-2,\alpha} - \tilde{a}^\dagger_{2,\alpha})\tilde{a}^\dagger_{0,\alpha} - (2\nu^2+1)\tilde{a}^\dagger_{-2,\alpha}\tilde{a}^\dagger_{2,\alpha}\right]|\Phi_{\text{vac}}\rangle \tag{37}$$

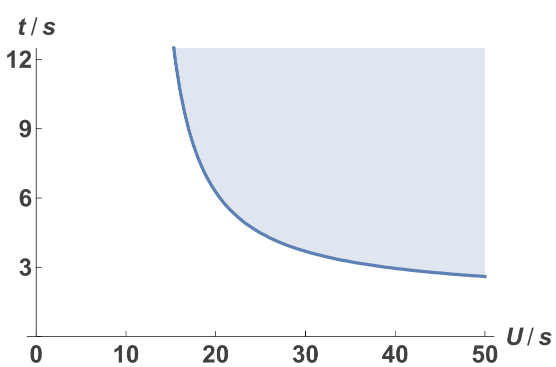

FIG. 3. The positive semidefiniteness of $h_x(n = 4)$ holds in the shaded region for $\nu = 1/\sqrt{2}$, $\kappa = 0$. The plot is obtained by diagonalizing h_x numerically. Lemma 1 says that the ground states of the full Hamiltonian H_2 are fully polarized states in the shaded region. For example, the ferromagnetism is established if $t/s \geqslant 4.5$ when $U/s = 25$.

and their corresponding eigenenergies are 0, 0, and $s(2\nu^2+1)$, respectively. In the sector (1,1), there are four eigenstates. Three of them can be obtained by applying $F^{\beta,\alpha}$ to the states $|\Phi_1\rangle$, $|\Phi_2\rangle$, and $|\Phi_3\rangle$. As a state orthogonal to them, we get a singlet state

$$|\Phi_4\rangle = (\tilde{a}^\dagger_{-2,\alpha}\tilde{a}^\dagger_{2,\beta} - \tilde{a}^\dagger_{-2,\beta}\tilde{a}^\dagger_{2,\alpha})|\Phi_{\rm vac}\rangle, \tag{38}$$

and we find that this state satisfies

$$Ph_0P\,|\Phi_4\rangle = s(2\nu^2 + 1)|\Phi_4\rangle. \tag{39}$$

Clearly, the state $|\Phi_4\rangle$ has a positive energy. Hence, we see that all the eigenvalues of h_0 are non-negative. Thus, we have proved Lemma 2. ■

IV. CONCLUSION

We have presented an extension of flat-band ferromagnetism to the SU(n) Hubbard model on the railroad-trestle lattice. Furthermore, we proved that in the nearly flat-band case, all the ground states are fully polarized if t and U are sufficiently large. One can similarly construct and analyze models in higher dimensions, in which the ground states are fully polarized if the lowest band is completely flat. The previous results for the SU(2) Hubbard models in higher dimensions suggest that the parameter ν has to be larger than a threshold value $\nu_c > 0$ when the lowest band is nearly flat [30,31]. The details will be discussed elsewhere.

Although we have focused on models with a nonzero band gap, it would be interesting to see if the method developed in this paper can be extended to include SU(n) Hubbard models with gapless flat or nearly flat bands [32]. It would also be interesting to study SU(n) ferromagnetism in systems with topological flat bands carrying nontrivial Chern number, as its SU(2) counterpart has been discussed in Ref. [33]. Another direction for future research is to explore ferromagnetism in multiorbital Hubbard models, including the one with SU(n) symmetry. In such systems, rigorous [34,35] and numerical results [36] about ferromagnetism, which are different from the flat-band scenario, have been obtained recently. It is thus interesting to see to what extent our results can be generalized to the multiorbital case.

ACKNOWLEDGMENTS

We would like to thank H. Tasaki and A. Tanaka for valuable discussions. H.K. was supported in part by JSPS Grant-in-Aid for Scientific Research on Innovative Areas No. JP18H04478 and JSPS KAKENHI Grant No. JP18K03445.

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